SOME RESULTS ON q-ANALOGUE OF THE BERNOULLI, EULER AND FIBONACCI MATRICES

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In this article, we study q-analogues of the Bernoulli, Euler and Fibonacci matrices. Some algebraic properties of these matrices are presented and proved. In particular, we show various factorizations of these matrices and their inverse matrices.

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1. INTRODUCTION

Matrix theory is recently used in the study of several combinatorial sequences. The matrix representation gives a powerful tool to obtain new or classical identities. In particular, Pascal type matrices [5, 7, 26, 36] have been used to obtain new interesting combinatorial identities involving sequences such as Fibonacci and Lucas sequence [22,23,32,37], k-Fibonacci numbers [21], Catalan numbers [31], Stirling numbers and their generalizations [8,11,22,24,25,27,28], Bernoulli numbers [8,35], among others. In the present paper, we are interested in lower-triangular matrices whose entries are q-Bernoulli numbers [1, 19], q-Euler numbers [20] and q-Fibonacci numbers [12]; for more details about these sequences see [13].

The q-Bernoulli polynomials, $B_{n,q}(x)$, are defined by the exponential generating function

$$F(x,t,q) := \frac{te_q(xt)}{e_q(t)-1} = \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{[n]_q!} \quad (|t| < 2\pi),$$

where the q-exponential function $e_q(x)$ is defined by

$$e_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{((1-q)x;q)_\infty} \quad (0 < |q| < 1, \quad x < |1-q|^{-1}),$$

with

$$(a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j), \quad [n]_q = 1+q+\cdots+q^{n-1} \quad \text{and} \quad [n]_q! = [1]_q[2]_q\cdots[n]_q.$$

In particular, if $x=0$ in $B_{n,q}(x)$, we obtain the q-Bernoulli numbers $B_{n,q}$ (cf. [13]). The q-exponential function satisfies the addition formula $e_q(x+y) = e_q(x)e_q(y)$ provided that $yx = qxy$ (q-commutative variables).

For a real or complex parameter α, the generalized q-Bernoulli polynomials $B_{n,q}^{(\alpha)}(x)$ are defined by the generating function

(1)
$$G(x,t,\alpha,q) := \left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(xt) = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x)\frac{t^n}{[n]_q!} \quad (|t|<2\pi,\ \alpha\in\mathbb{C}).$$

Clearly, $B_{n,q}^{(1)}(x) = B_{n,q}(x)$ and $B_{n,q}^{(1)}(0) = B_{n,q}$ are the q-analogue of the Bernoulli polynomials and numbers, respectively.

The q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}},$$

and

$$(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j).$$

Another way to write the q-binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

From (1) we can derive the following equation

$$B_{n,q}^{(\alpha+\beta)}(x+y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}^{(\alpha)}(x) B_{n-k,q}^{(\beta)}(y) \tag{2}$$

provided that x commutes with y, *i.e.*, if $yx = qxy$. Indeed,

$$\begin{aligned}
\sum_{i=0}^{\infty} B_{i,q}^{(\alpha+\beta)}(x+y)\frac{t^i}{[i]_q!} &= \left(\frac{t}{e_q(t)-1}\right)^{\alpha+\beta} e_q((x+y)t) \\
&= \left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(xt) \left(\frac{t}{e_q(t-1}\right)^{\beta} e_q(yt) \\
&= \sum_{i=0}^{\infty} B_{i,q}^{(\alpha)}(x)\frac{t^i}{[i]_q!} \sum_{i=0}^{\infty} B_{i,q}^{(\beta)}(y)\frac{t^i}{[i]_q!}
\end{aligned}$$

$$= \sum_{i=0}^{\infty} \left(\sum_{l=0}^{i} \begin{bmatrix} i \\ l \end{bmatrix}_q B_{l,q}^{(\alpha)}(x) B_{i-l,q}^{(\beta)}(y) \right) \frac{t^i}{[i]_q!}.$$

The result follows by comparing the coefficients.

