STRONGLY $\mathscr{C}$-COHERENT RINGS

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Let R be a ring and $\mathscr{C}$ be a class of some finitely presented left R-modules. R is called left strongly $\mathscr{C}$-coherent, if whenever $0 \to K \to P \to C \to 0$ is exact, where $C \in \mathscr{C}$ and P is finitely generated projective, then K is $\mathscr{C}$-projective. Characterizations of left strongly $\mathscr{C}$-coherent rings are given, $\mathscr{C}$-injective (resp., $\mathscr{C}$-projective, $\mathscr{C}$-flat) dimensions of modules over left strongly $\mathscr{C}$-coherent rings are studied, conditions under which left strongly $\mathscr{C}$-coherent rings are left $\mathscr{C}$-semihereditary are given.

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1. INTRODUCTION

Recall that a ring R is said to be *left coherent* [2] if every finitely generated left ideal of R is finitely presented. Coherent rings and their generalizations have been studied extensively by many authors. For example, Costa introduced the concept of *left n-coherent rings* in [3]. Let n be a nonnegative integer, then following [3], a ring R is said to be left n-coherent if every n-presented left R-module is $(n+1)$-presented, where a left R-module M is said to be n-presented in case there is an exact sequence of left R-modules $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ in which every F_i is a finitely generated free, equivalently projective left R-module. It is easy to see that a ring R is left 0-coherent if and only if R is left noetherian, R is left 1-coherent if and only if R is left coherent. We note that in some literatures such as [4, 9], Dobbs etc. called a ring R left n-coherent if every $(n-1)$-presented left ideal of R is n-presented, and they called R left strong n-coherent if every n-presented left R-modules is $(n+1)$-presented. Clearly, Dobbs's "strong n-coherent" implies Dobbs's "n-coherent", and Dobbs's "strong n-coherent" coincides with Costa's "n-coherent". We remark that the terminology of "n-coherence" in this paper is Costa's "n-coherence" but is not the same as that of Dobbs's.

In [18], we introduced a new generalization for left coherent rings. Let $\mathscr{C}$ be a class of some finitely presented left R-modules. Following [18], a ring

R is called *left $\mathscr{C}$-coherent* if every $C \in \mathscr{C}$ is 2-presented. We recall also that a left R-module M is called *$\mathscr{C}$-injective* [18] if $\mathrm{Ext}^1_R(C, M) = 0$ for every $C \in \mathscr{C}$; a right R-module M is called *$\mathscr{C}$-flat* [18] if $\mathrm{Tor}^R_1(M, C) = 0$ for every $C \in \mathscr{C}$. In this article, we will call a left R-module M $\mathscr{C}$-projective if $\mathrm{Ext}^1_R(M, N) = 0$ for any $\mathscr{C}$-injective module N, and we will call a ring R *left strongly $\mathscr{C}$-coherent*, if whenever $0 \to K \to P \to C \to 0$ is exact, where $C \in \mathscr{C}$ and P is finitely generated projective, then K is $\mathscr{C}$-projective. We will give some characterizations and properties of $\mathscr{C}$-projective modules and left strongly $\mathscr{C}$-coherent rings, left strongly $\mathscr{C}$-coherent rings will be characterized by $\mathscr{C}$-injective modules, $\mathscr{C}$-projective modules and $\mathscr{C}$-flat modules. Furthermore, we define $\mathscr{C}$-injective dimensions, $\mathscr{C}$-projective dimensions and $\mathscr{C}$-flat dimensions of modules, we will show that over a left strongly $\mathscr{C}$-coherent ring, the three classes of dimensions of modules have some nice properties.

Following [18], a ring R is called *left $\mathscr{C}$-semihereditary*, if whenever $0 \to K \to P \to C \to 0$ is exact, where $C \in \mathscr{C}$, P is finitely generated projective, then K is projective. Following [1], a pair $(\mathscr{A}, \mathscr{B})$ of classes of R-modules is called a *cotorsion pair* if $\mathscr{A}^{\perp} = \mathscr{B}$ and ${}^{\perp}\mathscr{B} = \mathscr{A}$. We note that a cotorsion pair is also called a cotorsion theory in some literatures such as [6, 15]. A cotorsion pair $(\mathscr{A}, \mathscr{B})$ is called *hereditary* [7, Definition 1.1] if whenever $0 \to A' \to A \to A'' \to 0$ is exact with $A, A'' \in \mathscr{A}$ then A' is also in $\mathscr{A}$. By [7, Proposition 1.2], a cotorsion pair $(\mathscr{A}, \mathscr{B})$ is hereditary if and only if whenever $0 \to B' \to B \to B'' \to 0$ is exact with $B', B \in \mathscr{B}$ then B'' is also in $\mathscr{B}$. In this paper, we will give some new characterizations of left $\mathscr{C}$-semihereditary rings.

Throughout this paper, R is an associative ring with identity and all modules considered are unitary, $\mathscr{C}$ is a class of some finitely presented left R-modules. For any R-module M, $M^+ = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ will be the character module of M. Given a class $\mathscr{L}$ of R-modules, we will denote by $\mathscr{L}^{\perp} = \{M : \mathrm{Ext}^1_R(L, M) = 0, L \in \mathscr{L}\}$ the right orthogonal class of $\mathscr{L}$, and by ${}^{\perp}\mathscr{L} = \{M : \mathrm{Ext}^1_R(M, L) = 0, L \in \mathscr{L}\}$ the left orthogonal class of $\mathscr{L}$. $\mathscr{C}\mathscr{I}$, $\mathscr{C}\mathscr{F}$, $\mathscr{F}\mathscr{P}\mathscr{I}$, $\mathscr{F}$ will denote the class of all $\mathscr{C}$-injective modules, $\mathscr{C}$-flat modules, FP-injective left R-modules and flat right R-modules, respectively.

