OPERATOR-SPLITTING METHOD FOR HIGH-DIMENSIONAL PARABOLIC EQUATION *VIA* FINITE ELEMENT METHOD

GANG PENG, JIANPING ZHAO and XINLONG FENG

Communicated by Dan Crişan

This work introduces a new operator-splitting method for solving the two-dimensional (2D) and three-dimensional (3D) parabolic equations. The aim is to reduce the computational complexity without loss of accuracy. Firstly, we split the 2D and 3D parabolic equations into a sequence of one-dimensional (1D) parabolic equations respectively, then we solve each 1D parabolic equation by using finite element method. In comparison with standard finite element method, the present method can save much CPU time. Furthermore, the stability analysis and error estimates for the proposed method are derived. Finally, numerical results of the 2D and 3D parabolic equations are presented to support our theoretical analysis.

AMS 2010 Subject Classification: 65M60, 65N12, 65N15.

Key words: high-dimensional parabolic equations, operator-splitting method, finite element method, error estimates.

1. INTRODUCTION

Since the high storage requirement and computational complexity, obtaining the numerical solution of high-dimensional parabolic equation is a hard work. To overcome this difficulty, many effective methods were presented [1, 3, 21, 23]. Among them, the sparse grid method [4–6] which is based on 1D hierarchical basis through tensor products, is a popular method for handling the high-dimensional parabolic equations. However, the theoretical analysis of this method is more difficult. The other popular way for solving high-dimensional parabolic equations is the operator-splitting method. In most of the previous studies, scholars mainly consider the operator-splitting method via finite difference method. The alternating direction implicit (ADI) method is a powerful operator-splitting method and it is first introduced in [7, 8] for solving the 2D heat equation. The ADI method is widely used for solving parabolic equations [9, 10, 24, 25], due to its unconditional stability and high efficiency. However, it is well known that finite difference methods [11–15, 22] are more difficult in solving the problem with irregular area and complex boundary conditions and suffer from a serious accuracy reduction in space for interface problems with different materials and nonsmooth solutions.

Recently, the heterogeneous alternating-direction method for Fokker-Planck equation has been studied by Knezevic and Sli [16]. Based on classical finite element method (FEM), it is well suited to implementation on a parallel computer and easy exploited to make largescale computations. Furthermore, Ganesan [17] discussed the mathematical methods for the population balance equations, where an operator-splitting Galerkin/SUPG finite element method is considered. The key idea of this method is to split the high-dimensional problem into two low-dimensional subproblems, and discretize the low-dimensional subproblems separately. In this article, the operator-splitting method via finite element method (OSFEM) is proposed to solve the 2D and 3D parabolic equations, respectively [2,18,19]. The basic idea of OSFEM is that we first split the 2D and 3D parabolic equations into a series of 1D parabolic equations, and then solve each 1D parabolic equation by using the classical FEM. The main advantages of this new method are problem simplification, low computational cost, and adaptability to the complex computation domain.

In this paper, we mainly study the following parabolic equation:

$$\begin{cases} \dfrac{\partial u(t,\mathbf{x})}{\partial t} - \Delta u(t,\mathbf{x}) = f(t,\mathbf{x}) & \text{in } \Omega \times J, \\ u(t,\mathbf{x}) = 0 & \text{on } \Gamma \times J, \\ u(0,\mathbf{x}) = u_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{1}$$

where $\mathbf{x} \in \Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) can be defined as a Cartesian product, Δ denotes the Laplace operator, Γ the boundary of Ω, u_0 the known initial value, f the known body force, and $t \in J = (0,T]$ ($T > 0$), respectively. In order to introduce the OSFEM, we take the 2D parabolic equation for instance, and present the stability analysis and error estimates. For the 3D parabolic equation, similar results can also be obtained.

The rest of this paper is organized as follows. In the next section, we present the finite element approximation for the parabolic equation (1). In Section 3, we will deduce the split form of diffusion equation and convection-diffusion equation by using the OSFEM, respectively. In Section 4, we present the stability analysis and error estimates of the OSFEM on regular domain. Furthermore, several numerical examples are given to show the effectiveness and practicability of the OSFEM in Section 5. Finally, we discuss the conclusions and suggestions of the OSFEM.

2. FINITE ELEMENT APPROXIMATION

For the mathematical setting of parabolic problem (1), we first introduce the L^2-inner product and norm over the domain Ω based on standard Sobolev spaces [1,23]:

$$(v,q)=\int_\Omega v(\mathbf{x})q(\mathbf{x})\mathrm{d}\mathbf{x},\ \ \|v\|_0^2=(v,v),\ \forall v,q\in L^2(\Omega).$$

Then, we have the following variational formulation of problem (1): given $u_0\in L^2(\Omega)$ and $f\in L^2((0,T]\times\Omega)$, find $u\in L^2(0,T;H_0^1(\Omega))$ such that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(u(t),v)+(\nabla u(t),\nabla v)=(f(t),v),\ \forall v\in H_0^1(\Omega),\\&(u(0),v)=(u_0,v).\end{aligned}\tag{2}$$

2.1. FINITE ELEMENT SPACES

Let $\Omega_x=[0,1]$, $\Omega_y=[0,1]$, and denotes, $\Omega=\Omega_x\times\Omega_y$. We know that $H^p(\Omega_x)$ and $H^p(\Omega_y)$ are the usual Sobolev spaces containing L^2-functions with weak derivative of order $p\geq 0$. Then, we define

$$H^{p,p}=H^p(\Omega_x;H^p(\Omega_y))\cap H^p(\Omega_y;H^p(\Omega_x)),$$

and the associated norm and seminorm

$$\|v\|_{p,p}^2:=\sum_{|\beta|\leq p}\sum_{|\alpha|\leq p}\|\partial_y^\beta\partial_x^\alpha v\|_{L^2(\Omega)}^2,\ \ |v|_{p,p}^2:=\sum_{|\beta|=p}\sum_{|\alpha|=p}\|\partial_y^\beta\partial_x^\alpha v\|_{L^2(\Omega)}^2.$$

Next, we introduce the finite element spaces $V_h\subset H_0^1(\Omega_x)$ and $W_h\subset H_0^1(\Omega_y)$, such that

$$\begin{aligned}V_h&=\mathbf{span}\{\phi_1(x),\phi_2(x),\ldots,\phi_M(x)\},\\W_h&=\mathbf{span}\{\psi_1(y),\psi_2(y),\ldots,\psi_N(y)\}.\end{aligned}$$

The finite element space $H^{1,1}(\Omega)$ is defined as follows:

$$V_h\times W_h=\{\varepsilon_h:\varepsilon_h=\sum_{j=1}^M\sum_{l=1}^N\varepsilon_{j,l}\phi_j(x)\psi_l(y),\ \varepsilon_{j,l}\in R\}\subset H^{1,1}(\Omega).$$