In particular, upon setting $\beta = 0$ in identity (2) and interchanging x and y, we obtain

$$B_{n,q}^{(\alpha)}(x+y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}^{(\alpha)}(y) x^{n-k}.$$

Specifically, if $\alpha = 1$

$$B_{n,q}(x+y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}(y) x^{n-k}, \tag{3}$$

and if $y = 0$

$$B_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{n-k,q} x^k.$$

The first few q-Bernoulli polynomials are

$$B_{0,q}(x) = 1, \quad B_{1,q}(x) = x - \frac{1}{q+1}, \quad B_{2,q}(x) = x^2 - x + \frac{q^2}{(q+1)(q^2+q+1)},$$
$$B_{3,q}(x) = x^3 - \frac{q^2+q+1}{q+1} x^2 + \frac{q^2}{q+1} x - \frac{(q-1)q^3}{(q+1)^2 (q^2+1)},$$
$$B_{4,q}(x) = x^4 - (q^2+1) x^3 + \frac{q^2(q^2+1)}{q+1} x^2 + \frac{q^3-q^4}{q+1} x + \frac{q^{10} - q^8 - 2q^7 - q^6 + q^4}{(q+1)^2 (q^2+q+1)(q^4+q^3+q^2+q+1)}.$$

The q-Euler polynomials $E_{n,q}(x)$ are defined by means of the following generating function (cf. [20])

$$\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_q!} := \frac{2}{e_q(t)+1} e_q(xt) \quad (|t| < \pi).$$

In the special case, $x = 0$, $E_{n,q}(0) := E_{n,q}$ are called the q-Euler numbers. The first few q-Euler numbers are

$$E_{0,q} = 1, \quad E_{1,q} = -\frac{1}{2}, \text{ï?‘}\frac{1}{2} \quad E_{2,q} = \frac{q-1}{4}, \quad E_{3,q} = -\frac{1}{8}(q+1)(q^2-3q+1),$$
$$E_{4,q} = \frac{1}{16}(q-1)(q+1)(q^2-4q+1)(q^2+q+1), \cdots.$$

On the other hand, there exist several slightly different q-analogues of the Fibonacci sequence, among other references, see, [2–4, 9, 12, 29]. In particular, we are interested in the polynomials introduced by Cigler [12]. The q-Fibonacci polynomials, $F_{n,q}(x,s)$, are defined as follows

$$F_{n,q}(x,s) := \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q q^{\binom{k+1}{2}} x^{n-1-2k} s^k.$$

In particular, if we take $x = s = 1$, we obtain the q-Fibonacci numbers $F_{n,q}$.

The first few q-Fibonacci number are

$$F_{0,q} = 0, \quad F_{1,q} = 1, \quad F_{2,q} = 1, \quad F_{3,q} = q+1, \quad F_{4,q} = q^2+q+1,$$
$$F_{5,q} = 2q^3+q^2+q+1, \quad F_{6,q} = q^5+2q^4+2q^3+q^2+q+1,$$
$$F_{7,q} = q^7+2q^6+3q^5+2q^4+2q^3+q^2+q+1, \ldots$$

It is clear that if $q = 1$ we recover the well-known Fibonacci sequence.

In the present article, we study three families of matrices, the q-Bernoulli matrix, the q-Euler matrix and the q-Fibonacci matrix. Then we obtain several results that generalize the classical case ($q = 1$). The outline of this paper is as follows. In Section 2, we introduce the generalize q-Bernoulli matrix, then we derive some basic identities, in particular we find its inverse matrix. In Section 3, we find a factorization of the q-Bernoulli matrix in terms of the q-Pascal matrix. In Section 4, we show a relation between the q-Pascal matrix plus one and the q-Euler matrix. Finally, in Section 5 we introduce the q-Fibonacci matrix and we find its inverse matrix and some interesting factorizations. In particular, we show a relation between the q-Bernoulli polynomial matrix and the q-Fibonacci matrix.