2. STRONGLY $\mathscr{C}$-COHERENT RINGS

Recall that a left R-module M is said to be *P-injective* [13] if $\mathrm{Ext}^1_R(R/Ra, M) = 0$ for any $a \in R$; a left R-module N is said to be *P-projective* [16] if $\mathrm{Ext}^1_R(N, M) = 0$ for any P-injective left R-module M; a left R-module M is said to be *FP-injective* [14] if $\mathrm{Ext}^1_R(V, M) = 0$ for any finitely presented left R-module V; a left R-module N is said to be *FP-projective* [11] if $\mathrm{Ext}^1_R(N, M) = 0$

for any FP-injective left R-module M. We extend the concepts of P-projective modules and FP-projective modules as follows.

Definition 1. Let R be a ring and $\mathscr{C}$ be a class of some finitely presented left R-modules. Then a left R-module M is called $\mathscr{C}$-projective if $\mathrm{Ext}^1_R(M,N)=0$ for any $\mathscr{C}$-injective module N.

We will denote the class of $\mathscr{C}$-projective modules by $\mathscr{C}\mathcal{P}$. The class of $\mathscr{C}$-projective modules is known in the sense that it is exactly the class ${}^{\perp}(\mathscr{C}^{\perp})$ for a class $\mathscr{C}$ of finitely presented modules, which is the first class of the cotorsion pair cogenerated by $\mathscr{C}$ (*e.g.* [11, Def. 1.10]). Clearly, the name $\mathscr{C}$-projective was inspired by the work of Mao and Ding [11] on FP-projective modules.

Definition 2. A ring R is called left strongly $\mathscr{C}$-coherent, if whenever $0\to K\to P\to C\to 0$ is exact, where $C\in\mathscr{C}$ and P is finitely generated projective, then K is $\mathscr{C}$-projective.

THEOREM 1. *The following are equivalent for a ring R:*

(1) *R is left strongly $\mathscr{C}$-coherent.*

(2) *If $C\in\mathscr{C}$, then there exists an exact sequence of left R-modules $0\to K\to P\to C\to 0$, where P is finitely generated projective, and K is $\mathscr{C}$-projective.*

(3) *If $C\in\mathscr{C}$, then there exists an exact sequence of left R-modules $0\to K\to P\to C\to 0$, where P is projective, and K is $\mathscr{C}$-projective.*

(4) *If $0\to K\to P\to C\to 0$ is exact, where $C\in\mathscr{C}$ and P is projective, then K is $\mathscr{C}$-projective.*

(5) $\mathrm{Ext}^{n+1}_R(C,N)=0$ *for any nonnegative integer n, any left R-module $C\in\mathscr{C}$ and any $\mathscr{C}$-injective left R-module N.*

(6) $\mathrm{Ext}^2_R(C,N)=0$ *for any left R-module $C\in\mathscr{C}$ and any $\mathscr{C}$-injective left R-module N.*

(7) *If N is a $\mathscr{C}$-injective left R-module, N_1 is a $\mathscr{C}$-injective submodule of N, then N/N_1 is $\mathscr{C}$-injective.*

(8) *For any $\mathscr{C}$-injective left R-module N, $E(N)/N$ is $\mathscr{C}$-injective.*

(9) *$(\mathscr{C}\mathcal{P},\mathscr{C}\mathcal{I})$ is a hereditary cotorsion pair.*

(10) *R is left $\mathscr{C}$-coherent, and $\mathrm{Tor}^R_{n+1}(M,C)=0$ for any nonnegative integer n, any left R-module $C\in\mathscr{C}$ and any $\mathscr{C}$-flat right R-module M.*

(11) *R is left $\mathscr{C}$-coherent, and $\mathrm{Tor}^R_2(M,C)=0$ for any left R-module $C\in\mathscr{C}$ and any $\mathscr{C}$-flat right R-module M.*

(12) $\mathrm{Ext}^{n+1}_R(P,N)=0$ *for any nonnegative integer n, any $\mathscr{C}$-projective left R-module P and any $\mathscr{C}$-injective left R-module N.*

Proof.

(1) $\Rightarrow$ (2) $\Rightarrow$ (3), and (4) $\Rightarrow$ (1) are obvious.

(3) $\Rightarrow$ (4) follows from Schanuel's Lemma.

(1) $\Rightarrow$ (5). Use induction on n. If $n = 0$, then it is clear that (5) holds. Assume that $\mathrm{Ext}_R^{k+1}(C, N) = 0$ for any $C \in \mathscr{C}$ and any $\mathscr{C}$-injective left R-module N. Then, by the isomorphism $\mathrm{Ext}_R^{k+1}(C, N) \cong \mathrm{Ext}_R^1(C, L^k)$, where L^k is the kth cosyzygy of N, we have that $\mathrm{Ext}_R^1(C, L^k) = 0$ and so L^k is $\mathscr{C}$-injective. Then, $\mathrm{Ext}_R^{k+1}(K, N) \cong \mathrm{Ext}_R^1(K, L^k) = 0$ for any $\mathscr{C}$-projective module K and any $\mathscr{C}$-injective left R-module N. Now, let $C \in \mathscr{C}$ and N be any $\mathscr{C}$-injective left R-module. Then there exists an exact sequence of left R-modules $0 \to K \to P \to C \to 0$, where P is finitely generated projective. Since R is left strongly $\mathscr{C}$-coherent, K is $\mathscr{C}$-projective, and so $\mathrm{Ext}_R^{k+1}(K, N) = 0$. Thus, from the exact sequence

$$0 = \mathrm{Ext}_R^{k+1}(P, N) \to \mathrm{Ext}_R^{k+1}(K, N) \to \mathrm{Ext}_R^{k+2}(C, N) \to \mathrm{Ext}_R^{k+2}(P, N) = 0$$

we have $\mathrm{Ext}_R^{k+2}(C, N) \cong \mathrm{Ext}_R^{k+1}(K, N) = 0$. Therefore, (5) holds by induction axioms.

