STRONGLY C-COHERENT RINGS

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Let $R$ be a ring and $C$ be a class of some finitely presented left $R$-modules. $R$ is called left strongly $C$-coherent, if whenever $0 \rightarrow K \rightarrow P \rightarrow C \rightarrow 0$ is exact, where $C \in C$ and $P$ is finitely generated projective, then $K$ is $C$-projective. Characterizations of left strongly $C$-coherent rings are given, $C$-injective (resp., $C$-projective, $C$-flat) dimensions of modules over left strongly $C$-coherent rings are studied, conditions under which left strongly $C$-coherent rings are left $C$-semihereditary are given.

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1. INTRODUCTION

Recall that a ring $R$ is said to be left coherent [2] if every finitely generated left ideal of $R$ is finitely presented. Coherent rings and their generalizations have been studied extensively by many authors. For example, Costa introduced the concept of left $n$-coherent rings in [3]. Let $n$ be a nonnegative integer, then following [3], a ring $R$ is said to be left $n$-coherent if every $n$-presented left $R$-module is $(n+1)$-presented, where a left $R$-module $M$ is said to be $n$-presented in case there is an exact sequence of left $R$-modules $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ in which every $F_i$ is a finitely generated free, equivalently projective left $R$-module. It is easy to see that a ring $R$ is left $0$-coherent if and only if $R$ is left noetherian, $R$ is left 1-coherent if and only if $R$ is left coherent. We note that in some literatures such as [4, 9], Dobbs etc. called a ring $R$ left $n$-coherent if every $(n-1)$-presented left ideal of $R$ is $n$-presented, and they called $R$ left strong $n$-coherent if every $n$-presented left $R$-modules is $(n+1)$-presented. Clearly, Dobbs’s “strong $n$-coherent” implies Dobbs’s “$n$-coherent”, and Dobbs’s “strong $n$-coherent” coincides with Costa’s “$n$-coherent”. We remark that the terminology of “$n$-coherence” in this paper is Costa’s “$n$-coherence” but is not the same as that of Dobbs’s.

In [18], we introduced a new generalization for left coherent rings. Let $C$ be a class of some finitely presented left $R$-modules. Following [18], a ring
$R$ is called \emph{left $\mathcal{C}$-coherent} if every $C \in \mathcal{C}$ is 2-presented. We recall also that a left $R$-module $M$ is called $\mathcal{C}$-injective \cite{18} if $\text{Ext}^1_R(C, M) = 0$ for every $C \in \mathcal{C}$; a right $R$-module $M$ is called $\mathcal{C}$-flat \cite{18} if $\text{Tor}^1_R(M, C) = 0$ for every $C \in \mathcal{C}$. In this article, we will call a left $R$-module $M$ $\mathcal{C}$-projective if $\text{Ext}^1_R(M, N) = 0$ for any $\mathcal{C}$-injective module $N$, and we will call a ring $R$ \emph{left strongly $\mathcal{C}$-coherent}, if whenever $0 \to K \to P \to C \to 0$ is exact, where $C \in \mathcal{C}$ and $P$ is finitely generated projective, then $K$ is $\mathcal{C}$-projective. We will give some characterizations and properties of $\mathcal{C}$-projective modules and left strongly $\mathcal{C}$-coherent rings, left strongly $\mathcal{C}$-coherent rings will be characterized by $\mathcal{C}$-injective modules, $\mathcal{C}$-projective modules and $\mathcal{C}$-flat modules. Furthermore, we define $\mathcal{C}$-injective dimensions, $\mathcal{C}$-projective dimensions and $\mathcal{C}$-flat dimensions of modules, we will show that over a left strongly $\mathcal{C}$-coherent ring, the three classes of dimensions of modules have some nice properties.

Following \cite{18}, a ring $R$ is called \emph{left $\mathcal{C}$-semihereditary}, if whenever $0 \to K \to P \to C \to 0$ is exact, where $C \in \mathcal{C}$, $P$ is finitely generated projective, then $K$ is projective. Following \cite{1}, a pair $(\mathcal{A}, \mathcal{B})$ of classes of $R$-modules is called a \emph{cotorsion pair} if $\mathcal{A}^\perp = \mathcal{B}$ and $\perp \mathcal{B} = \mathcal{A}$. We note that a cotorsion pair is also called a cotorsion theory in some literatures such as \cite{6, 15}. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called \emph{hereditary} \cite[Definition 1.1]{7} if whenever $0 \to A' \to A \to A'' \to 0$ is exact with $A, A'' \in \mathcal{A}$ then $A'$ is also in $\mathcal{A}$. By \cite[Proposition 1.2]{7}, a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is hereditary if and only if whenever $0 \to B' \to B \to B'' \to 0$ is exact with $B', B \in \mathcal{B}$ then $B''$ is also in $\mathcal{B}$. In this paper, we will give some new characterizations of left $\mathcal{C}$-semihereditary rings.

Throughout this paper, $R$ is an associative ring with identity and all modules considered are unitary, $\mathcal{C}$ is a class of some finitely presented left $R$-modules. For any $R$-module $M$, $M^+ = \text{Hom}(M, \mathbb{Q}/\mathbb{Z})$ will be the character module of $M$. Given a class $\mathcal{L}$ of $R$-modules, we will denote by $\mathcal{L}^\perp = \{M : \text{Ext}^1_R(L, M) = 0, L \in \mathcal{L}\}$ the right orthogonal class of $\mathcal{L}$, and by $\perp \mathcal{L} = \{M : \text{Ext}^1_R(M, L) = 0, L \in \mathcal{L}\}$ the left orthogonal class of $\mathcal{L}$. $\mathcal{C}I$, $\mathcal{C}F$, $\mathcal{F}I$, $\mathcal{F}P$ will denote the class of all $\mathcal{C}$-injective modules, $\mathcal{C}$-flat modules, FP-injective left $R$-modules and flat right $R$-modules, respectively.