2. THE q-ANALOGUE OF THE GENERALIZED BERNOULLI MATRIX

We define the q-analogue of the generalized $(n+1)\times(n+1)$ Bernoulli matrix $\mathscr{B}_{n,q}^{(\alpha)}(x) := \mathscr{B}_q^{(\alpha)}(x) := [B_{i,j}^{(\alpha)}(x)](0 \le i, j \le n)$, where

$$B_{i,j}^{(\alpha)}(x) = \begin{cases} \begin{bmatrix} i \\ j \end{bmatrix}_q B_{i-j,q}^{(\alpha)}(x), & \text{if } i \ge j; \\ 0, & \text{otherwise.} \end{cases}$$

The matrices $\mathscr{B}_{n,q}^{(1)}(x) := \mathscr{B}_q^{(1)}(x) := \mathscr{B}_q(x)$ and $\mathscr{B}_{n,q}^{(1)}(0) := \mathscr{B}_q(0) := \mathscr{B}_q$ are called the q-Bernoulli polynomial matrix and the q-Bernoulli matrix, respectively. For more information about this matrix see [13, 16]. Note that this matrix is different from one recently studied by Tuglu and Kuş, [33].

In particular, if $q = 1$ we recover the generalized Bernoulli matrix [35].

Example 1. For $n=3$, we have

$$\mathscr{B}_{3,q}(x)$$
$$=\begin{bmatrix} 1 & 0 & 0 & 0 \\ x-\frac{1}{q+1} & 1 & 0 & 0 \\ x^2-x+\frac{q^2}{q^3+2q^2+2q+1} & (q+1)x-1 & 1 & 0 \\ x^3-\frac{(q^2+q+1)}{q+1}x^2+\frac{q^2}{q+1}x-\frac{(q-1)q^3}{(q+1)^2(q^2+1)} & (q^2+q+1)(x-x^2)+\frac{q^2}{1+q} & (q^2+q+1)x-\frac{q^2+q+1}{q+1} & 1 \end{bmatrix}$$

and

$$\mathscr{B}_{3,q}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{q+1} & 1 & 0 & 0 \\ \frac{q^2}{q^3+2q^2+2q+1} & -1 & 1 & 0 \\ -\frac{(q-1)q^3}{(q+1)^2(q^2+1)} & \frac{q^2}{q+1} & -\frac{q^2+q+1}{q+1} & 1 \end{bmatrix}.$$

THEOREM 2. *If x commutes with y, then the following equalities hold for any α and β*

$$\mathscr{B}_q^{(\alpha+\beta)}(x+y)=\mathscr{B}_q^{(\alpha)}(x)\mathscr{B}_q^{(\beta)}(y)=\mathscr{B}_q^{(\beta)}(y)\mathscr{B}_q^{(\alpha)}(x). \tag{4}$$

We need the following q-binomial identity.

LEMMA 3. *The following identity holds for any positive integers i,j,k*

$$\begin{bmatrix} i \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q = \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} i-j \\ k-j \end{bmatrix}_q.$$

Proof. From the definition of q-binomial coefficient we get

$$\begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} i-j \\ k-j \end{bmatrix}_q = \frac{[i]_q!}{[j]_q![i-j]_q!}\frac{[i-j]_q!}{[k-j]_q![i-j-k+j]_q!}$$
$$=\frac{[i]_q![k]_q!}{[j]_q![k-j]_q![i-k]_q![k]_q!}=\frac{[i]_q!}{[k]_q![i-k]_q!}\frac{[k]_q!}{[j]_q![k-j]_q!}=\begin{bmatrix} i \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q. \qquad \square$$