(5) $\Rightarrow$ (6) is obvious.

(6) $\Rightarrow$ (7). For any left R-module $C \in \mathscr{C}$. The exact sequence $0 \to N_1 \to N \to N/N_1 \to 0$ induces the exactness of the sequence

$$0 = \mathrm{Ext}_R^1(C, N) \to \mathrm{Ext}_R^1(C, N/N_1) \to \mathrm{Ext}_R^2(C, N_1) = 0.$$

This follows that $\mathrm{Ext}_R^1(C, N/N_1) = 0$, as desired.

(7) $\Rightarrow$ (8) is obvious.

(8) $\Rightarrow$ (1). Let $C \in \mathscr{C}$. If $0 \to K \to P \to C \to 0$ is an exact sequence of left R-modules, where P is finitely generated projective. Then for any $\mathscr{C}$-injective module N, $E(N)/N$ is $\mathscr{C}$-injective by (7). From the exactness of the two sequences

$$0 = \mathrm{Ext}_R^1(P, N) \to \mathrm{Ext}_R^1(K, N) \to \mathrm{Ext}_R^2(C, N) \to \mathrm{Ext}_R^2(P, N) = 0$$

and

$$0 = \mathrm{Ext}_R^1(C, E(N)) \to \mathrm{Ext}_R^1(C, E(N)/N) \to \mathrm{Ext}_R^2(C, N) \to \mathrm{Ext}_R^2(C, E(N)) = 0,$$

we have $\mathrm{Ext}_R^1(K, N) \cong \mathrm{Ext}_R^2(C, N) \cong \mathrm{Ext}_R^1(C, E(N)/N) = 0$. Thus, K is $\mathscr{C}$-projective, as required.

(6) $\Leftrightarrow$ (9) $\Leftrightarrow$ (11). It follows from [18, Proposition 3.11].

(5), (11) $\Rightarrow$ (10). By (11), R is left $\mathscr{C}$-coherent. Let M be any $\mathscr{C}$-flat right R-module, $C \in \mathscr{C}$ and n be any nonnegative integer. Then by [18, Theorem 2.7], M^+ is $\mathscr{C}$-injective. So, by (5), we have $\mathrm{Ext}_R^{n+1}(C, M^+) = 0$, and thus $\mathrm{Tor}_{n+1}^R(M, C) = 0$ by the isomorphism $\mathrm{Tor}_{n+1}^R(M, C)^+ \cong \mathrm{Ext}_R^{n+1}(C, M^+)$.

(10) $\Rightarrow$ (11) is obvious.

(9) $\Leftrightarrow$ (12) follows from [7, Proposition 1.2]. $\square$

Recall that if n is a positive integer, then a left R-module M is called $(n,0)$-*injective* [17] if $\mathrm{Ext}^1_R(A,M)=0$ for every n-presented left R-module A.

PROPOSITION 1. *Let n be a positive integer and $\mathscr{C}$ be the class of all n-presented left R-modules. Then R is left strongly $\mathscr{C}$-coherent if and only if R is left n-coherent.*

Proof. $\Rightarrow$. Let V be an n-presented left R-module. Then there is an exact sequence $0\to K\to P\to V\to 0$, where P is finitely generated projective and K is $(n-1)$-presented. Since R is left strongly $\mathscr{C}$-coherent, K is $\mathscr{C}$-projective, that is, $\mathrm{Ext}^1_R(K,M)=0$ for every $(n,0)$-injective module M. Consequently, by [17, Theorem 2.6], K is n-presented, and thus V is $(n+1)$-presented.

$\Leftarrow$. Obvious. $\square$

COROLLARY 1. *Let $\mathscr{C}$ be the class of all finitely presented left R-modules. Then R is left strongly $\mathscr{C}$-coherent if and only if R is left $\mathscr{C}$-coherent if and only if R is left coherent.*

Let A be a ring and ${}_AE_A$ a bimodule, the trivial ring extension of A by E is the ring $R=A\propto E$ whose underlying group is $A\times E=\{(a,e):a\in A,e\in E\}$ with addition defined componentwise and multiplication defined by

$$(a,e)(a',e')=(aa',ae'+ea'),\quad \text{where } a,a'\in A,e,e'\in E.$$

Example 1. Let (V,M) be a nondiscrete valuation domain and let $R=V\propto V/M$ be the trivial ring extension of V by V/M . Then by [9, Example 3.8], R is not Dobbs's "2-coherent", and so it is not 2-coherent. Let $\mathscr{C}$ be the class of all 2-presented left R-modules. Then R is clearly left $\mathscr{C}$-coherent but it is not left strongly $\mathscr{C}$-coherent by Proposition 1.

Recall that a left R-module M is called *cyclically presented* if $M\cong R/Ra$ for some $a\in R$; a ring R is called *left strongly P-coherent* [12] if every principal left ideal of R is cyclically presented; a ring R is called *left generalized morphic* [16] if for every $a\in R$, there exists $b\in R$ such that $\mathbf{l}(a)\cong R/Rb$. By [16, Corollary 2.3], a ring R is left generalized morphic if and only if $\mathbf{l}(a)$ is a principal left ideal for each $a\in R$. It is easy to see that a left generalized morphic ring is left strongly P-coherent. We recall also that a ring R is called *left Lee n-coherent* [10] (for integers $n>0$ or $n=\infty$) if every finitely generated submodule of a free left R-module whose projective dimension is $\le n-1$ is finitely presented. It is easy to see that a ring R is left Lee n-coherent if and only if every finitely presented left R-module with projective dimension $\le n$ is 2-presented.