Further, the finite element functions can be written as:

$$u_h(t,x,y)=\sum_{j=1}^M\sum_{l=1}^N u_{j,l}(t)\phi_j(x)\psi_l(y).\tag{3}$$

Hence, we can obtain

$$\nabla_x u_h=\sum_{j=1}^M\sum_{l=1}^N u_{j,l}(\nabla_x\phi_j)\psi_l,\ \nabla_y u_h=\sum_{j=1}^M\sum_{l=1}^N u_{j,l}\phi_j(\nabla_y\psi_l).\tag{4}$$

With the above definitions, the weak formulation of problem (1) can be written as: given u_0 and f, find $u_h(t) \in V_h \times W_h$ such that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(u_h(t), v_h) + (\nabla u_h(t), \nabla v_h) = (f(t), v_h),\ \forall v_h \in V_h \times W_h,\\
&(u_h(0), v_h) = (u_0, v_h).
\end{aligned} \tag{5}
$$

2.2. TIME DISCRETIZATION: BACKWARD EULER SCHEME

Let $m > 0$ be a fixed integer number and $0 = t^0 < t^1 < \ldots < t^m = T$ be the time discretization. Then $\delta t = t^n - t^{n-1}, 1 \leq n \leq m$, denote the discrete time step. The backward Euler scheme of problem (5) is as follows:

$$
\begin{aligned}
&\Big(\frac{u_h^n - u_h^{n-1}}{\delta t}, v_h\Big) + (\nabla u_h^n, \nabla v_h) = (f^n, v_h),\ \forall v_h \in V_h \times W_h,\\
&(u_h^0, v_h) = (u_0, v_h).
\end{aligned} \tag{6}
$$

Define the four matrices $M_x,\ M_y,\ A_x$ and A_y as follows:

$$
[M_x]_{ij} = \int_{\Omega_x} \phi_i\phi_j \mathrm{d}x, \quad [A_x]_{ij} = \int_{\Omega_x} \nabla_x\phi_i \cdot \nabla_x\phi_j \mathrm{d}x, \ \ 1 \leq i, j \leq M,
$$

$$
[M_y]_{kl} = \int_{\Omega_y} \psi_k\psi_l \mathrm{d}y, \quad [A_y]_{kl} = \int_{\Omega_y} \nabla_y\psi_k \cdot \nabla_y\psi_l \mathrm{d}y, \ \ 1 \leq k, l \leq N,
$$

where $\otimes$ denotes the kronecker product of two matrices. Then, the mass matrix $[M]_{MN\times MN}$ can be written as:

$$
[M]_{MN\times MN} = M_x \otimes M_y = \begin{bmatrix} [\Phi_{1,1}]_{k,l} & \cdots & [\Phi_{1,M}]_{k,l} \\ \vdots & \ddots & \vdots \\ [\Phi_{M,1}]_{k,l} & \ldots & [\Phi_{M,M}]_{k,l} \end{bmatrix}.
$$

Moreover, the element of $M_x \otimes M_y$ can be written as follows:

$$
[\Phi_{i,j}]_{k,l} := \int_{\Omega_x} \phi_i\phi_j \mathrm{d}x \int_{\Omega_y} \psi_k\psi_l \mathrm{d}y, \ \ 1 \leq k, l \leq N.
$$

Thus, the algebraic form of problem (6) reads:

$$
\begin{aligned}
&\{(M_x \otimes M_y) + \delta t((A_x \otimes M_y) + (M_x \otimes A_y))\}U^n\\
&\qquad\qquad = \delta t F^n + (M_x \otimes M_y)U^{n-1},
\end{aligned} \tag{7}
$$

where

$$
[F^n]_{N(i-1)+k,1} = \int_{\Omega} f^n \phi_i \psi_k \mathrm{d}x\mathrm{d}y,\ 1 \leq i \leq M,\ 1 \leq k \leq N,
$$

and U^n is the vectorization of matrix $[u_{j,l}^n]^{\mathrm{T}}$.

3. THE OSFEM FOR PARABOLIC EQUATIONS

In this section, we will present the split form of parabolic equations. Firstly, we use the OSFEM to solve 2D parabolic equation. Secondly, the split form of 3D parabolic equation will be presented. Furthermore, we use the OSFEM to deal with the convection-diffusion equation.

3.1. THE OSFEM FOR 2D PARABOLIC EQUATION

Setting

$$\Delta = \Delta_x + \Delta_y, \quad \Delta_x = \frac{\partial^2}{\partial x^2}, \quad \Delta_y = \frac{\partial^2}{\partial y^2}.$$

After time discretization, we can split the 2D parabolic problem into two subproblems in (t^{n-1}, t^n) as follows:

Sp 1.1. We solve the 1D parabolic problem of $Y-$direction, that is: find $\hat{u}(t, x, y)$ for all $x \in \Omega_x$ such that

$$(8) \qquad \begin{cases} \dfrac{\partial \hat{u}}{\partial t} - \Delta_y \hat{u} = f & \text{in } (t^{n-1}, t^n) \times \Omega_y, \\ \hat{u} = 0 & \text{on } (t^{n-1}, t^n) \times \Gamma_y, \\ \hat{u}(t^{n-1}, x, y) = u(t^{n-1}, x, y). \end{cases}$$

Sp 1.2. We solve the 1D parabolic problem of $X-$direction, that is: find $\widetilde{u}(t, x, y)$ for all $y \in \Omega_y$ such that

$$(9) \qquad \begin{cases} \dfrac{\partial \widetilde{u}}{\partial t} - \Delta_x \widetilde{u} = 0 & \text{in } (t^{n-1}, t^n) \times \Omega_x, \\ \widetilde{u} = 0 & \text{on } (t^{n-1}, t^n) \times \Gamma_x, \\ \widetilde{u}(t^{n-1}, x, y) = \hat{u}(t^n, x, y). \end{cases}$$

With the fixed $x_j \in \Omega_x, j = 1, \cdots, M$, and $y_l \in \Omega_y, l = 1, \cdots, N$ of the above subproblems, we define the finite element functions $\hat{u}_h(t, x_j, y) \in W_h$ and $\widetilde{u}_h(t, x, y_l) \in V_h$ as:

$$\hat{u}_h(t, x_j, y) = \sum_{l=1}^{N} \hat{u}_{j,l}(t)\psi_l(y), \ \ \widetilde{u}_h(t, x, y_l) = \sum_{j=1}^{M} \widetilde{u}_{j,l}(t)\phi_j(x).$$