Proof of Theorem 2. From above lemma and Identity (2) we have

$$(\mathscr{B}_q^{(\alpha)}(x)\mathscr{B}_q^{(\beta)}(y))_{i,j}=\sum_{k=j}^{i}\begin{bmatrix} i \\ k \end{bmatrix}_q B_{i-k,q}^{(\alpha)}(x)\begin{bmatrix} k \\ j \end{bmatrix}_q B_{k-j,q}^{(\beta)}(y)$$
$$=\sum_{k=j}^{i}\begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} i-j \\ k-j \end{bmatrix}_q B_{i-k,q}^{(\alpha)}(x)B_{k-j,q}^{(\beta)}(y)=\begin{bmatrix} i \\ j \end{bmatrix}_q \sum_{k=0}^{i-j}\begin{bmatrix} i-j \\ k \end{bmatrix}_q B_{i-j-k,q}^{(\alpha)}(x)B_{k,q}^{(\beta)}(y)$$
$$=\begin{bmatrix} i \\ j \end{bmatrix}_q B_{i-j,q}^{(\alpha+\beta)}(x+y)=(\mathscr{B}_q^{(\alpha+\beta)}(x+y))_{i,j}.$$

Then Equation (4) follows. $\square$

If $q = 1$ we obtain Theorem 2.1 of [35].

Corollary 4. *If the variables commute then the following equality holds for any integer* $k \geq 2$

$$\mathscr{B}_q^{(\alpha_1+\alpha_2+\cdots+\alpha_k)}(x_1 + x_2 + \cdots + x_k) = \mathscr{B}_q^{(\alpha_1)}(x_1)\mathscr{B}_q^{(\alpha_2)}(x_2)\cdots\mathscr{B}_q^{(\alpha_k)}(x_k). \tag{5}$$

Proof. We proceed by induction on k. From Theorem 2 the equality clearly holds for $k = 2$. Now suppose that the result is true for all $i < k$. We prove it for k.

$$\begin{aligned}&\mathscr{B}_q^{(\alpha_1+\alpha_2+\cdots+\alpha_k)}(x_1 + x_2 + \cdots + (x_{k-1} + x_k))\\ &\qquad\qquad = \mathscr{B}_q^{(\alpha_1)}(x_1)\mathscr{B}_q^{(\alpha_2)}(x_2)\cdots\mathscr{B}_q^{(\alpha_{k-1}+\alpha_k)}(x_{k-1} + x_k).\end{aligned}$$

By Theorem 2, we get

$$\begin{aligned}&\mathscr{B}_q^{(\alpha_1)}(x_1)\mathscr{B}_q^{(\alpha_2)}(x_2)\cdots\mathscr{B}_q^{(\alpha_{k-1}+\alpha_k)}(x_{k-1} + x_k)\\ &\qquad\qquad = \mathscr{B}_q^{(\alpha_1)}(x_1)\mathscr{B}_q^{(\alpha_2)}(x_2)\cdots\mathscr{B}_q^{(\alpha_{k-1})}(x_{k-1})\mathscr{B}^{(\alpha_k)}(x_k). \qquad \square\end{aligned}$$

If we take $x_1 = x_2 = \cdots = x_k = x$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \alpha$, then $(\mathscr{B}_q^{(\alpha)}(x))^k = \mathscr{B}_q^{(k\alpha)}(kx)$. In particular, if $\alpha = 1$, then $(\mathscr{B}_q(x))^k = \mathscr{B}_q^{(k)}(kx)$; if $x = 0$, $(\mathscr{B}_q^{(\alpha)})^k = \mathscr{B}_q^{(k\alpha)}$. Finally, if $\alpha = 1, x = 0$ we obtain $\mathscr{B}_q^k = \mathscr{B}_q^{(k)}$, *i.e.*, a formula of the powers of the q-Bernoulli matrix.

Now, we study the inverse matrix of the q-Bernoulli matrix. Let $\mathscr{D}_q = [d_{i,j}](0 \leq i, j \leq n)$ be the $(n + 1) \times (n + 1)$ matrix defined by

$$d_{i,j} = \begin{cases} \frac{1}{[i-j+1]_q}\begin{bmatrix} i \\ j \end{bmatrix}_q, & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases}$$

We need the following lemma [20].