2. STRONGLY $\mathcal{C}$-COHERENT RINGS

Recall that a left $R$-module $M$ is said to be $P$-injective \cite{13} if $\text{Ext}^1_R(R/Ra, M) = 0$ for any $a \in R$; a left $R$-module $N$ is said to be $P$-projective \cite{16} if $\text{Ext}^1_R(N, M) = 0$ for any $P$-injective left $R$-module $M$; a left $R$-module $M$ is said to be $FP$-injective \cite{14} if $\text{Ext}^1_R(V, M) = 0$ for any finitely presented left $R$-module $V$; a left $R$-module $N$ is said to be $FP$-projective \cite{11} if $\text{Ext}^1_R(N, M) = 0$.
for any FP-injective left $R$-module $M$. We extend the concepts of P-projective modules and FP-projective modules as follows.

**Definition 1.** Let $R$ be a ring and $\mathcal{C}$ be a class of some finitely presented left $R$-modules. Then a left $R$-module $M$ is called $\mathcal{C}$-projective if $\text{Ext}^1_R(M, N) = 0$ for any $\mathcal{C}$-injective module $N$.

We will denote the class of $\mathcal{C}$-projective modules by $\mathcal{C}P$. The class of $\mathcal{C}$-projective modules is known in the sense that it is exactly the class $\bot(\bot)$ for a class $\mathcal{C}$ of finitely presented modules, which is the first class of the cotorsion pair cogenerated by $\mathcal{C}$ (e.g. [11, Def. 1.10]). Clearly, the name $\mathcal{C}$-projective was inspired by the work of Mao and Ding [11] on FP-projective modules.

**Definition 2.** A ring $R$ is called left strongly $\mathcal{C}$-coherent, if whenever $0 \to K \to P \to C \to 0$ is exact, where $C \in \mathcal{C}$ and $P$ is finitely generated projective, then $K$ is $\mathcal{C}$-projective.

**Theorem 1.** The following are equivalent for a ring $R$:

1. $R$ is left strongly $\mathcal{C}$-coherent.
2. If $C \in \mathcal{C}$, then there exists an exact sequence of left $R$-modules $0 \to K \to P \to C \to 0$, where $P$ is finitely generated projective, and $K$ is $\mathcal{C}$-projective.
3. If $C \in \mathcal{C}$, then there exists an exact sequence of left $R$-modules $0 \to K \to P \to C \to 0$, where $P$ is projective, and $K$ is $\mathcal{C}$-projective.
4. If $0 \to K \to P \to C \to 0$ is exact, where $C \in \mathcal{C}$ and $P$ is projective, then $K$ is $\mathcal{C}$-projective.
5. $\text{Ext}_R^{n+1}(C, N) = 0$ for any nonnegative integer $n$, any left $R$-module $C \in \mathcal{C}$ and any $\mathcal{C}$-injective left $R$-module $N$.
6. $\text{Ext}_R^2(C, N) = 0$ for any left $R$-module $C \in \mathcal{C}$ and any $\mathcal{C}$-injective left $R$-module $N$.
7. If $N$ is a $\mathcal{C}$-injective left $R$-module, $N_1$ is a $\mathcal{C}$-injective submodule of $N$, then $N/N_1$ is $\mathcal{C}$-injective.
8. For any $\mathcal{C}$-injective left $R$-module $N$, $E(N)/N$ is $\mathcal{C}$-injective.
9. $(\mathcal{C}P, \mathcal{T})$ is a hereditary cotorsion pair.
10. $R$ is left $\mathcal{C}$-coherent, and $\text{Tor}_R^{n+1}(M, C) = 0$ for any nonnegative integer $n$, any left $R$-module $C \in \mathcal{C}$ and any $\mathcal{C}$-flat right $R$-module $M$.
11. $R$ is left $\mathcal{C}$-coherent, and $\text{Tor}_R^2(M, C) = 0$ for any left $R$-module $C \in \mathcal{C}$ and any $\mathcal{C}$-flat right $R$-module $M$.
12. $\text{Ext}_R^{n+1}(P, N) = 0$ for any nonnegative integer $n$, any $\mathcal{C}$-projective left $R$-module $P$ and any $\mathcal{C}$-injective left $R$-module $N$.

**Proof.**

(1) $\Rightarrow$ (2) $\Rightarrow$ (3), and (4) $\Rightarrow$ (1) are obvious.
(3) ⇒ (4) follows from Schanuel’s Lemma.

(1) ⇒ (5). Use induction on \( n \). If \( n = 0 \), then it is clear that (5) holds. Assume that \( \text{Ext}_{R}^{k+1}(C, N) = 0 \) for any \( C \in \mathcal{C} \) and any \( \mathcal{C} \)-injective left \( R \)-module \( N \). Then, by the isomorphism \( \text{Ext}_{R}^{k+1}(C, N) \cong \text{Ext}_{R}^{1}(C, L^{k}) \), where \( L^{k} \) is the \( k \)th cosyzygy of \( N \), we have that \( \text{Ext}_{R}^{1}(C, L^{k}) = 0 \) and so \( L^{k} \) is \( \mathcal{C} \)-injective. Then, \( \text{Ext}_{R}^{k+1}(K, N) \cong \text{Ext}_{R}^{1}(K, L^{k}) = 0 \) for any \( \mathcal{C} \)-projective module \( K \) and any \( \mathcal{C} \)-injective left \( R \)-module \( N \). Now, let \( C \in \mathcal{C} \) and \( N \) be any \( \mathcal{C} \)-injective left \( R \)-module. Then there exists an exact sequence of left \( R \)-modules \( 0 \to K \to P \to C \to 0 \), where \( P \) is finitely generated projective. Since \( R \) is left strongly \( \mathcal{C} \)-coherent, \( K \) is \( \mathcal{C} \)-projective, and so \( \text{Ext}_{R}^{k+1}(K, N) = 0 \). Thus, from the exact sequence

\[
0 = \text{Ext}_{R}^{k+1}(P, N) \to \text{Ext}_{R}^{k+1}(K, N) \to \text{Ext}_{R}^{k+2}(C, N) \to \text{Ext}_{R}^{k+2}(P, N) = 0
\]

we have \( \text{Ext}_{R}^{k+2}(C, N) \cong \text{Ext}_{R}^{k+1}(K, N) = 0 \). Therefore, (5) holds by induction axioms.