Hence, the discrete form of subproblems (8) and (9) in (t^{n-1}, t^n) can be written as:

Dsp 1.1. Given $\hat{u}_h^{n-1}(x_j,y)=u_h^{n-1}(x_j,y)$ and $f^n(x_j,y)$, find $\hat{u}_h^n(x_j,y)\in W_h$ such that for all $v_h\in W_h$ with $j=1,\cdots,M$ satisfying

$$\Big(\frac{\hat{u}_h^n-\hat{u}_h^{n-1}}{\delta t},v_h\Big)_{\Omega_y}+(\nabla_y\hat{u}_h^n,\nabla_y v_h)_{\Omega_y}=(f^n,v_h)_{\Omega_y}. \tag{10}$$

Dsp 1.2. Given $\widetilde{u}_h^{n-1}(x,y_l)=\hat{u}_h^n(x,y_l)$, find $\widetilde{u}_h^n(x,y_l)\in V_h$ such that for all $v_h\in V_h$ with $l=1,\cdots,N$ satisfying

$$\Big(\frac{\widetilde{u}_h^n-\widetilde{u}_h^{n-1}}{\delta t},v_h\Big)_{\Omega_x}+(\nabla_x\widetilde{u}_h^n,\nabla_x v_h)_{\Omega_x}=0. \tag{11}$$

Then, we obtain the numerical solution

$$\widetilde{u}_h^n(x,y)=\sum_{j=1}^{M}\sum_{l=1}^{N}\widetilde{u}_{j,l}^n\phi_j(x)\psi_l(y),$$

and then let $u_h^n(x,y)=\widetilde{u}_h^n(x,y)$ to compute $u_h^{n+1}(x,y)$ further.

Now we have the algebraic form of (10) and (11) as follows:

$$(M_y+\delta tA_y)\hat{U}^n=\delta tF_y^n+M_yU^{n-1}, \tag{12}$$

$$(M_x+\delta tA_x)\widetilde{U}^n=M_x(\hat{U}^n)^{\mathrm{T}}. \tag{13}$$

Where

$$[F_y^n]_{k,i}=\int_{\Omega_y}f^n\psi_k\mathrm{d}y,\quad 1\le k\le N,\quad 1\le i\le M.$$

Next, multiply (12) by $M_x\otimes I_N$, and (13) by $I_M\otimes(M_y+\delta tA_y)$ respectively, we obtain

$$\{(M_x\otimes M_y)+\delta t(M_x\otimes A_y)\}\hat{U}^n=\delta t(M_x\otimes I_N)F_y^n+(M_x\otimes M_y)U^{n-1}, \tag{14}$$

and

$$\begin{aligned}&\{(M_x\otimes M_y)+\delta t\{(A_x\otimes M_y)+(M_x\otimes A_y)\}+\delta t^2(A_x\otimes A_y)\}\widetilde{U}^n\\&\qquad=\{(M_x\otimes M_y)+\delta t(M_x\otimes A_y)\}\hat{U}^n.\end{aligned} \tag{15}$$

From (14) and (15), we have

$$\begin{aligned}&\{(M_x\otimes M_y)+\delta t\{(A_x\otimes M_y)+(M_x\otimes A_y)\}+(\delta t)^2(A_x\otimes A_y)\}\widetilde{U}^n\\&\qquad=\delta t(M_x\otimes I_N)F_y^n+(M_x\otimes M_y)U^{n-1}.\end{aligned} \tag{16}$$

The element of vector $(A_x\otimes A_y)\widetilde{U}^n$ is as follows

$$\begin{aligned}\int_\Omega\sum_{j=1}^{M}\sum_{l=1}^{N}\widetilde{u}_{j,l}^n\frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x}\frac{\partial\psi_k}{\partial y}\frac{\partial\psi_l}{\partial y}\mathrm{d}x\mathrm{d}y&=\Big(\sum_{j=1}^{M}\sum_{l=1}^{N}\widetilde{u}_{j,l}^n\frac{\partial\phi_j}{\partial x}\frac{\partial\psi_l}{\partial y},\frac{\partial\phi_i}{\partial x}\frac{\partial\psi_k}{\partial y}\Big)\\&=(\nabla_x\nabla_y\widetilde{u}_h^n,\nabla_x\nabla_y v_h),\ \forall v_h\in V_h\times W_h.\end{aligned}$$

With the property of finite element space V_h, we have

$$\int_{\Omega_x} \sum_{j=1}^{M} \phi_j \mathrm{d}y = 1.$$

Hence, the element of $(M_x \otimes I_N)F_y^n$ can be written as:

$$\begin{aligned}\int_{\Omega_x \times \Omega_y} f^n \phi_i \psi_k \sum_{j=1}^{M} \phi_j \mathrm{d}x\mathrm{d}y &= \int_{\Omega} f^n \phi_i \psi_k \mathrm{d}x\mathrm{d}y \Big(\int_{\Omega_x} \sum_{j=1}^{M} \phi_j \mathrm{d}x \Big) \\ &= \int_{\Omega} f^n \phi_i \psi_k \mathrm{d}x\mathrm{d}y.\end{aligned} \tag{17}$$

Using (7) and (17), we have $(M_x \otimes I_N)F_y^n = F^n$. As shown in [2], (16) can be derived from

$$\begin{aligned}\Big(\frac{u_h^n - u_h^{n-1}}{\delta t}, v_h\Big) + (\nabla u_h^n, \nabla v_h) + \delta t(\nabla_x \nabla_y u_h^n, \nabla_x \nabla_y v_h) \\ = (f^n, v_h),\ \forall v_h \in V_h \times W_h.\end{aligned} \tag{18}$$

From (7) and (16), we know that the solution U^n of (7) is approximated by $\widetilde{U}^n$ of (16).