Example 2. (1) Let $\mathscr{C}=\{R/I : I \text{ is a finitely generated left ideal of } R\}$. Then R is left strongly $\mathscr{C}$-coherent if and only if R is left $\mathscr{C}$-coherent if and only if R is left coherent.

(2) Let $\mathscr{C}$ be the class of all finitely presented left R-modules with projective dimensions $\leq n$. Then R is left strongly $\mathscr{C}$-coherent if and only if R is left $\mathscr{C}$-coherent if and only if R is left Lee n-coherent.

(3) Let $\mathscr{C} = \{R/Ra : a \in R\}$. Then R is left strongly $\mathscr{C}$-coherent if and only every principal left ideal is P-projective.

(4) Let R be a left strongly P-coherent ring and $\mathscr{C} = \{R/Ra : a \in R\}$. Then R is left strongly $\mathscr{C}$-coherent. In particular, every left generalized morphic ring is left strongly $\mathscr{C}$-coherent.

(5) Every left $\mathscr{C}$-semihereditary ring is left strongly $\mathscr{C}$-coherent.

Proof. (1) follows from Theorem 1(2) and [5, Proposition]. The others are easy. □

Recall that a right R-module M is said to be *cotorsion* [6, Definition 5.3.22] if $\mathrm{Ext}^1_R(F, M) = 0$ for all flat right R-modules F. We call a right R-module M *$\mathscr{C}$-cotorsion* if $\mathrm{Ext}^1_R(F, M) = 0$ for all $\mathscr{C}$-flat right R-modules F.

Proposition 2. *The following are equivalent for a left $\mathscr{C}$-coherent ring R:*

(1) *Every $\mathscr{C}$-flat right R-module is flat.*

(2) *Every $\mathscr{C}$-injective left R-module is FP-injective.*

(3) *Every cotorsion right R-module is $\mathscr{C}$-cotorsion.*

(4) *Every finitely presented left R-module is $\mathscr{C}$-projective.*

In this case, R is left coherent and left strongly $\mathscr{C}$-coherent.

Proof. (1) $\Leftrightarrow$ (2). By [18, Corollary 3.8].

(1) $\Rightarrow$ (3). Assume (1). Then $\mathscr{F} = \mathscr{C}\mathscr{F}$. And so, if M is a cotorsion right R-module, then $M \in \mathscr{F}^{\perp} = (\mathscr{C}\mathscr{F})^{\perp}$.

(3) $\Rightarrow$ (1). Since $\mathscr{F} \subseteq \mathscr{C}\mathscr{F}$, we have $(\mathscr{C}\mathscr{F})^{\perp} \subseteq \mathscr{F}^{\perp}$. But $\mathscr{F}^{\perp} \subseteq (\mathscr{C}\mathscr{F})^{\perp}$ by (3), so $(\mathscr{C}\mathscr{F})^{\perp} = \mathscr{F}^{\perp}$. Now let M be a $\mathscr{C}$-flat right R-module. Then $M \in {}^{\perp}((\mathscr{C}\mathscr{F})^{\perp}) = {}^{\perp}(\mathscr{F}^{\perp}) = \mathscr{F}$ since $(\mathscr{F}, \mathscr{F}^{\perp})$ is a cotorsion pair, and thus (1) follows.

(2) $\Rightarrow$ (4). By (2), $\mathscr{C}\mathscr{I} \subseteq \mathscr{F}\mathscr{P}\mathscr{I}$. But $\mathscr{F}\mathscr{P}\mathscr{I} \subseteq \mathscr{C}\mathscr{I}$, so $\mathscr{F}\mathscr{P}\mathscr{I} = \mathscr{C}\mathscr{I}$. Let V be any finitely presented left R-module. Then $V \in {}^{\perp}(\mathscr{F}\mathscr{P}\mathscr{I}) = {}^{\perp}(\mathscr{C}\mathscr{I}) = \mathscr{C}\mathscr{P}$.

(4) $\Rightarrow$ (2). It is obvious.

Finally, let N be a $\mathscr{C}$-injective left R-module and N_1 be a $\mathscr{C}$-injective submodule of N. Then by (2), N_1 is FP-injective. Since R is left $\mathscr{C}$-coherent, by [18, Theorem 3.5], N/N_1 is $\mathscr{C}$-injective. And so, by Theorem 1(7), R is left strongly $\mathscr{C}$-coherent. Moreover, in this case, R is also left coherent by [18, Corollary 3.8]. □

Definition 3. (1). The $\mathscr{C}$-injective dimension of a module ${}_RM$ is defined by

$$\mathscr{C}\mathcal{I}-dim({}_RM) = inf\{n : \mathrm{Ext}_R^{n+1}(C, M) = 0 \quad \text{for every } C \in \mathscr{C}\}$$

(2). The $\mathscr{C}$-injective global dimension of a ring R is defined by

$$\mathscr{C}\mathcal{I}-GLD(R) = sup\{\mathscr{C}\mathcal{I}-dim(M) : M \text{ is a left } R\text{-module}\}$$

THEOREM 2. *Let R be a left strongly $\mathscr{C}$-coherent ring, M a left R-module and n a nonnegative integer. Then the following are equivalent:*

(1) $\mathscr{C}\mathcal{I}$-$dim({}_RM) \leq n$.

(2) $\mathrm{Ext}_R^{n+k}(C, M) = 0$ *for all* $C \in \mathscr{C}$ *and all positive integers* k .

(3) $\mathrm{Ext}_R^{n+1}(C, M) = 0$ *for all* $C \in \mathscr{C}$.