(5) ⇒ (6) is obvious.

(6) ⇒ (7). For any left \( R \)-module \( C \in \mathcal{C} \). The exact sequence \( 0 \to N_{1} \to N \to N/N_{1} \to 0 \) induces the exactness of the sequence

\[
0 = \text{Ext}_{R}^{1}(C, N) \to \text{Ext}_{R}^{1}(C, N/N_{1}) \to \text{Ext}_{R}^{2}(C, N_{1}) = 0.
\]

This follows that \( \text{Ext}_{R}^{1}(C, N/N_{1}) = 0 \), as desired.

(7) ⇒ (8) is obvious.

(8) ⇒ (1). Let \( C \in \mathcal{C} \). If \( 0 \to K \to P \to C \to 0 \) is an exact sequence of left \( R \)-modules, where \( P \) is finitely generated projective. Then for any \( \mathcal{C} \)-injective module \( N \), \( E(N)/N \) is \( \mathcal{C} \)-injective by (7). From the exactness of the two sequences

\[
0 = \text{Ext}_{R}^{1}(P, N) \to \text{Ext}_{R}^{1}(K, N) \to \text{Ext}_{R}^{2}(C, N) \to \text{Ext}_{R}^{2}(P, N) = 0
\]

and

\[
0 = \text{Ext}_{R}^{1}(C, E(N)) \to \text{Ext}_{R}^{1}(C, E(N)/N) \to \text{Ext}_{R}^{2}(C, N) \to \text{Ext}_{R}^{2}(C, E(N)) = 0,
\]

we have \( \text{Ext}_{R}^{1}(K, N) \cong \text{Ext}_{R}^{2}(C, N) \cong \text{Ext}_{R}^{1}(C, E(N)/N) = 0 \). Thus, \( K \) is \( \mathcal{C} \)-projective, as required.

(6) ⇔ (9) ⇔ (11). It follows from [18, Proposition 3.11].

(5),(11) ⇒ (10). By (11), \( R \) is left \( \mathcal{C} \)-coherent. Let \( M \) be any \( \mathcal{C} \)-flat right \( R \)-module, \( C \in \mathcal{C} \) and \( n \) be any nonnegative integer. Then by [18, Theorem 2.7], \( M^{+} \) is \( \mathcal{C} \)-injective. So, by (5), we have \( \text{Ext}_{R}^{n+1}(C, M^{+}) = 0 \), and thus \( \text{Tor}_{n+1}^{R}(M, C) = 0 \) by the isomorphism \( \text{Tor}_{n+1}^{R}(M, C)^{+} \cong \text{Ext}_{R}^{n+1}(C, M^{+}) \).

(10) ⇒ (11) is obvious.

(9) ⇔ (12) follows from [7, Proposition 1.2]. □
Recall that if \( n \) is a positive integer, then a left \( R \)-module \( M \) is called \((n,0)\)-injective \([17]\) if \( \text{Ext}^1_R(A,M) = 0 \) for every \( n \)-presented left \( R \)-module \( A \).

**Proposition 1.** Let \( n \) be a positive integer and \( \mathcal{C} \) be the class of all \( n \)-presented left \( R \)-modules. Then \( R \) is left strongly \( \mathcal{C} \)-coherent if and only if \( R \) is left \( n \)-coherent.

**Proof.** \( \Rightarrow. \) Let \( V \) be an \( n \)-presented left \( R \)-module. Then there is an exact sequence \( 0 \to K \to P \to V \to 0 \), where \( P \) is finitely generated projective and \( K \) is \((n−1)\)-presented. Since \( R \) is left strongly \( \mathcal{C} \)-coherent, \( K \) is \( \mathcal{C} \)-projective, that is, \( \text{Ext}^1_R(K,M) = 0 \) for every \((n,0)\)-injective module \( M \). Consequently, by \([17, \text{Theorem 2.6}]\), \( K \) is \( n \)-presented, and thus \( V \) is \((n + 1)\)-presented.

\( \Leftarrow. \) Obvious. \( \square \)

**Corollary 1.** Let \( \mathcal{C} \) be the class of all finitely presented left \( R \)-modules. Then \( R \) is left strongly \( \mathcal{C} \)-coherent if and only if \( R \) is left \( \mathcal{C} \)-coherent if and only if \( R \) is left coherent.

Let \( A \) be a ring and \( _AE_A \) a bimodule, the trivial ring extension of \( A \) by \( E \) is the ring \( R = A \times E \) whose underlying group is \( A \times E = \{(a,e) : a \in A, e \in E\} \) with addition defined componentwise and multiplication defined by

\[
(a,e)(a',e') = (aa', ae' + ea'), \quad \text{where } a,a' \in A, e,e' \in E.
\]

**Example 1.** Let \((V,M)\) be a nondiscrete valuation domain and let \( R = V \times V/M \) be the trivial ring extension of \( V \) by \( V/M \). Then by \([9, \text{Example 3.8}]\), \( R \) is not Dobbs’s “2-coherent”, and so it is not 2-coherent. Let \( \mathcal{C} \) be the class of all 2-presented left \( R \)-modules. Then \( R \) is clearly left \( \mathcal{C} \)-coherent but it is not left strongly \( \mathcal{C} \)-coherent by Proposition 1.