3.2. THE OSFEM FOR 3D PARABOLIC EQUATION

Assume that the domain $\Omega \subset \mathbb{R}^3$ can be defined as a Cartesian product of three subdomains $\Omega = \Omega_x \times \Omega_y \times \Omega_z$. Setting

$$\Delta = \Delta_x + \Delta_y + \Delta_z, \quad \Delta_x = \frac{\partial^2}{\partial x^2}, \quad \Delta_y = \frac{\partial^2}{\partial y^2}, \quad \Delta_z = \frac{\partial^2}{\partial z^2}.$$

After the time discretization, we split the 3D parabolic problem (1) into three subproblems in (t^{n-1}, t^n) as follows:

Sp 2.1. We solve the 1D parabolic problem of $Z-$direction, that is: find $\hat{u}(t, x, y, z)$ for all $(x, y) \in \Omega_x \times \Omega_y$ such that

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t} - \Delta_z \hat{u} = f & \text{in } (t^{n-1}, t^n) \times \Omega_z, \\ \hat{u} = 0 & \text{on } (t^{n-1}, t^n) \times \Gamma_z, \\ \hat{u}(t^{n-1}, x, y, z) = u(t^{n-1}, x, y, z). \end{cases} \tag{19}$$

Sp 2.2. We solve the 1D parabolic problem of $Y-$direction, that is: find $\widetilde{u}(t,x,y,z)$ for all $(x,z)\in\Omega_x\times\Omega_z$ such that

$$\begin{cases}\dfrac{\partial\widetilde{u}}{\partial t}-\Delta_y\widetilde{u}=0 & \text{in } (t^{n-1},t^n)\times\Omega_y,\\ \widetilde{u}=0 & \text{on } (t^{n-1},t^n)\times\Gamma_y,\\ \widetilde{u}(t^{n-1},x,y,z)=\hat{u}(t^n,x,y,z). & \end{cases}\tag{20}$$

Sp 2.3. We solve the 1D parabolic problem of $X-$direction, that is: find $\overline{u}(t,x,y,z)$ for all $(y,z)\in\Omega_y\times\Omega_z$ such that

$$\begin{cases}\dfrac{\partial\overline{u}}{\partial t}-\Delta_x\overline{u}=0 & \text{in } (t^{n-1},t^n)\times\Omega_x,\\ \overline{u}=0 & \text{on } (t^{n-1},t^n)\times\Gamma_x,\\ \overline{u}(t^{n-1},x,y,z)=\widetilde{u}(t^n,x,y,z). & \end{cases}\tag{21}$$

Here, $u(t^n,x,y,z)=\overline{u}(t^n,x,y,z)$, and the discrete form of (19)–(21) can be presented as the 2D parabolic equation. And then, the algebraic form of the 3D parabolic equation will be obtained. In addition, we can also split the 3D parabolic equation into the 2D and 1D subproblems in one time step.

3.3. THE OSFEM FOR CONVECTION-DIFFUSION EQUATION

In this subsection, we use the OSFEM to deal with the following 2D convection-diffusion equation

$$\begin{cases}\dfrac{\partial u(t,\mathbf{x})}{\partial t}-\Delta u(t,\mathbf{x})+\beta\cdot\nabla u(t,\mathbf{x})=f(t,\mathbf{x}) & \text{in } \Omega\times J,\\ u(t,\mathbf{x})=0 \qquad \text{on } \Gamma\times J, & \\ u(0,\mathbf{x})=u_0(\mathbf{x}), \qquad \text{in } \Omega, & \end{cases}\tag{22}$$

where $\Omega\subset\mathbb{R}^2$ can be defined as a Cartesian product of two subdomains $\Omega=\Omega_x\times\Omega_y$, $\beta=(\alpha_1,\alpha_2)$ the known convection coefficient. Let $\beta_1=(0,\alpha_1)$ and $\beta_2=(\alpha_2,0)$. After the time discretization, we split the problem (22) into two subproblems as follows:

Sp 3.1. We solve the 1D parabolic problem of $Y-$direction, that is: find $\hat{u}(t,x,y)$ for all $x\in\Omega_x$ such that

$$\begin{cases}\dfrac{\partial\hat{u}}{\partial t}-\Delta_y\hat{u}+\beta_1\cdot\nabla\hat{u}=f & \text{in } (t^{n-1},t^n)\times\Omega_y,\\ \hat{u}=0 & \text{on } (t^{n-1},t^n)\times\Gamma_y,\\ \hat{u}(t^{n-1},x,y)=u(t^{n-1},x,y). & \end{cases}\tag{23}$$

Sp 3.2. We solve the 1D parabolic problem of $X-$direction, that is: find $\widetilde{u}(t,x,y)$ for all $y \in \Omega_y$ such that

$$
\begin{cases}
\dfrac{\partial \widetilde{u}}{\partial t} - \Delta_x \widetilde{u} + \beta_2 \cdot \nabla \widetilde{u} = 0 & \text{in } (t^{n-1}, t^n) \times \Omega_x, \\
\widetilde{u} = 0 & \text{on } (t^{n-1}, t^n) \times \Gamma_x, \\
\widetilde{u}(t^{n-1}, x, y) = \hat{u}(t^n, x, y).
\end{cases} \tag{24}
$$

Here, $u(t^n,x,y) = \widetilde{u}(t^n,x,y)$, and the discrete form of (23) and (24) can also be presented as the 2D parabolic equation. Furthermore, the 3D convection-diffusion equation can also be solved similar to that.

4. STABILITY ANALYSIS AND ERROR ESTIMATES

In this section, we will present stability analysis and error estimates of the 2D parabolic problem. To do this, let

$$
d_t u_h^n = \frac{u_h^n - u_h^{n-1}}{\delta t}, \ \hat{f}^n = \frac{1}{\delta t} \int_{t^{n-1}}^{t^n} f(t) \mathrm{d}t.
$$

4.1. STABILITY ANALYSIS OF THE OSFEM

THEOREM 4.1. *The numerical scheme (8)–(9) is unconditionally stable, that is,*

$$
\begin{aligned}
\|u_h^m\|_0^2 + \sum_{n=1}^{m} \delta t^2 \|d_t u_h^n\|_0^2 + 2\delta t^2 \sum_{n=1}^{m} \|\nabla_x \nabla_y u_h^n\|_0^2 \\
\le \|u_h^0\|_0^2 + C \int_0^T \|f(t)\|_0^2 \mathrm{d}t.
\end{aligned} \tag{25}
$$

Proof. Taking $v_h = 2\delta t u_h^n$ in (18) yields

$$
\begin{aligned}
&2(u_h^n - u_h^{n-1}, u_h^n) + 2\delta t(\nabla u_h^n, \nabla u_h^n) \\
&\qquad + 2\delta t^2 (\nabla_x \nabla_y u_h^n, \nabla_x \nabla_y u_h^n) = 2\delta t(\hat{f}^n, u_h^n).
\end{aligned}
$$

With the identity $2(a-b,a) = |a|^2 - |b|^2 + |a-b|^2$ and the Cauchy-Schwarz's inequality, we deduce that

$$
\begin{aligned}
\|u_h^n\|_0^2 - \|u_h^{n-1}\|_0^2 + \delta t^2 \|d_t u_h^n\|_0^2 + 2\delta t \|\nabla u_h^n\|_0^2 + 2\delta t^2 \|\nabla_x \nabla_y u_h^n\|_0^2 \\
\le 2\delta t \|\nabla u_h^n\|_0^2 + C \int_{t^{n-1}}^{t^n} \|f(t)\|_0^2 \mathrm{d}t.
\end{aligned} \tag{26}
$$