(4) *If the sequence* $0 \to M \xrightarrow{\varepsilon} E_0 \xrightarrow{d_0} \cdots \to E_{n-1} \xrightarrow{d_{n-1}} E_n \to 0$ *is exact with* $E_0, \cdots, E_{n-1}$ $\mathscr{C}$*-injective, then* E_n *is also* $\mathscr{C}$*-injective.*

(5) *There exists an exact sequence of left R-modules* $0 \to M \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0$ *such that* $E_0, \cdots, E_{n-1}, E_n$ *are* $\mathscr{C}$*-injective.*

Proof. (1) $\Rightarrow$ (2). Use induction on n. If $n = 0$, then (2) holds by Theorem 1(5). Assume that $\mathrm{Ext}_R^{n-1+k}(C, N) = 0$ for any $C \in \mathscr{C}$, any positive integer k and any left R-module N with $\mathscr{C}\mathcal{I}$-$dim(N) \leq n-1$. Then for any left R-module M with $\mathscr{C}\mathcal{I}$-$dim(M) \leq n$. If $\mathscr{C}\mathcal{I}$-$dim(M) = 0$, then (2) holds by Theorem 1(5). If $\mathscr{C}\mathcal{I}$-$dim(M) > 0$, then there exists a positive integer $m \leq n$ such that $\mathrm{Ext}_R^{m+1}(C, M) = 0$ for any $C \in \mathscr{C}$, which implies that $\mathrm{Ext}_R^{m}(C, E(M)/M) = 0$ for any $C \in \mathscr{C}$. So $\mathscr{C}\mathcal{I}$-$dim(E(M)/M) \leq m-1$, and hence $\mathscr{C}\mathcal{I}$-$dim(E(M)/M) \leq n-1$. By hypothesis, we have $\mathrm{Ext}_R^{n-1+k}(C, E(M)/M) = 0$ for any $C \in \mathscr{C}$ and any positive integer k, it follows that $\mathrm{Ext}_R^{n+k}(C, M) = 0$. Therefore, (2) holds by induction axioms.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5) are obvious.

(3) $\Rightarrow$ (4). Since R is left strongly $\mathscr{C}$-coherent and $E_0, \cdots, E_{n-1}$ is $\mathscr{C}$-injective, by Theorem 1(5), we have $\mathrm{Ext}_R^{n+1}(C, M) \cong \mathrm{Ext}_R^{n}(C, im(d_0)) \cong \mathrm{Ext}_R^{n-1}(C, im(d_1)) \cong \cdots \cong \mathrm{Ext}_R^{1}(C, im(d_{n-1})) = \mathrm{Ext}_R^{1}(C, E_n)$. So (4) follows from (3).

(5) $\Rightarrow$ (3). It follows from the above isomorphism $\mathrm{Ext}_R^{n+1}(C, M) \cong \mathrm{Ext}_R^{1}(C, E_n)$. $\square$

Definition 4. (1). The $\mathscr{C}$-flat dimension of a module M_R is defined by

$$\mathscr{C}\mathcal{F}\text{-}dim(M_R) = inf\{n : \mathrm{Tor}_{n+1}^R(M, C) = 0 \text{ for every } C \in \mathscr{C}\}$$

(2). The $\mathscr{C}$-weak global dimension of a ring R is defined by

$$\mathscr{C}\text{-WD(R)}=\sup\{\mathscr{C}\mathcal{F}\text{-dim(M): M is a right } R\text{-module}\}$$

THEOREM 3. *Let R be a left strongly $\mathscr{C}$-coherent ring, M a right R-module and n a non-negative integer. Then the following statements are equivalent:*

(1) $\mathscr{C}\mathcal{F}$-*dim*$(M_R) \leq n$.

(2) $\mathrm{Tor}^R_{n+k}(M, C) = 0$ *for all* $C \in \mathscr{C}$ *and all positive integers* k .

(3) $\mathrm{Tor}^R_{n+1}(M, C) = 0$ *for all* $C \in \mathscr{C}$.

(4) *If the sequence* $0 \to F_n \xrightarrow{\varepsilon} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$ *is exact with* $F_0, \cdots, F_{n-1}$ $\mathscr{C}$*-flat, then* F_n *is also* $\mathscr{C}$*-flat.*

(5) *There exists an exact sequence of right R-modules* $0 \to F_n \xrightarrow{\varepsilon} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$ *such that* $F_0, \cdots, F_{n-1}, F_n$ *are* $\mathscr{C}$*-flat.*

Proof. (1) $\Rightarrow$ (2). Let $C \in \mathscr{C}$ and k be any positive integer. By (1), there exists a nonnegative integer $m \leq n$ such that $\mathrm{Tor}^R_{m+1}(M, C) = 0$. And so, by the isomorphism $\mathrm{Tor}^R_{m+1}(M, C)^+ \cong \mathrm{Ext}^{m+1}_R(C, M^+)$, we have $\mathrm{Ext}^{m+1}_R(C, M^+) = 0$. Since R is left strongly $\mathscr{C}$-coherent, by Theorem 2, we have $\mathrm{Ext}^{n+k}_R(C, M^+) = 0$, and hence $\mathrm{Tor}^R_{n+k}(M, C) = 0$ by the isomorphism $\mathrm{Tor}^R_{n+k}(M, C)^+ \cong \mathrm{Ext}^{n+k}_R(C, M^+)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5) are obvious.

(3) $\Rightarrow$ (4). Since R is left strongly $\mathscr{C}$-coherent and $F_0, \cdots, F_{n-1}$ is $\mathscr{C}$-flat, by Theorem 1(10), we have $\mathrm{Tor}^R_{n+1}(M, C) \cong \mathrm{Tor}^R_n(Ker(d_0), C) \cong \mathrm{Tor}^R_{n-1}(Ker(d_1), C) \cong \cdots \cong \mathrm{Tor}^R_1(Ker(d_{n-1}), C) = \mathrm{Tor}^R_1(F_n, C)$. So (4) follows from (3).