Recall that a left \( R \)-module \( M \) is called *cyclically presented* if \( M \cong R/Ra \) for some \( a \in R \); a ring \( R \) is called *left strongly P-coherent* \([12]\) if every principal left ideal of \( R \) is cyclically presented; a ring \( R \) is called *left generalized morphic* \([16]\) if for every \( a \in R \), there exists \( b \in R \) such that \( I(a) \cong R/Rb \). By \([16, \text{Corollary 2.3}]\), a ring \( R \) is left generalized morphic if and only if \( I(a) \) is a principal left ideal for each \( a \in R \). It is easy to see that a left generalized morphic ring is left strongly P-coherent. We recall also that a ring \( R \) is called *left Lee n-coherent* \([10]\) (for integers \( n > 0 \) or \( n = \infty \)) if every finitely generated submodule of a free left \( R \)-module whose projective dimension is \( \leq n − 1 \) is finitely presented. It is easy to see that a ring \( R \) is left Lee \( n \)-coherent if and only if every finitely presented left \( R \)-module with projective dimension \( \leq n \) is 2-presented.

**Example 2.** (1) Let \( \mathcal{C} = \{R/I : I \text{ is a finitely generated left ideal of } R\} \). Then \( R \) is left strongly \( \mathcal{C} \)-coherent if and only if \( R \) is left \( \mathcal{C} \)-coherent if and only if \( R \) is left coherent.
(2) Let \( C \) be the class of all finitely presented left \( R \)-modules with projective dimensions \( \leq n \). Then \( R \) is left strongly \( C \)-coherent if and only if \( R \) is left \( C \)-coherent if and only if \( R \) is left Lee \( n \)-coherent.

(3) Let \( C = \{ R/Ra : a \in R \} \). Then \( R \) is left strongly \( C \)-coherent if and only if every principal left ideal is P-projective.

(4) Let \( R \) be a left strongly P-coherent ring and \( C = \{ R/Ra : a \in R \} \). Then \( R \) is left strongly \( C \)-coherent. In particular, every left generalized morphic ring is left strongly \( C \)-coherent.

(5) Every left \( C \)-semihereditary ring is left strongly \( C \)-coherent.

Proof. (1) follows from Theorem 1(2) and [5, Proposition]. The others are easy. □

Recall that a right \( R \)-module \( M \) is said to be cotorsion \([6, Definition 5.3.22]\) if \( \text{Ext}^1_R(F, M) = 0 \) for all flat right \( R \)-modules \( F \). We call a right \( R \)-module \( M \) \( C \)-cotorsion if \( \text{Ext}^1_R(F, M) = 0 \) for all \( C \)-flat right \( R \)-modules \( F \).

**Proposition 2.** The following are equivalent for a left \( C \)-coherent ring \( R \):

1. Every \( C \)-flat right \( R \)-module is flat.
2. Every \( C \)-injective left \( R \)-module is FP-injective.
3. Every cotorsion right \( R \)-module is \( C \)-cotorsion.
4. Every finitely presented left \( R \)-module is \( C \)-projective.

In this case, \( R \) is left coherent and left strongly \( C \)-coherent.

Proof. (1) \( \iff \) (2). By [18, Corollary 3.8].

(1) \( \Rightarrow \) (3). Assume (1). Then \( \mathcal{F} = C \mathcal{F} \). And so, if \( M \) is a cotorsion right \( R \)-module, then \( M \in \mathcal{F} \perp = (C \mathcal{F}) \perp \).

(3) \( \Rightarrow \) (1). Since \( \mathcal{F} \subseteq C \mathcal{F} \), we have \((C \mathcal{F}) \perp \subseteq \mathcal{F} \perp \). But \( \mathcal{F} \perp \subseteq (C \mathcal{F}) \perp \) by (3), so \((C \mathcal{F}) \perp = \mathcal{F} \perp \). Now let \( M \) be a \( C \)-flat right \( R \)-module. Then \( M \in \perp ((C \mathcal{F}) \perp) = (\mathcal{F} \perp) = \mathcal{F} \) since \((\mathcal{F}, \mathcal{F} \perp)\) is a cotorsion pair, and thus (1) follows.

(2) \( \Rightarrow \) (4). By (2), \( C \mathcal{I} \subseteq \mathcal{F} \mathcal{P} \mathcal{I} \). But \( \mathcal{F} \mathcal{P} \mathcal{I} \subseteq C \mathcal{I} \), so \( \mathcal{F} \mathcal{P} \mathcal{I} = C \mathcal{I} \). Let \( V \) be any finitely presented left \( R \)-module. Then \( V \in \perp (\mathcal{F} \mathcal{P} \mathcal{I}) = \perp (C \mathcal{I}) = C \mathcal{P} \).

(4) \( \Rightarrow \) (2). It is obvious.

Finally, let \( N \) be a \( C \)-injective left \( R \)-module and \( N_1 \) be a \( C \)-injective submodule of \( N \). Then by (2), \( N_1 \) is FP-injective. Since \( R \) is left \( C \)-coherent, by [18, Theorem 3.5], \( N/N_1 \) is \( C \)-injective. And so, by Theorem 1(7), \( R \) is left strongly \( C \)-coherent. Moreover, in this case, \( R \) is also left coherent by [18, Corollary 3.8]. □
\textit{Definition 3.} (1). The $\mathcal{C}$-injective dimension of a module $RM$ is defined by
\[\mathcal{C}I - \text{dim}(RM) = \inf \{n : \text{Ext}^{n+1}_R(C, M) = 0 \text{ for every } C \in \mathcal{C}\}\]

(2). The $\mathcal{C}$-injective global dimension of a ring $R$ is defined by
\[\mathcal{C}I - \text{GLD}(R) = \sup \{\mathcal{C}I - \text{dim}(M) : M \text{ is a left } R\text{-module}\}\]

\textit{Theorem 2.} Let $R$ be a left strongly $\mathcal{C}$-coherent ring, $M$ a left $R$-module and $n$ a nonnegative integer. Then the following are equivalent:

(1) $\mathcal{C}I$-$\text{dim}(RM) \leq n$.

(2) $\text{Ext}^{n+k}_R(C, M) = 0$ for all $C \in \mathcal{C}$ and all positive integers $k$.