Lemma 5. *For any positive integer* n

$$\sum_{k=0}^{n} \frac{1}{[k+1]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q B_{n-k,q} = [n]_q!\delta_{n,0},$$

where $\delta_{n,m}$ *is the Kronecker delta symbol.*

Theorem 6. *The inverse matrix of the q-Bernoulli matrix* $\mathscr{B}_q$ *is the matrix* $\mathscr{D}_q$, *i.e,* $\mathscr{B}_q^{-1} = \mathscr{D}$. *Furthermore,* $(\mathscr{B}_q^{(k)})^{-1} = \mathscr{D}_q^k$.

Proof. From above lemma, we get

$$(\mathscr{B}_q\mathscr{D}_q)_{i,j} = \sum_{k=j}^{i} \begin{bmatrix} i \\ k \end{bmatrix}_q B_{i-k,q} \cdot \frac{1}{[k-j+1]_q}\begin{bmatrix} k \\ j \end{bmatrix}_q$$

$$= \begin{bmatrix} i \\ j \end{bmatrix}_q \sum_{k=j}^{i} \frac{1}{[k-j+1]_q} \begin{bmatrix} i-j \\ k-j \end{bmatrix}_q B_{i-k,q}$$

$$= \begin{bmatrix} i \\ j \end{bmatrix}_q \sum_{k=0}^{i-j} \frac{1}{[k+1]_q} \begin{bmatrix} i-j \\ k \end{bmatrix}_q B_{i-j-k,q} = \begin{bmatrix} i \\ j \end{bmatrix}_q [n]_q! \delta_{i-j,0}.$$

This is 1 only when $i = j$, and 0 otherwise. Therefore $\mathscr{B}_q \mathscr{D}_q = I$, *i.e.*, $\mathscr{B}_q^{-1} = \mathscr{D}_q$. Moreover, if $\alpha = 1$ and $x = 0$ we get

$$(\mathscr{B}_q^{(k)})^{-1} = (\mathscr{B}_q^k)^{-1} = (\mathscr{B}_q^{-1})^k = \mathscr{D}_q^k. \quad \square$$

If $q = 1$ we obtain Theorem 2.4 of [35].

3. THE q-ANALOGUE OF THE GENERALIZED BERNOULLI MATRIX AND THE GENERALIZED q-PASCAL MATRIX

In this section, we show some relations between the q-analogues of the generalized Bernoulli matrix and the generalized Pascal matrix. The Pascal matrix is one of the most important matrices of mathematics. It arises in many different areas such as combinatorics, number theory, etc. Many kinds of generalizations of the Pascal matrix have been presented in the literature. In particular, we are interested in its q-analogue; see, *e.g.*, [13–17, 38].

The $(n+1) \times (n+1)$ q-Pascal matrix $\mathcal{P}_n := \mathcal{P}_{n,q} := [p_{i,j}]\,(0 \le i, j \le n)$ is defined with the q-binomial coefficients as follows

$$p_{i,j} := \begin{cases} \left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]_q, & \text{if } i \geqslant j; \\ 0, & \text{otherwise.} \end{cases}$$

For any nonnegative integers n and k, the $(n+1) \times (n+1)$ matrices $\mathcal{I}_n, \mathcal{S}_n^{(k)}, \mathcal{D}_n^{(k)}$ and $\mathcal{P}_n^{(k)}$ are defined by

$$(\mathcal{I}_n)_{i,j} := \operatorname{diag}(1, 1, \ldots, 1),$$

$$(\mathcal{S}_n^{(k)})_{i,j} := \begin{cases} q^{(i-j)k}, & \text{if } i \ge j; \\ 0, & \text{otherwise.} \end{cases} \quad (0 \le i, j \le n),$$

$$(D_n^{(k)})_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ -q^k, & \text{if } j = i-1; \\ 0, & \text{otherwise.} \end{cases}$$