(5) $\Rightarrow$ (3). It follows from the above isomorphism $\mathrm{Tor}^R_{n+1}(M, C) \cong \mathrm{Tor}^R_1(F_n, C)$. $\square$

Next, we give some new characterizations of left $\mathscr{C}$-semihereditary rings.

THEOREM 4. *The following are equivalent for a ring R:*

(1) *R is left $\mathscr{C}$-semihereditary.*

(2) *R is left strongly $\mathscr{C}$-coherent and $\mathscr{C}\mathcal{I}$-GLD(R)≤ 1.*

(3) *R is left strongly $\mathscr{C}$-coherent and $\mathscr{C}$-WD(R)≤ 1.*

(4) *Every torsionless right R-module is $\mathscr{C}$-flat.*

Proof. (1) $\Rightarrow$ (2). Suppose that R is left $\mathscr{C}$-semihereditary. Then it is clear that R is left strongly $\mathscr{C}$-coherent. Now let M be any left R-module. Then for any $C \in \mathscr{C}$, we have an exact sequence $0 \to K \to P \to C \to 0$ of left R-modules, where P is finitely generated projective. By condition, K is projective. Thus the exact sequence $0 = \mathrm{Ext}^1_R(K, M) \to \mathrm{Ext}^2_R(C, M) \to \mathrm{Ext}^2_R(P, M) = 0$ implies that $\mathrm{Ext}^2_R(C, M) = 0$. This follows that $\mathscr{C}\mathcal{I}$-GLD(R)$\leq$ 1 by Definition 3.

(2) $\Rightarrow$ (3). It follows from Theorem 2 and the isomorphism

$$\mathrm{Tor}_2^R(M,C)^+ \cong \mathrm{Ext}_R^2(C,M^+).$$

(3) $\Rightarrow$ (1). Assume (3). Then R is clearly left $\mathscr{C}$-coherent. Let A be a submodule of a $\mathscr{C}$-flat right R-module B and let $C \in \mathscr{C}$. Since R is left strongly $\mathscr{C}$-coherent and $\mathscr{C}$-WD(R)≤ 1, by Theorem 3, we have $\mathrm{Tor}_2^R(B/A,C) = 0$. Then, from the exactness of the sequence $0 = \mathrm{Tor}_2^R(B/A,C) \to \mathrm{Tor}_1^R(A,C) \to \mathrm{Tor}_1^R(B,C) = 0$ we have $\mathrm{Tor}_1^R(A,C) = 0$, and so A is $\mathscr{C}$-flat. Therefore, by [18, Theorem 4.3(2)], R is left $\mathscr{C}$-semihereditary.

(1) $\Rightarrow$ (4). Let M be a torsionless right R-module. Then there exists an exact sequence $0 \to M \to \prod R_R$. Since R is left $\mathscr{C}$-semihereditary, by [18, Theorem 4.3(2)], R is left $\mathscr{C}$-coherent and every submodule of a $\mathscr{C}$-flat right R-module is $\mathscr{C}$-flat, so M is $\mathscr{C}$-flat by [18, Theorem 3.3(4)].

(4) $\Rightarrow$ (1). Assume (4). Then $\prod R_R$ is $\mathscr{C}$-flat, and hence R is left $\mathscr{C}$-coherent by [18, Theorem 3.3(4)]. Moreover, every right ideal of R is torsionless and so is $\mathscr{C}$-flat. Thus, R is left $\mathscr{C}$-semihereditary by [18, Theorem 4.3(3)]. $\square$

By taking $\mathscr{C}$ to be the class of all finitely presented left R-modules, we have

Corollary 2. *The following are equivalent for a ring R:*
(1) *R is left semihereditary.*
(2) *R is left coherent and left $\mathscr{F}\mathscr{P}\mathscr{I}$-GLD(R)$\leq 1$.*
(3) *R is left coherent and $WD(R) \leq 1$.*
(4) *Every torsionless right R-module is flat.*

Recall that a ring R is called *left PP* [8] if every principal left ideal of R is projective.

Our following Examples show that strongly $\mathscr{C}$-coherent rings need not be $\mathscr{C}$-semihereditary.

Example 3. (1) Let K be a field and let $R = K \propto K$. Then R is a commutative ring with only three ideals: 0, R, and (0,K)=R(0,1), so R is generalized morphic. Since $\mathbf{l}_R(0,1) = (0,K)$ is not a direct summand of ${}_RR$, it is not a PP ring. So, let $\mathscr{C} = \{R/Ra : a \in R\}$. Then R is strongly $\mathscr{C}$-coherent by Example 2(4), but it is not $\mathscr{C}$-semihereditary by [18, Example 4.2(3)].

(2) Let $R = \mathbb{Z}_4 = \{\overline{0},\overline{1},\overline{2},\overline{3}\}$. Then it is easy to see that R is a commutative generalized morphic ring. Since $\mathbf{l}(\overline{2}) = \{\overline{0},\overline{2}\}$ contains no nonzero idempotent elements, it is not a direct summand of ${}_RR$, so that $R\overline{2}$ is not projective, and thus R is not PP. So, let $\mathscr{C} = \{R/Ra : a \in R\}$. Then R is strongly $\mathscr{C}$-coherent but it is not a $\mathscr{C}$-semihereditary ring.

We remark that the above two examples show that commutative artinian rings need not be PP.

3. $\mathscr{C}$-PROJECTIVE MODULES

Next, we give some characterizations of $\mathscr{C}$-projective modules.