(3) $\text{Ext}^{n+1}_R(C, M) = 0$ for all $C \in \mathcal{C}$.

(4) If the sequence $0 \to M \to E_0 \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0$ is exact with $E_0, \cdots, E_{n-1}$ $\mathcal{C}$-injective, then $E_n$ is also $\mathcal{C}$-injective.

(5) There exists an exact sequence of left $R$-modules $0 \to M \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0$ such that $E_0, \cdots, E_{n-1}, E_n$ are $\mathcal{C}$-injective.

\textit{Proof.} (1) $\Rightarrow$ (2). Use induction on $n$. If $n = 0$, then (2) holds by Theorem 1(5). Assume that $\text{Ext}^{n-1+k}_R(C, N) = 0$ for any $C \in \mathcal{C}$, any positive integer $k$ and any left $R$-module $N$ with $\mathcal{C}I$-$\text{dim}(N) \leq n-1$. Then for any left $R$-module $M$ with $\mathcal{C}I$-$\text{dim}(M) \leq n$. If $\mathcal{C}I$-$\text{dim}(M) = 0$, then (2) holds by Theorem 1(5). If $\mathcal{C}I$-$\text{dim}(M) > 0$, then there exists a positive integer $m \leq n$ such that $\text{Ext}^{m+1}_R(C, M) = 0$ for any $C \in \mathcal{C}$, which implies that $\text{Ext}^{m}_R(C, E(M)/M) = 0$ for any $C \in \mathcal{C}$. So $\mathcal{C}I$-$\text{dim}(E(M)/M) \leq m - 1$, and hence $\mathcal{C}I$-$\text{dim}(E(M)/M) \leq n - 1$. By hypothesis, we have $\text{Ext}^{n+k}_R(C, E(M)/M) = 0$ for any $C \in \mathcal{C}$ and any positive integer $k$, it follows that $\text{Ext}^{n+k}_R(C, M) = 0$. Therefore, (2) holds by induction axioms.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5) are obvious.

(3) $\Rightarrow$ (4). Since $R$ is left strongly $\mathcal{C}$-coherent and $E_0, \cdots, E_{n-1}$ is $\mathcal{C}$-injective, by Theorem 1(5), we have $\text{Ext}^{n+1}_R(C, M) \cong \text{Ext}^{n}_R(C, \text{im}(d_0)) \cong \text{Ext}^{n-1}_R(C, \text{im}(d_1)) \cong \cdots \cong \text{Ext}^{1}_R(C, \text{im}(d_{n-1})) = \text{Ext}^{1}_R(C, E_n)$. So (4) follows from (3).

(5) $\Rightarrow$ (3). It follows from the above isomorphism $\text{Ext}^{n+1}_R(C, M) \cong \text{Ext}^{1}_R(C, E_n)$. \hfill \Box

\textit{Definition 4.} (1). The $\mathcal{C}$-flat dimension of a module $MR$ is defined by
\[\mathcal{C}F$-dim$(MR) = \inf \{n : \text{Tor}^{n+1}_R(M, C) = 0 \text{ for every } C \in \mathcal{C}\}\]

(2). The $\mathcal{C}$-weak global dimension of a ring $R$ is defined by
\[\mathcal{C}$-WD$(R) = \sup \{\mathcal{C}F$-dim$(M) : M \text{ is a right } R\text{-module}\}\]
Theorem 3. Let $R$ be a left strongly $\mathcal{C}$-coherent ring, $M$ a right $R$-module and $n$ a non-negative integer. Then the following statements are equivalent:

1. $\mathcal{C}\mathcal{F}$-$\text{dim}(M_R) \leq n$.
2. $\text{Tor}^R_{n+k}(M, C) = 0$ for all $C \in \mathcal{C}$ and all positive integers $k$.
3. $\text{Tor}^R_{n+1}(M, C) = 0$ for all $C \in \mathcal{C}$.
4. If the sequence $0 \to F_n \xrightarrow{\varepsilon} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$ is exact with $F_0, \cdots, F_{n-1}$ $\mathcal{C}$-flat, then $F_n$ is also $\mathcal{C}$-flat.
5. There exists an exact sequence of right $R$-modules $0 \to F_n \xrightarrow{\varepsilon} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$ such that $F_0, \cdots, F_{n-1}, F_n$ are $\mathcal{C}$-flat.

Proof. (1) $\Rightarrow$ (2). Let $C \in \mathcal{C}$ and $k$ be any positive integer. By (1), there exists a nonnegative integer $m \leq n$ such that $\text{Tor}^R_{m+1}(M, C) = 0$. And so, by the isomorphism $\text{Tor}^R_{m+1}(M, C)^+ \cong \text{Ext}^{m+1}_R(C, M^+)$, we have $\text{Ext}^{m+1}_R(C, M^+) = 0$. Since $R$ is left strongly $\mathcal{C}$-coherent, by Theorem 2, we have $\text{Ext}^{n+k}_R(C, M^+) = 0$, and hence $\text{Tor}^R_{n+k}(M, C) = 0$ by the isomorphism $\text{Tor}^R_{n+k}(M, C)^+ \cong \text{Ext}^{n+k}_R(C, M^+)$. 

(2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5) are obvious.

(3) $\Rightarrow$ (4). Since $R$ is left strongly $\mathcal{C}$-coherent and $F_0, \cdots, F_{n-1}$ is $\mathcal{C}$-flat, by Theorem 1(10), we have $\text{Tor}^R_{n+1}(M, C) \cong \text{Tor}^R_{n}(\text{Ker}(d_0), C) \cong \text{Tor}^R_{n-1}(\text{Ker}(d_1), C) \cong \cdots \cong \text{Tor}^R_{1}(\text{Ker}(d_{n-1}), C) = \text{Tor}^R_{1}(F_n, C)$. So (4) follows from (3).

(5) $\Rightarrow$ (3). It follows from the above isomorphism $\text{Tor}^R_{n+1}(M, C) \cong \text{Tor}^R_{1}(F_n, C)$.