$$(\mathcal{P}_n^{(k)})_{i,j} := (\mathcal{P}_n)_{i,j} q^{(i-j)k}.$$

Clearly, $\mathcal{P}_n^{(0)} = \mathcal{P}_n$. Furthermore we need the $(n+1)\times(n+1)$ matrices

$$\overline{\mathcal{P}}_n^{(k)} := \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{P}_{n-1}^{(k)} \end{bmatrix}$$

$$\mathcal{F}_k := \begin{bmatrix} \mathcal{I}_{n-k-1} & 0 \\ 0 & \mathcal{D}_k^{(n-k)} \end{bmatrix}, (1 \leq k \leq n-1) \text{ and } \mathcal{F}_n := \mathcal{D}_n^{(0)},$$

$$\mathcal{G}_k := \begin{bmatrix} \mathcal{I}_{n-k-1} & 0 \\ 0 & \mathcal{S}_k^{(n-k)} \end{bmatrix}, (1 \leq k \leq n-1) \text{ and } \mathcal{G}_n := \mathcal{S}_n^{(0)}.$$

It is not difficult to see that $(\mathcal{D}_n^{(k)})^{-1} = \mathcal{S}_n^{(k)}$ and $\mathcal{F}_k^{-1} = \mathcal{G}_k$. Moreover, for any nonnegative integer k, we have the following factorization [38]

$$\mathcal{D}_n^{(k)}\mathcal{P}_n^{(k)} = \overline{\mathcal{P}}_n^{(k+1)}.$$

By the above equation and the definition of the matrix $\mathcal{F}_k$, we get

$$\mathcal{F}_1\mathcal{F}_2\cdots\mathcal{F}_n\mathcal{P}_n = \mathcal{I}_n \quad \text{or} \quad \mathcal{P}_n = \mathcal{F}_n^{-1}\mathcal{F}_{n-1}^{-1}\cdots\mathcal{F}_1^{-1}.$$

Therefore, the q-Pascal matrix $\mathcal{P}_n$ can be factorized as follows

$$\mathcal{P}_n = \mathcal{G}_n\mathcal{G}_{n-1}\cdots\mathcal{G}_1.$$

Moreover, it is clear that the inverse of the q-Pascal matrix can be factorized as

$$\mathcal{P}_n^{-1} = \mathcal{F}_1\mathcal{F}_2\cdots\mathcal{F}_n = \overline{\mathcal{P}_n},$$

where $(\overline{\mathcal{P}_n})_{i,j} := (-1)^{i-j}(\mathcal{P}_n)_{i,j}q^{\binom{i-j}{2}}$.

The $(n+1)\times(n+1)$ generalized q-Pascal matrix $\mathcal{P}_n[x] := [p_{i,j}]$ $(0 \leq i,j \leq n)$, is defined by [38]

$$p_{i,j} = \begin{cases} \begin{bmatrix} i \\ j \end{bmatrix}_q x^{i-j}, & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases}$$

Example 7. For $n = 4$, the generalized q-Pascal matrix is

$$\mathcal{P}_4[x] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ x^2 & (q+1)x & 1 & 0 & 0 \\ x^3 & (q^2+q+1)x^2 & (q^2+q+1)x & 1 & 0 \\ x^4 & (1+q)(1+q^2)x^3 & (1+q^2)(1+q+q^2)x^2 & (1+q)(1+q^2)x & 1 \end{bmatrix}$$

In [17], Ernst studied a generalization of this matrix.

Theorem 8. *If x commutes with y, then we have the following identity*

$$\mathscr{B}_q(x+y) = \mathcal{P}_n[x]\mathscr{B}_q(y) = \mathcal{P}_n[y]\mathscr{B}_q(x).$$

In particular, $\mathscr{B}_q(x) = \mathcal{P}_n[x]\mathscr{B}_q$.