Further, we have

$$\|u_h^n\|_0^2 + \delta t^2\|d_t u_h^n\|_0^2 + 2\delta t^2\|\nabla_x\nabla_y u_h^n\|_0^2 \le \|u_h^{n-1}\|_0^2 + C\int_{t^{n-1}}^{t^n}\|f(t)\|_0^2 \mathrm{d}t. \tag{27}$$

The initial solution u_0 is approximated by u_h^0 and sum (27) with respect to n from 1 to m. Then, the proof is completed. $\square$

4.2. ERROR ESTIMATES OF THE BACKWARD EULER SCHEME

First, we introduce the approximation properties of the finite element spaces V_h and W_h (Theorem 4.8 in [20]). Assume that there are interpolation operators $I_x \in \zeta(H_0^1(\Omega_x); V_h)$ and $I_y \in \zeta(H_0^1(\Omega_y); W_h)$ such that for all $1 \le s \le r+1$

$$\begin{aligned}
&\|u - I_x u\|_{L^2(\Omega_x)} + h\|u - I_x u\|_{H^1(\Omega_x)} \le Ch^s\|u\|_{H^s(\Omega_x)},\\
&\|I_x u\|_{H^s(\Omega_x)} \le C\|u\|_{H^s(\Omega_x)}, \ \forall u \in H_0^1(\Omega_x)\cap H^s(\Omega_x),\\
&\|u - I_y u\|_{L^2(\Omega_y)} + h\|u - I_y u\|_{H^1(\Omega_y)} \le Ch^s\|u\|_{H^s(\Omega_y)},\\
&\|I_y u\|_{H^s(\Omega_y)} \le C\|u\|_{H^s(\Omega_y)}, \ \forall u \in H_0^1(\Omega_y)\cap H^s(\Omega_y).
\end{aligned} \tag{28}$$

Here, r is the order of finite element space $V_h \times W_h$ and $\zeta(X;Y)$ denotes the set of continuous linear mappings from X to Y. Then, given an interpolation operator $I_h \in \zeta(H_0^{1,1}(\Omega)\cap H^{r+1,r+1}(\Omega), V_h \times W_h)$ as

$$I_h := I_x I_y = I_y I_x. \tag{29}$$

For the error estimates, we divide it into two parts and define

$$e_h^n = u(t^n) - u_h^n = (u(t^n) - I_h u(t^n)) + (I_h u(t^n) - u_h^n) = \eta^n + \xi^n. \tag{30}$$

THEOREM 4.2. *Assume that $0 < \delta t \le \frac{1}{2}$ and assumptions (28)–(29) hold, then we have the following estimate:*

$$\|e_h^n\|_0^2 \le C(h^{2r} + \delta t^2), \ n = 1, 2, \cdots m. \tag{31}$$

Proof. Taking $v_h = \xi^n$ in (18) yields

$$\begin{aligned}
&\Big(\frac{\xi^n - \xi^{n-1}}{\delta t}, \xi^n\Big) + (\nabla\xi^n, \nabla\xi^n) + \delta t(\nabla_x\nabla_y\xi^n, \nabla_x\nabla_y\xi^n)\\
&= \Big(\frac{u(t^n) - u(t^{n-1})}{\delta t} - \frac{\eta^n - \eta^{n-1}}{\delta t}, \xi^n\Big) + (\nabla(u(t^n) - \eta^n), \nabla\xi^n)\\
&+ \delta t(\nabla_x\nabla_y(u(t^n) - \eta^n), \nabla_x\nabla_y\xi^n) - (f^n, \xi^n).
\end{aligned} \tag{32}$$

Since $\xi^n|_{\partial\Omega_x} = 0$ and $\xi^n|_{\partial\Omega_y} = 0$, we have

$$(\nabla_x\nabla_y u(t^n), \nabla_x\nabla_y\xi^n) = (\Delta_x\Delta_y u(t^n), \xi^n).$$

Using (2) for $t = t^n$ and $v = \xi^n$ and setting

$$S^n = \frac{u(t^n) - u(t^{n-1})}{\delta t} - \frac{\partial u(t^n)}{\partial t} - \frac{\eta^n - \eta^{n-1}}{\delta t} + \delta t \Delta_x \Delta_y u(t^n),$$

then (32) can be written as

$$\begin{aligned}(33)\qquad &\Big(\frac{\xi^n - \xi^{n-1}}{\delta t}, \xi^n\Big) + (\nabla \xi^n, \nabla \xi^n) + \delta t(\nabla_x \nabla_y \xi^n, \nabla_x \nabla_y \xi^n) \\ &\qquad = (S^n, \xi^n) - \delta t(\nabla_x \nabla_y \eta^n, \nabla_x \nabla_y \xi^n) - (\nabla \eta^n, \nabla \xi^n).\end{aligned}$$

Further, we have

$$\begin{aligned}(34)\qquad &\|\xi^n\|_0^2 + \|\xi^n - \xi^{n-1}\|_0^2 + 2\delta t \|\nabla \xi^n\|_0^2 + 2\delta t^2 \|\nabla_x \nabla_y \xi^n\|_0^2 \le \|\xi^{n-1}\|_0^2 \\ &\qquad + 2\delta t |(S^n, \xi^n)| + 2\delta t |(\nabla \eta^n, \nabla \xi^n)| + 2\delta t^2 |(\nabla_x \nabla_y \eta^n, \nabla_x \nabla_y \xi^n)|.\end{aligned}$$

Using the Cauchy-Schwarz's and Young's inequality, we obtain

$$\begin{aligned}&2\delta t |(S^n, \xi^n)| \le \delta t \|S^n\|_0^2 + \delta t \|\xi^n\|_0^2, \\ &2\delta t |(\nabla \eta^n, \nabla \xi^n)| \le \delta t \|\nabla \eta^n\|_0^2 + \delta t \|\nabla \xi^n\|_0^2, \\ &2\delta t^2 |(\nabla_x \nabla_y \eta^n, \nabla_x \nabla_y \xi^n)| \le \delta t^2 \|\nabla_x \nabla_y \eta^n\|_0^2 + \delta t^2 \|\nabla_x \nabla_y \xi^n\|_0^2.\end{aligned}$$

Combining the above inequalities with (34) yields

$$\begin{aligned}(35)\qquad &(1 - \delta t)\|\xi^n\|_0^2 + \delta t \|\nabla \xi^n\|_0^2 \le \|\xi^{n-1}\|_0^2 + \delta t \|S^n\|_0^2 \\ &\qquad + \delta t \|\nabla \eta^n\|_0^2 + \delta t^2 \|\nabla_x \nabla_y \eta^n\|_0^2.\end{aligned}$$