Next, we give some new characterizations of left $\mathcal{C}$-semihereditary rings.

Theorem 4. The following are equivalent for a ring $R$:

1. $R$ is left $\mathcal{C}$-semihereditary.
2. $R$ is left strongly $\mathcal{C}$-coherent and $\mathcal{C}\mathcal{I}$-$\text{GLD}(R) \leq 1$.
3. $R$ is left strongly $\mathcal{C}$-coherent and $\mathcal{C}$-$\text{WD}(R) \leq 1$.
4. Every torsionless right $R$-module is $\mathcal{C}$-flat.

Proof. (1) $\Rightarrow$ (2). Suppose that $R$ is left $\mathcal{C}$-semihereditary. Then it is clear that $R$ is left strongly $\mathcal{C}$-coherent. Now let $M$ be any left $R$-module. Then for any $C \in \mathcal{C}$, we have an exact sequence $0 \to K \to P \to C \to 0$ of left $R$-modules, where $P$ is finitely generated projective. By condition, $K$ is projective. Thus the exact sequence $0 = \text{Ext}^1_R(K, M) \to \text{Ext}^2_R(C, M) \to \text{Ext}^2_R(P, M) = 0$ implies that $\text{Ext}^2_R(C, M) = 0$. This follows that $\mathcal{C}\mathcal{I}$-$\text{GLD}(R) \leq 1$ by Definition 3.
(2) $\Rightarrow$ (3). It follows from Theorem 2 and the isomorphism
\[
\text{Tor}_2^R(M, C)^+ \cong \text{Ext}_R^2(C, M^+).
\]

(3) $\Rightarrow$ (1). Assume (3). Then $R$ is clearly left $\mathcal{C}$-coherent. Let $A$ be a submodule of a $\mathcal{C}$-flat right $R$-module $B$ and let $C \in \mathcal{C}$. Since $R$ is left strongly $\mathcal{C}$-coherent and $\mathcal{C}$-WD(R)$\leq 1$, by Theorem 3, we have $\text{Tor}_2^R(B/A, C) = 0$. Then, from the exactness of the sequence $0 = \text{Tor}_2^R(B/A, C) \rightarrow \text{Tor}_1^R(A, C) \rightarrow \text{Tor}_1^R(B, C) = 0$ we have $\text{Tor}_1^R(A, C) = 0$, and so $A$ is $\mathcal{C}$-flat. Therefore, by [18, Theorem 4.3(2)], $R$ is left $\mathcal{C}$-semihereditary.

(1) $\Rightarrow$ (4). Let $M$ be a torsionless right $R$-module. Then there exists an exact sequence $0 \rightarrow M \rightarrow \prod R_R$. Since $R$ is left $\mathcal{C}$-semihereditary, by [18, Theorem 4.3(2)], $R$ is left $\mathcal{C}$-coherent and every submodule of a $\mathcal{C}$-flat right $R$-module is $\mathcal{C}$-flat, so $M$ is $\mathcal{C}$-flat by [18, Theorem 3.3(4)].

(4) $\Rightarrow$ (1). Assume (4). Then $\prod R_R$ is $\mathcal{C}$-flat, and hence $R$ is left $\mathcal{C}$-coherent by [18, Theorem 3.3(4)]. Moreover, every right ideal of $R$ is torsionless and so is $\mathcal{C}$-flat. Thus, $R$ is left $\mathcal{C}$-semihereditary by [18, Theorem 4.3(3)].

By taking $\mathcal{C}$ to be the class of all finitely presented left $R$-modules, we have

**Corollary 2.** The following are equivalent for a ring $R$:

1. $R$ is left semihereditary.
2. $R$ is left coherent and left $\mathcal{FP}_I$-GLD($R$)$\leq 1$.
3. $R$ is left coherent and $\mathcal{WD}(R) \leq 1$.
4. Every torsionless right $R$-module is flat.

Recall that a ring $R$ is called left $PP$ [8] if every principal left ideal of $R$ is projective.

Our following Examples show that strongly $\mathcal{C}$-coherent rings need not be $\mathcal{C}$-semihereditary.

**Example 3.** (1) Let $K$ be a field and let $R = K \ltimes K$. Then $R$ is a commutative ring with only three ideals: $0$, $R$, and $(0,K) = R(0,1)$, so $R$ is generalized morphic. Since $\mathbf{I}_R(0,1) = (0,K)$ is not a direct summand of $R_R$, it is not a PP ring. So, let $\mathcal{C} = \{R/Ra : a \in R\}$. Then $R$ is strongly $\mathcal{C}$-coherent by Example 2(4), but it is not $\mathcal{C}$-semihereditary by [18, Example 4.2(3)].

(2) Let $R = \mathbb{Z}_4 = \{0, 1, 2, 3\}$. Then it is easy to see that $R$ is a commutative generalized morphic ring. Since $\mathbf{I}(2) = \{0, 2\}$ contains no nonzero idempotent elements, it is not a direct summand of $R_R$, so that $R_2$ is not projective, and thus $R$ is not PP. So, let $\mathcal{C} = \{R/Ra : a \in R\}$. Then $R$ is strongly $\mathcal{C}$-coherent but it is not a $\mathcal{C}$-semihereditary ring.

We remark that the above two examples show that commutative artinian rings need not be PP.


3. \( C \)-PROJECTIVE MODULES

Next, we give some characterizations of \( C \)-projective modules.

**Proposition 3.** Let \( M \) be a left \( R \)-module. Then the following are equivalent:

1. \( M \) is \( C \)-projective.
2. \( M \) is projective with respect to every exact sequence \( 0 \to A \to B \to C \to 0 \) of left \( R \)-modules with \( A \) \( C \)-injective.
3. \( M \) is projective with respect to the exact sequence \( 0 \to A \to E(A) \xrightarrow{\pi} E(A)/A \to 0 \) of left \( R \)-modules with \( A \) \( C \)-injective.