PROPOSITION 3. *Let M be a left R-module. Then the following are equivalent:*

(1) *M is $\mathscr{C}$-projective.*

(2) *M is projective with respect to every exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with A $\mathscr{C}$-injective.*

(3) *M is projective with respect to the exact sequence $0 \to A \to E(A) \xrightarrow{\pi} E(A)/A \to 0$ of left R-modules with A $\mathscr{C}$-injective.*

Proof. (1) $\Rightarrow$ (2). By the exact sequence $\mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to \mathrm{Ext}^1_R(M, A) = 0$.

(2) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (1). For any $\mathscr{C}$-injective module A, we get an exact sequence $\mathrm{Hom}(M, E(A)) \xrightarrow{\pi_*} \mathrm{Hom}(M, E(A)/A) \to \mathrm{Ext}^1_R(M, A) \to \mathrm{Ext}^1_R(M, E(A)) = 0$. By (3), π_* is epic, and so $\mathrm{Ext}^1_R(M, A) = 0$. Therefore, M is $\mathscr{C}$-projective. $\square$

Let $\mathscr{L}$ be a class of R-modules and M an R-module. Following [6], we say that a homomorphism $\varphi : M \to L$ where $L \in \mathscr{L}$ is a $\mathscr{L}$-preenvelope of M if for any homomorphism $f : M \to L'$ with $L' \in \mathscr{L}$, there is a $g : L \to L'$ such that $g\varphi = f$.

PROPOSITION 4. *Let R be a $\mathscr{C}$-injective ring and M be a left R-module. Then the following statements are equivalent:*

(1) *M is $\mathscr{C}$-projective.*

(2) *For every exact sequence $0 \to K \to A \to M \to 0$, where A is $\mathscr{C}$-injective, $K \to A$ is a $\mathscr{C}$-injective preenvelope of K.*

(3) *M is a cokernet of a $\mathscr{C}$-injective preenvelope $K \to P$ with P projective.*

Proof. (1) $\Rightarrow$ (2). It follows from the exact sequence $\mathrm{Hom}(A, N) \to \mathrm{Hom}(K, N) \to \mathrm{Ext}^1_R(M, N) = 0$, where N is $\mathscr{C}$-injective.

(2) $\Rightarrow$ (3). Let $0 \to K \to P \to M \to 0$ be an exact sequence of left R-modules with P projective. Since R is a $\mathscr{C}$-injective ring, by [18, Proposition 2.5], P is $\mathscr{C}$-injective. And so, by (2), $K \to P$ is a $\mathscr{C}$-injective preenvelope.

(3) $\Rightarrow$ (1). By (3), there exists an exact sequence $0 \to K \to P \to M \to 0$, where P is projective and $K \to P$ is a $\mathscr{C}$-injective preenvelope. Thus we get an exact sequence $\mathrm{Hom}(P, N) \to \mathrm{Hom}(K, N) \to \mathrm{Ext}^1_R(M, N) \to 0$ for every $\mathscr{C}$-injective module N. Note that the map $\mathrm{Hom}(P, N) \to \mathrm{Hom}(K, N)$ is epic, we have that $\mathrm{Ext}^1_R(M, N) = 0$, as required. $\square$

Definition 5. (1). The $\mathscr{C}$-projective dimension of a module ${}_RM$ is defined by

$$\mathscr{CP}\text{-}dim({}_RM) = inf\{n : \mathrm{Ext}_R^{n+1}(M,N) = 0 \text{ for every } \mathscr{C}\text{-injective module N}\}$$

(2). The $\mathscr{C}$-projective global dimension of a ring R is defined by

$$\mathscr{CP}\text{-GLD}(R) = \sup\{\mathscr{CP} - \dim(M) : M \text{ is a left } R\text{-module}\}$$

THEOREM 5. *Let R be a left strongly $\mathscr{C}$-coherent ring, M be a left R-module and n be a nonnegative integer. Then the following statements are equivalent:*

(1) $\mathscr{CP}\text{-}dim({}_RM) \leq n$.

(2) $\mathrm{Ext}_R^{n+k}(M,N) = 0$ *for all $\mathscr{C}$-injective modules N and all positive integers k .*

(3) $\mathrm{Ext}_R^{n+1}(M,N) = 0$ *for all $\mathscr{C}$-injective modules N.*

(4) *If the sequence* $0 \to P_n \xrightarrow{\varepsilon} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ *is exact with $P_0, \cdots, P_{n-1}$ $\mathscr{C}$-projective, then P_n is also $\mathscr{C}$-projective.*

(5) *There exists an exact sequence of right R-modules* $0 \to P_n \xrightarrow{\varepsilon} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ *such that $P_0, \cdots, P_{n-1}, P_n$ are $\mathscr{C}$-projective.*

Proof. (1) $\Rightarrow$ (2) . Use induction on n. If $n = 0$, then M is $\mathscr{C}$-projective. Since R is left strongly $\mathscr{C}$-coherent, by Theorem 1(12), (2) holds. Now assume that $\mathrm{Ext}_R^{n-1+k}(L,N) = 0$ for any $\mathscr{C}$-injective module N, any positive integer k and any left R-module L with $\mathscr{CP}\text{-}dim(L) \leq n-1$. Then for any left R-module M with $\mathscr{CP}\text{-}dim(M) \leq n$. If $\mathscr{CP}\text{-}dim(M) = 0$, then (2) holds by Theorem 1(12). If $\mathscr{CP}\text{-}dim(M) > 0$, then there exists a positive integer $m \leq n$ such that $\mathrm{Ext}_R^{m+1}(M,N) = 0$ for any $\mathscr{C}$-injective module N. Let $0 \to K \to P \to M \to 0$ be exact with P projective. Then it is easy to see that $\mathrm{Ext}_R^{m}(K,N) = 0$ for any $\mathscr{C}$-injective module N. So $\mathscr{CP}\text{-}dim(K) \leq m-1$, and hence $\mathscr{CP}\text{-}dim(K) \leq n-1$. By hypothesis, we have $\mathrm{Ext}_R^{n-1+k}(K,N) = 0$ for any $\mathscr{C}$-injective module N and any positive integer k, it follows that $\mathrm{Ext}_R^{n+k}(M,N) = 0$. Therefore, (2) holds by induction axioms.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5) are obvious.