Applying Taylor's theorem with integral remainder for S^n, we have

$$\begin{aligned}\delta t \|S^n\|_0^2 \le C\Big(&\delta t^2 \int_{t^{n-1}}^{t^n} \Big\|\frac{\partial^2 u(s)}{\partial s^2}\Big\|_0^2 \mathrm{d}s \\ &+ \int_{t^{n-1}}^{t^n} \Big\|\frac{\partial \eta(s)}{\partial s^2}\Big\|_0^2 \mathrm{d}s + \delta t^3 \|\Delta_x \Delta_y u(t^n)\|_0^2\Big).\end{aligned}$$

Using $1 \le \frac{1}{1-\delta t} \le 2$ and combining the above inequality with (35), we obtain

$$\begin{aligned}(36)\qquad \|\xi^n\|_0^2 + \delta t \|\nabla \xi^n\|_0^2 &\le (1 + 2\delta t)\|\xi^{n-1}\|_0^2 \\ &+ C\Big(\delta t^2 \int_{t^{n-1}}^{t^n} \Big\|\frac{\partial^2 u(s)}{\partial s^2}\Big\|_0^2 \mathrm{d}s + \int_{t^{n-1}}^{t^n} \Big\|\frac{\partial \eta(s)}{\partial s^2}\Big\|_0^2 \mathrm{d}s \\ &+ \delta t \|\nabla \eta^n\|_0^2 + \delta t^3 \|\Delta_x \Delta_y u(t^n)\|_0^2 + \delta t^2 \|\nabla_x \nabla_y \eta^n\|_0^2\Big).\end{aligned}$$

Summing (36) with respect to n from 1 to m, we have

$$\|\xi^m\|_0^2 + \sum_{n=1}^{m} \delta t \|\nabla \xi^n\|_0^2 \le \|\xi^0\|_0^2 + 2\sum_{n=1}^{m} \delta t \|\xi^{n-1}\|_0^2$$

$$
\begin{aligned}
&+C\Big(\delta t^2\int_0^T\Big\|\frac{\partial^2 u(s)}{\partial s^2}\Big\|_0^2\mathrm{d}s+\sum_{n=1}^m\delta t^3\|\Delta_x\Delta_y u(t^n)\|_0^2\\
&+\int_0^T\Big\|\frac{\partial\eta(s)}{\partial s}\Big\|_0^2\mathrm{d}s+\sum_{n=1}^m\delta t\|\nabla\eta^n\|_0^2+\sum_{n=1}^m\delta t^2\|\nabla_x\nabla_y\eta^n\|_0^2\Big).
\end{aligned}
$$

Finally, using the discrete Gronwall's lemma, we deduce that

$$
\begin{aligned}
\|\xi^m\|_0^2+\sum_{n=1}^m\delta t\|\nabla\xi^n\|_0^2\le C\exp(2T)\Big(&\|\xi^0\|_0^2+\delta t^2\Big\|\frac{\partial^2 u(s)}{\partial s^2}\Big\|_{L^2(0,T;L^2(\Omega))}\\
&+\sum_{n=1}^m\delta t\|\nabla\eta^n\|_0^2+\delta t^2\|\Delta_x\Delta_y u\|^2_{L^2(0,T;L^2(\Omega))}\\
&+\int_0^T\Big\|\frac{\partial\eta(s)}{\partial s}\Big\|_0^2\mathrm{d}s+\delta t^2\sum_{n=1}^m\|\nabla_x\nabla_y\eta^n\|_0^2\Big).
\end{aligned}\tag{37}
$$

Moreover, we deduce from (28)–(30) that

$$
\begin{aligned}
&\|\nabla\eta\|_0^2=\|\nabla_x\eta\|_0^2+\|\nabla_y\eta\|_0^2,\\
&\Big\|\frac{\partial\eta}{\partial s}\Big\|_0\le Ch^r\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{L^2(\Omega_y;H^r(\Omega_x))}+\Big\|\frac{\partial u}{\partial t}\Big\|_{L^2(\Omega_x;H^r(\Omega_y))}\Big),\\
&\|\nabla_x\eta\|_0\le\|\nabla_x u-\nabla_x I_x u\|_0+\|\nabla_x I_x u-I_y\nabla_x I_x u\|_0\\
&\qquad\le Ch^r\big(\|u\|_{L^2(\Omega_y;H^{r+1}(\Omega_x))}+\|u\|_{H^1(\Omega_x;H^r(\Omega_y))}\big),\\
&\|\nabla_y\eta\|_0\le\|\nabla_y u-\nabla_y I_y u\|_0+\|\nabla_y I_y u-I_x\nabla_y I_y u\|_0\\
&\qquad\le Ch^r\big(\|u\|_{L^2(\Omega_x;H^{r+1}(\Omega_y))}+\|u\|_{H^1(\Omega_y;H^r(\Omega_x))}\big),\\
&\|\nabla_x\nabla_y\eta\|_0\le\|\nabla_x\nabla_y u-\nabla_x I_x\nabla_y u\|_0+\|\nabla_y\nabla_x I_x u-\nabla_y I_y\nabla_x I_x u\|_0\\
&\qquad\qquad\le Ch^r\big(\|u\|_{H^1(\Omega_y;H^{r+1}(\Omega_x))}+\|u\|_{H^1(\Omega_x;H^{r+1}(\Omega_x))}\big).
\end{aligned}
$$

Combining the above inequalities with (37), we get the L^2-error estimate for ξ^n. It remains the L^2-error estimate for η^n:

$$
\begin{aligned}
\|\eta^n\|_0&=\|u(t^n)-I_xI_y u(t^n)\|_0\\
&\le\|u(t^n)-I_x u(t^n)\|_0+\|I_x u(t^n)-I_yI_x u(t^n)\|_0\\
&\le\Big(\int_{\Omega_y}\|u(t^n)-I_x u(t^n)\|^2_{L^2(\Omega_x)}\mathrm{d}y\Big)^{\frac12}\\
&+\Big(\int_{\Omega_x}\|I_x u(t^n)-I_yI_x u(t^n)\|^2_{L^2(\Omega_y)}\mathrm{d}x\Big)^{\frac12}\\
&\le Ch^r\Big(\int_{\Omega_y}\|u(t^n)\|^2_{H^r(\Omega_x)}\mathrm{d}y\Big)^{\frac12}+Ch^r\Big(\int_{\Omega_x}\|I_x u(t^n)\|^2_{H^r(\Omega_y)}\mathrm{d}x\Big)^{\frac12}\\
&\le Ch^r(\|u(t^n)\|_{L^2(\Omega_y;H^r(\Omega_x))}+\|u(t^n)\|_{L^2(\Omega_x;H^r(\Omega_y))}).
\end{aligned}
$$