**Proof.** (1) \( \Rightarrow \) (2). By the exact sequence \( \text{Hom}(M,B) \to \text{Hom}(M,C) \to \text{Ext}^1_R(M,A) = 0 \).

(2) \( \Rightarrow \) (3). It is clear.

(3) \( \Rightarrow \) (1). For any \( C \)-injective module \( A \), we get an exact sequence \( \text{Hom}(M,E(A)) \xrightarrow{\pi_*} \text{Hom}(M,E(A)/A) \to \text{Ext}^1_R(M,A) \to \text{Ext}^1_R(M,E(A)) = 0 \).

By (3), \( \pi_* \) is epic, and so \( \text{Ext}^1_R(M,A) = 0 \). Therefore, \( M \) is \( C \)-projective. \( \square \)

Let \( L \) be a class of \( R \)-modules and \( M \) an \( R \)-module. Following [6], we say that a homomorphism \( \phi : M \to L \) where \( L \in L \) is a \( L \)-preenvelope of \( M \) if for any homomorphism \( f : M \to L' \) with \( L' \in L \), there is a \( g : L \to L' \) such that \( g\phi = f \).

**Proposition 4.** Let \( R \) be a \( C \)-injective ring and \( M \) be a left \( R \)-module. Then the following statements are equivalent:

1. \( M \) is \( C \)-projective.
2. For every exact sequence \( 0 \to K \to A \to M \to 0 \), where \( A \) is \( C \)-injective, \( K \to A \) is a \( C \)-injective preenvelope of \( K \).
3. \( M \) is a cokernel of a \( C \)-injective preenvelope \( K \to P \) with \( P \) projective.

**Proof.** (1) \( \Rightarrow \) (2). It follows from the exact sequence \( \text{Hom}(A,N) \to \text{Hom}(K,N) \to \text{Ext}^1_R(M,N) = 0 \), where \( N \) is \( C \)-injective.

(2) \( \Rightarrow \) (3). Let \( 0 \to K \to P \to M \to 0 \) be an exact sequence of left \( R \)-modules with \( P \) projective. Since \( R \) is a \( C \)-injective ring, by [18, Proposition 2.5], \( P \) is \( C \)-injective. And so, by (2), \( K \to P \) is a \( C \)-injective preenvelope. By (3), there exists an exact sequence \( 0 \to K \to P \to M \to 0 \), where \( P \) is projective and \( K \to P \) is a \( C \)-injective preenvelope. Thus we get an exact sequence \( \text{Hom}(P,N) \to \text{Hom}(K,N) \to \text{Ext}^1_R(M,N) \to 0 \) for every \( C \)-injective module \( N \). Note that the map \( \text{Hom}(P,N) \to \text{Hom}(K,N) \) is epic, we have that \( \text{Ext}^1_R(M,N) = 0 \), as required. \( \square \)

**Definition 5.** (1). The \( C \)-projective dimension of a module \( _RM \) is defined by
\(\mathcal{CP}-\dim(RM) = \inf\{n : \Ext_R^{n+1}(M, N) = 0 \text{ for every } \mathcal{C}\text{-injective module } N\}\)

(2). The \(\mathcal{C}\)-projective global dimension of a ring \(R\) is defined by

\[\mathcal{CP}\text{-GLD}(R) = \sup\{\mathcal{CP} - \dim(M) : M \text{ is a left } R\text{-module}\}\]

**Theorem 5.** Let \(R\) be a left strongly \(\mathcal{C}\)-coherent ring, \(M\) be a left \(R\)-module and \(n\) be a nonnegative integer. Then the following statements are equivalent:

(1) \(\mathcal{CP}\text{-dim}(RM) \leq n\).

(2) \(\Ext_R^{n+k}(M, N) = 0\) for all \(\mathcal{C}\text{-injective modules } N\) and all positive integers \(k\).

(3) \(\Ext_R^{n+1}(M, N) = 0\) for all \(\mathcal{C}\text{-injective modules } N\).

(4) If the sequence \(0 \to P_n \xrightarrow{\varepsilon} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0\) is exact with \(P_0, \cdots, P_{n-1}\) \(\mathcal{C}\)-projective, then \(P_n\) is also \(\mathcal{C}\)-projective.

(5) There exists an exact sequence of right \(R\)-modules \(0 \to P_n \xrightarrow{\varepsilon} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0\) such that \(P_0, \cdots, P_{n-1}, P_n\) are \(\mathcal{C}\)-projective.

**Proof.** (1) \(\Rightarrow\) (2). Use induction on \(n\). If \(n = 0\), then \(M\) is \(\mathcal{C}\)-projective. Since \(R\) is left strongly \(\mathcal{C}\)-coherent, by Theorem 1(12), (2) holds. Now assume that \(\Ext_R^{n-1+k}(L, N) = 0\) for any \(\mathcal{C}\text{-injective module } N\), any positive integer \(k\) and any left \(R\)-module \(L\) with \(\mathcal{CP}\text{-dim}(L) \leq n - 1\). Then for any left \(R\)-module \(M\) with \(\mathcal{CP}\text{-dim}(M) \leq n\). If \(\mathcal{CP}\text{-dim}(M) = 0\), then (2) holds by Theorem 1(12). \(\mathcal{CP}\text{-dim}(M) > 0\), then there exists a positive integer \(m \leq n\) such that \(\Ext_R^{m+1}(M, N) = 0\) for any \(\mathcal{C}\text{-injective module } N\). Let \(0 \to K \to P \to M \to 0\) be exact with \(P\) projective. Then it is easy to see that \(\Ext_R^m(K, N) = 0\) for any \(\mathcal{C}\text{-injective module } N\). So \(\mathcal{CP}\text{-dim}(K) \leq m - 1\), and hence \(\mathcal{CP}\text{-dim}(K) \leq n - 1\). By hypothesis, we have \(\Ext_R^{n-1+k}(K, N) = 0\) for any \(\mathcal{C}\text{-injective module } N\) and any positive integer \(k\), it follows that \(\Ext_R^{n+k}(M, N) = 0\). Therefore, (2) holds by induction axioms.