(3) $\Rightarrow$ (4). Since R is left strongly $\mathscr{C}$-coherent, by Theorem 1(12), we have $\mathrm{Ext}_R^{i}(P,N) = 0$ for any positive integer i, any $\mathscr{C}$-projective left R-module P and any $\mathscr{C}$-injective left R-module N. Thus, $\mathrm{Ext}_R^{n+1}(M,N) \cong \mathrm{Ext}_R^{n}(Ker(d_0),N) \cong \mathrm{Ext}_R^{n-1}(Ker(d_1),N) \cong \cdots \cong \mathrm{Ext}_R^{1}(Ker(d_{n-1}),N) = \mathrm{Ext}_R^{1}(P_n,N)$. So (4) follows from (3).

(5) $\Rightarrow$ (3). It follows from the above isomorphism $\mathrm{Ext}_R^{n+1}(M,N) \cong \mathrm{Ext}_R^{1}(P_n,N)$. $\square$

Recall that a cotorsion pair $(\mathscr{A},\mathscr{B})$ is called complete (see [6, Definition 7.16] and [15, Lemma 1.13]) if for any R-module M, there is an exact sequence $0 \to M \to B \to A \to 0$ with $A \in \mathscr{A}$ and $B \in \mathscr{B}$.

THEOREM 6. *The following are equivalent for a ring R:*

(1) *$\mathscr{CP}$-GLD(R)=0.*

(2) *Every left R-module is $\mathscr{C}$-projective.*

(3) *Every $\mathscr{C}$-injective module is injective.*

(4) *R is left strongly $\mathscr{C}$-coherent and every $\mathscr{C}$-injective module is $\mathscr{C}$-projective.*

(5) *Every cyclic left R-module is $\mathscr{C}$-projective.*

In this case, R is left noetherian.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) and (2) $\Rightarrow$ (4), (5) are obvious.

(4) $\Rightarrow$ (2). Let M be a left R-module. By [18, Theorem 2.10(1)], $(\mathscr{CP}, \mathscr{CI})$ is a complete cotorsion pair, so there is a short exact sequence $0 \to M \to E \to P \to 0$, where E is $\mathscr{C}$-injective and P is $\mathscr{C}$-projective. By hypothesis, every $\mathscr{C}$-injective module is $\mathscr{C}$-projective, so E $\mathscr{C}$-projective. Since R is left strongly $\mathscr{C}$-coherent, by Theorem 1(12), we have $\mathrm{Ext}^1_R(E, N) = \mathrm{Ext}^2_R(P, N) = 0$ for any $\mathscr{C}$-injective module N. It follows that $\mathrm{Ext}^1_R(M, N) = 0$ for any $\mathscr{C}$-injective module N, and so M is $\mathscr{C}$-projective.

(5) $\Rightarrow$ (3). Let N be a $\mathscr{C}$-injective module. Then for any left ideal I, by (5), R/I is $\mathscr{C}$-projective, so $\mathrm{Ext}^1_R(R/I, N) = 0$, and thus N is injective.

In this case, any direct sum of injective left R-modules is injective since any direct sum of $\mathscr{C}$-injective modules is $\mathscr{C}$-injective by [18, Proposition 2.5(3)], and hence R is left noetherian. $\square$

Observing that every FP-injective left module over a left noetherian ring is injective, by taking $\mathscr{C}$ to be the class of all finitely presented left R-modules in Theorem 6, we have immediately the following results.

COROLLARY 3. *The following are equivalent for a ring R:*

(1) *R is left noetherian.*

(2) *Every left R-module is FP-projective.*

(3) *Every FP-injective module is injective.*

(4) *R is left coherent and every FP-injective left R-module is FP-projective.*

(5) *Every cyclic left R-module is FP-projective.*

THEOREM 7. *Let R be a ring and consider the following conditions:*

(1) *Every submodule of a $\mathscr{C}$-projective module is $\mathscr{C}$-projective.*

(2) *Every submodule of a projective left R-module is $\mathscr{C}$-projective.*

(3) *Every left ideal of R is $\mathscr{C}$-projective.*

(4) $\mathscr{CP} - GLD(R) \leq 1$.

Then we always have the following implications:

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).$$

Furthermore, if R is left strongly $\mathscr{C}$-coherent, then the four conditions are equivalent.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Let N be any $\mathscr{C}$-injective module. Then for any left ideal I, we have $\mathrm{Ext}^1_R(I, N) = 0$ by (3), it follows that $\mathrm{Ext}^2_R(R/I, N) = 0$. So $id(N) \leq 1$, and then $\mathrm{Ext}^2_R(M, N) = 0$ for every left R-module M. Therefore (4) holds.

Now suppose that R is left strongly $\mathscr{C}$-coherent and (4) holds. Then we will prove (1), and thus, in this case, the four conditions are equivalent. In fact, let K be a submodule of a $\mathscr{C}$-projective module P. Then, for any $\mathscr{C}$-injective module N, we get an exact sequence

$$0 = \mathrm{Ext}^1_R(P, N) \to \mathrm{Ext}^1_R(K, N) \to \mathrm{Ext}^2_R(P/K, N).$$

Since R is left strongly $\mathscr{C}$-coherent, by (4) and Theorem 5, we have $\mathrm{Ext}^2_R(P/K, N) = 0$, and so $\mathrm{Ext}^1_R(K, N) = 0$, which shows that K is $\mathscr{C}$-projective, as required. $\square$

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