Combining the above error estimate for η^n and ξ^n with

$$\|e_h^n\|_0^2 = \|\eta^n + \xi^n\|_0^2 \leq 2(\|\eta^n\|_0^2 + \|\xi^n\|_0^2), \tag{38}$$

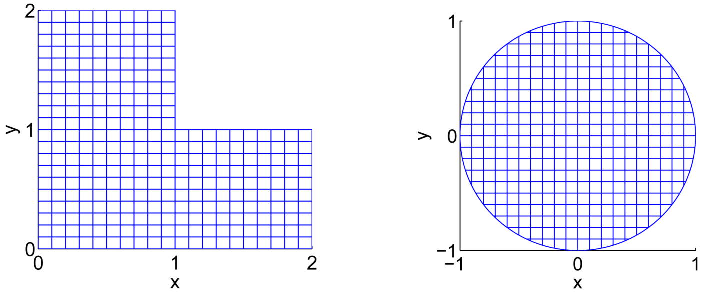
and using $\|\xi^0\|_0 \leq \|\eta^0\|_0$, the desired result is obtained.

5. NUMERICAL EXPERIMENTS

In this section, we present some numerical examples to show the efficiency and accuracy of the OSFEM. We first deal with the 2D parabolic equation on complex region. Further, we provide the numerical result of the 3D parabolic equation and give a comparison between the OSFEM and the classical FEM. Finally, the numerical result of 2D convection-diffusion equation is presented. For these numerical examples, we first split the parabolic equation into a series of 1D parabolic equations, then we solve each 1D parabolic equation based on P_1-conforming element. These results suggest that the numerical scheme is first-order accuracy in time and second-order accuracy in spatial. In order to calculate the error, we define

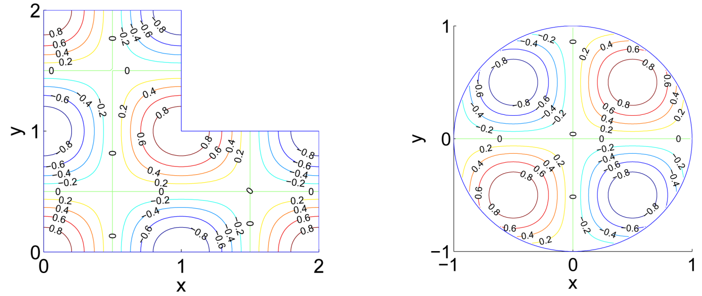
$$\|u - u_h\|_\infty = \max |u(t^n) - u_h^n|, \ \|u - u_h\|_0 = \Big(\int_\Omega |u^n - u_h^n|^2 d\mathbf{x} \Big)^{\frac{1}{2}}.$$

Example 5.1. The exact solution of 2D parabolic problem (1) is given as $u(t, x, y) = t\cos(\pi x)\cos(\pi y)$ with the final time $T = 1$. Let the computation be carried out on L-type region $\Omega = [0, 2]^2 \setminus [1, 2]^2$ with the Neumann boundary condition. Using the OSFEM to split this 2D parabolic equation into a sequence of 1D parabolic equation ($Y-$direction and $X-$direction), each 1D parabolic equation can be solved on P_1-conforming element. We get the numerical solution in Fig. 2(a) with the mesh grid as in Fig. 1(a). From Table 1, we can see that the OSFEM work well and consistent with the theoretical



(a) (b)

Fig. 1 – Mesh grid of L-type region (a), mesh grid of circular region (b).



(a) (b)

Fig. 2 – Numerical solutions of L-type region (a) and circular region (b) at $T = 1$ with $\delta t = h^2$ and $h = 1/64$.

TABLE 1

Numerical error and convergence order of Example 5.1 on L-type region with $\delta t = h^2$

$1/h$	$\|u - u_h\|_\infty$	order	$\|u - u_h\|_0$	order
8	6.848E-2		6.901E-2	
16	1.787E-2	1.938	1.675E-2	2.042
32	4.517E-3	1.984	4.074E-3	2.039
64	1.132E-3	1.996	1.001E-3	2.024
128	2.833E-4	1.998	2.479E-4	2.013

TABLE 2

Numerical error and convergence order of Example 5.1 on circular region with $\delta t = h^2$

$1/h$	$\|u - u_h\|_\infty$	order	$\|u - u_h\|_0$	order
8	6.920E-2		6.430E-2	
16	1.810E-2	1.934	1.730E-3	1.894
32	4.600E-3	1.976	4.400E-3	1.975
64	1.200E-3	1.938	1.100E-3	2.000
128	2.877E-4	2.055	2.798E-4	1.974

analysis. Besides, we consider the 2D parabolic problem (1) on a circular region $\Omega = \{(x, y)|\ x^2 + y^2 \leq 1\}$ with final time $T = 1$. And, the exact solution is also given by $u(t, x, y) = t\sin(\pi x)\sin(\pi y)$. Of course, this problem can also be solved by the OSFEM. The numerical solution in Fig. 2(b) is obtained with the mesh grid as in Fig. 1(b). The numerical errors are listed in Table 2, which shows that the convergence rate of the OSFEM is first-order accuracy in time and second-order accuracy in spatial.

Example 5.2. Given the exact solution of the 3D parabolic equation (1) as $u(t, x, y) = t\sin(\pi x)\sin(\pi y)\sin(\pi z)$. And, we take the final time $T = 1$ and domain $\Omega = [0, 1]^3$. Table 3 shows numerical results of the OSFEM with the backward Euler scheme in time discretization. Besides, in order to illustrate the efficiency of OSFEM, we also solve this problem by applying the Q_1-FEM for spatial discretization and the backward Euler scheme in time discretization. The CPU time of this method and the OSFEM in one time-step simulation are presented in Table 4. It is clear that the OSFEM is more efficient.

TABLE 3

Numerical error and convergence order of Example 5.2 with $\delta t = h^2$

$1/h$	$\|u-u_h\|_\infty$	order	$\|u-u_h\|_0$	order
8	1.367E-1		6.000E-2	
16	3.640E-2	1.909	1.620E-2	1.888
32	9.200E-3	1.984	4.100E-3	1.982
64	2.300E-3	2.000	1.000E-3	2.035
128	5.577E-4	2.044	2.460E-4	2.022

TABLE 4

Computational time for one time step of the 3D parabolic problem

$1/h$	8	16	32	64	128
Q_1-FEM	2.28s	37.83s	396.50s	>1h	>3h
OSFEM	0.03s	0.26s	1.96s	15.19s	130.80s

Example 5.3. The exact solution of the 2D convection-diffusion equation (22) is given by $u(t,x,y) = t\sin(\pi x)\sin(\pi y)$ with the final time $T = 1$. Let the computed region Ω be the unit square $[0,1]^2$ in $\mathbb{R}^2$ and the convection coefficient $\beta = (1,1)$. The time discretization is based on the backward Euler scheme. The numerical error and convergence rate of this numerical simulation are presented in Table 5.