(2) \(\Rightarrow\) (3) \(\Rightarrow\) (1) and (4) \(\Rightarrow\) (5) are obvious.

(3) \(\Rightarrow\) (4). Since \(R\) is left strongly \(\mathcal{C}\)-coherent, by Theorem 1(12), we have \(\Ext_R^i(P, N) = 0\) for any positive integer \(i\), any \(\mathcal{C}\text{-projective left } R\text{-module } P\) and any \(\mathcal{C}\text{-injective left } R\text{-module } N\). Thus, \(\Ext_R^{n+1}(M, N) \cong \Ext_R^1(Ker(d_0), N) \cong \Ext_R^{n-1}(Ker(d_1), N) \cong \cdots \cong \Ext_R^1(Ker(d_{n-1}), N) = \Ext_R^1(P_n, N)\). So (4) follows from (3).

(5) \(\Rightarrow\) (3). It follows from the above isomorphism \(\Ext_R^{n+1}(M, N) \cong \Ext_R^1(P_n, N)\). \(\Box\)

Recall that a cotorsion pair \((\mathcal{A}, \mathcal{B})\) is called complete (see [6, Definition 7.16] and [15, Lemma 1.13]) if for any \(R\)-module \(M\), there is an exact sequence \(0 \to M \to B \to A \to 0\) with \(A \in \mathcal{A}\) and \(B \in \mathcal{B}\).
Theorem 6. The following are equivalent for a ring $R$:

1. $\mathcal{C}P-\text{GLD}(R)=0$.
2. Every left $R$-module is $\mathcal{C}$-projective.
3. Every $\mathcal{C}$-injective module is injective.
4. $R$ is left strongly $\mathcal{C}$-coherent and every $\mathcal{C}$-injective module is $\mathcal{C}$-projective.
5. Every cyclic left $R$-module is $\mathcal{C}$-projective.

In this case, $R$ is left noetherian.

Proof. (1) $\iff$ (2) $\iff$ (3) and (2) $\implies$ (4), (5) are obvious.

(4) $\implies$ (2). Let $M$ be a left $R$-module. By [18, Theorem 2.10(1)], $(\mathcal{C}P, \mathcal{CI})$ is a complete cotorsion pair, so there is a short exact sequence $0 \rightarrow M \rightarrow E \rightarrow P \rightarrow 0$, where $E$ is $\mathcal{C}$-injective and $P$ is $\mathcal{C}$-projective. By hypothesis, every $\mathcal{C}$-injective module is $\mathcal{C}$-projective, so $E$ is $\mathcal{C}$-projective. Since $R$ is left strongly $\mathcal{C}$-coherent, by Theorem 1(12), we have $\text{Ext}_R^1(E,N) = \text{Ext}_R^2(P,N) = 0$ for any $\mathcal{C}$-injective module $N$. It follows that $\text{Ext}_R^1(M,N) = 0$ for any $\mathcal{C}$-injective module $N$, and so $M$ is $\mathcal{C}$-projective.

(5) $\implies$ (3). Let $N$ be a $\mathcal{C}$-injective module. Then for any left ideal $I$, by (5), $R/I$ is $\mathcal{C}$-projective, so $\text{Ext}_R^1(R/I,N) = 0$, and thus $N$ is injective.

In this case, any direct sum of injective left $R$-modules is injective since any direct sum of $\mathcal{C}$-injective modules is $\mathcal{C}$-injective by [18, Proposition 2.5(3)], and hence $R$ is left noetherian. $\square$

Observing that every $FP$-injective left module over a left noetherian ring is injective, by taking $\mathcal{C}$ to be the class of all finitely presented left $R$-modules in Theorem 6, we have immediately the following results.

Corollary 3. The following are equivalent for a ring $R$:

1. $R$ is left noetherian.
2. Every left $R$-module is $FP$-projective.
3. Every $FP$-injective module is injective.
4. $R$ is left coherent and every $FP$-injective left $R$-module is $FP$-projective.
5. Every cyclic left $R$-module is $FP$-projective.

Theorem 7. Let $R$ be a ring and consider the following conditions:

1. Every submodule of a $\mathcal{C}$-projective module is $\mathcal{C}$-projective.
2. Every submodule of a projective left $R$-module is $\mathcal{C}$-projective.
3. Every left ideal of $R$ is $\mathcal{C}$-projective.
4. $\mathcal{C}P - \text{GLD}(R) \leq 1$.

Then we always have the following implications:

$$(1) \implies (2) \implies (3) \implies (4).$$
Furthermore, if $R$ is left strongly $\mathcal{C}$-coherent, then the four conditions are equivalent.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Let $N$ be any $\mathcal{C}$-injective module. Then for any left ideal $I$, we have $\text{Ext}^1_R(I, N) = 0$ by (3), it follows that $\text{Ext}^2_R(R/I, N) = 0$. So $id(N) \leq 1$, and then $\text{Ext}^2_R(M, N) = 0$ for every left $R$-module $M$. Therefore (4) holds.

Now suppose that $R$ is left strongly $\mathcal{C}$-coherent and (4) holds. Then we will prove (1), and thus, in this case, the four conditions are equivalent. In fact, let $K$ be a submodule of a $\mathcal{C}$-projective module $P$. Then, for any $\mathcal{C}$-injective module $N$, we get an exact sequence

$$0 = \text{Ext}^1_R(P, N) \rightarrow \text{Ext}^1_R(K, N) \rightarrow \text{Ext}^2_R(P/K, N).$$

Since $R$ is left strongly $\mathcal{C}$-coherent, by (4) and Theorem 5, we have $\text{Ext}^2_R(P/K, N) = 0$, and so $\text{Ext}^1_R(K, N) = 0$, which shows that $K$ is $\mathcal{C}$-projective, as required. $\square$

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