TABLE 5

Numerical error and convergence order of Example 5.3 with $\delta t = h^2$

$1/h$	$\|u-u_h\|_\infty$	order	$\|u-u_h\|_0$	order
8	7.006E-2		4.701E-2	
16	1.841E-2	1.927	1.237E-2	1.926
32	4.686E-3	1.974	3.135E-3	1.980
64	1.175E-3	1.994	7.868E-4	1.994
128	2.942E-4	1.998	1.969E-4	1.998

6. CONCLUSIONS

In this work, we successfully apply the OSFEM to deal with the high-dimensional parabolic equations. The main feature of the proposed method is that the high-dimensional parabolic equation can be simplified to a sequence of 1D parabolic equations. Besides, each 1D parabolic equation can be solved by the classical FEM separately. The OSFEM is successfully applied in convection-diffusion equation. So we can use this method to solve more complicated parabolic equation. In addition, numerical examples illustrate that the proposed method agree with the theoretical analysis. In the future, we will pursue the higher-order operator-splitting method for time discretization scheme.

Acknowledgments. The authors would like to thank the editor and referees for their valuable comments and suggestions which helped us to improve the results of this paper. This work was in parts supported by the Graduate Student Research Innovation Program of Xinjiang (No. XJGRI2015008), the Natural Science Foundation of Xinjiang Province (No. 2015211C289) and the NSF of China (No. 11671345).

REFERENCES

[1] V. Thomee, *Galerkin Finite Element Methods for Parabolic Problems.* Third Ed. Springer, 2008.

[2] S. Ganesan and L. Tobiska, *Operator-splitting finite element algorithms for computations of high-dimensional parabolic problems.* Appl. Math. Comput. **219** (2013), 6182–6196.

[3] Z. Weng, X. Feng and P. Huang, *A new mixed finite element method based on the Crank-Nicolson scheme for the parabolic problems.* Appl. Math. Model. **36** (2012), 5068–5079.

[4] H. Bungartz and M. Griebel, *Sparse grids.* Acta Numer. **13** (2004), 1–123.

[5] M. Griebel and D. Oeltz, *A sparse grid space-time discretization scheme for parabolic problems.* Computing **81** (2007), 1–34.

[6] M. Griebel, D. Oeltz and P. Vassilevski, *Space-time approximation with sparse grids.* SIAM J. Sci. Comput. **28** (2005), 701–727.

[7] J. Douglas, *On the numerical integration of* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$ *by implicit methods.* J. Indust. Appl. Math. **3** (1955), 42–65.

[8] D. Peaceman, *The numerical solution of parabolic andelliptic differential equations.* J. Indust. Appl. Math. **3** (1955), 28–41.

[9] R. Huilgol and G. Kefayati, *Natural convection problem in a Bingham fluid using the operator-splitting method.* J. Non-Newton. Fluid Mech. (2014).

[10] L. Li and M. Luo, *Alternating direction implicit Galerkin finite element method for the two-dimensional fractional diffusion-wave equation.* J. Comput. Phys. **255** (2013), 471–485.

[11] S. Zhai, Z. Weng, D. Gui and X. Feng, *High-order compact operator splitting method for three-dimensional fractional equation with subdiffusion.* Int. J. Heat Mass Trans. **84** (2015), 440–447.

[12] S. Zhai, X. Feng and Y. He, *Numerical simulation of the three dimensional Allen-Cahn equation by the high-order compact ADI method.* Comput. Phys. Commun. **185** (2014), 2449–2455.

[13] S. Zhai, Z. Weng and X. Feng, *Investigations on several numerical methods for the non-local Allen-Cahn equation.* Int. J. Heat Mass Trans. **87** (2015), 111–118.

[14] V. Bokil and A. Leung, *Operator splitting methods for Maxwells equations in dispersive media with orientational polarization.* J. Comput. Appl. Math. **263** (2014), 160–188.

[15] G. Jannounb, R. Toumaa and F. Brockb, *Convergence of two-dimensional staggered central schemes on unstructured triangular grids.* Appl. Numer. Math. **92** (2015), 1–20.

[16] D. Knezevic and E. Sli, *A heterogeneous alternating-direction method for a micro-macro dilute polymeric fluid model.* ESAIM Math. Model. Numer. Anal. **43** (2009), 1117–1156.

[17] S. Ganesan, *An operator-splitting Galerkin/SUPG finite element method for population balance equations: stability and convergence.* ESAIM Math. Model. Numer. Anal. **46** (2012), 1447–1465.

[18] Y. Hou and H. Wei, *Dimension splitting algorithm for a three-dimensional elliptic equation.* Int. J. Comput. Math. (2011), 112–127.

[19] K. Li and H. Jia, *Dimension split method for the Navier-Stokes equations on flow surface.* Acta Math. Sci. Ser. A Chin. Ed. **58** (2008), 266-282

[20] S. Brenner and L. Scott, *The Mathematical Theory of Finite Element Methods.* Third Ed. Springer, 2008.

[21] Z. Weng, X. Feng and D. Liu, *A fully discrete stabilized mixed finite element method for parabolic problems.* Numer. Heat Trans. A **63(10)** (2013), 755–775.

[22] S. Zhai and X. Feng, *A block-centered finite-difference method for the time-fractional diffusion equation on nonuniform grids.* Numer. Heat Trans. B **69** (2016), 217–233.

[23] X. Feng and Y. He, *Convergence of the Crank-Nicolson/Newton scheme for nonlinear parabolic problem.* Acta Math. Sci. **36** (2016), 124–138.

[24] X. Xiao, D. Gui and X. Feng, *A highly efficient operator-splitting finite element method for 2D/3D nonlinear Allen-Cahn equation.* Internat. J. Numer. Methods Heat Fluid Flow **27** (2017), 530–542.

[25] S. Zhai, Z. Weng and X. Feng, *Fast explicit operator splitting method and time-step adaptivity for fractional non-local Allen-Cahn model.* Appl. Math. Model. **40** (2016), 1315–1324.

Received 9 January 2016

Xinjiang University,
College of Mathematics and System Sciences,
Urumqi, 830046, P.R. China
gangpengmath@foxmail.com
zhaojianping@126.com

Xinjiang University,
Institute of Mathematics and Physics,
Urumqi, 830046, P.R. China
fxlmath@xju.edu.cn