Proof. From Equation (3), we have

$$
\begin{aligned}
(\mathcal{P}_n[x]\mathscr{B}_q(y))_{i,j} &= \sum_{k=j}^{i} \begin{bmatrix} i \\ k \end{bmatrix}_q x^{i-k} \begin{bmatrix} k \\ j \end{bmatrix}_q B_{k-j,q}(y) \\
&= \begin{bmatrix} i \\ j \end{bmatrix}_q \sum_{k=j}^{i} \begin{bmatrix} i-j \\ k-j \end{bmatrix}_q B_{k-j,q}(y) x^{i-k} = \begin{bmatrix} i \\ j \end{bmatrix}_q \sum_{k=0}^{i-j} \begin{bmatrix} i-j \\ k \end{bmatrix}_q B_{k,q}(y) x^{i-j-k} \\
&= \begin{bmatrix} i \\ j \end{bmatrix}_q B_{i-j,q}(x+y) = (\mathscr{B}_q(x+y))_{i,j}.
\end{aligned}
$$

Similarly, we have $\mathscr{B}_q(x+y) = \mathcal{P}_n[y]\mathscr{B}_q(x)$. ☐

In particular, if $q=1$ we obtain Theorem 3.1 of [35].

Example 9. If $n=2$ we obtain the following factorization

$$
\mathscr{B}_{2,q}(x) = \begin{bmatrix} 1 & 0 & 0 \\ x - \frac{1}{q+1} & 1 & 0 \\ x^2 - x + \frac{q^2}{q^3+2q^2+2q+1} & (q+1)x - 1 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^2 & (q+1)x & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{q+1} & 1 & 0 \\ \frac{q^2}{q^3+2q^2+2q+1} & -1 & 1 \end{bmatrix} = \mathcal{P}_2[x]\mathscr{B}_{2,q}.
$$

Let $\mathcal{S}_n^{(k)}[x]$ and $\mathcal{D}_n^{(k)}[x]$ be the $(n+1)\times(n+1)$ matrices defined by

$$
(\mathcal{S}_n^{(k)}[x])_{i,j} = (\mathcal{S}_n^{(k)})_{i,j} x^{i-j} \quad \text{and} \quad (\mathcal{D}_n^{(k)}[x])_{i,j} = (\mathcal{D}_n^{(k)})_{i,j} x^{i-j}.
$$

For $1 \le k \le n-1$, we define the following matrices

$$
\mathcal{F}_k[x] := \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & \mathcal{D}_{n-k}^{(k)}[x] \end{bmatrix}, (1 \le k \le n-1) \quad \text{and} \quad \mathcal{F}_n[x] := \mathcal{D}_n^{(0)}[x],
$$

$$
\mathcal{G}_k[x] := \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & \mathcal{S}_{n-k}^{(k)}[x] \end{bmatrix}, (1 \le k \le n-1) \quad \text{and} \quad \mathcal{G}_n[x] := \mathcal{S}_n^{(0)}[x].
$$

Clearly, $\mathcal{F}_k[x] = \mathcal{G}_k^{-1}[x]$, $1 \le k \le n$. We need the $(n+1)\times(n+1)$ matrix

$$
\mathcal{I}_n[x] := \operatorname{diag}(1, x, \ldots, x^n).
$$

By definition, we have [38]

$$
\begin{aligned}
\mathcal{S}_n^{(k)}[x] &= \mathcal{I}_n[x]\mathcal{S}_n^{(k)}\mathcal{I}_n^{-1}[x], \\
\mathcal{G}_k[x] &= \mathcal{I}_n[x]\mathcal{G}_k\mathcal{I}_n^{-1}[x], \\
\mathcal{P}_n[x] &= \mathcal{I}_n[x]\mathcal{P}_n\mathcal{I}_n^{-1}[x].
\end{aligned}
$$

Moreover, the generalized q-Pascal matrix $\mathcal{P}_n[x]$ can be factorized by the matrices $\mathcal{G}_k[x]$ as follows

$$\begin{aligned}\mathcal{P}_n[x] &= \mathcal{I}_n[x]\mathcal{G}_n\mathcal{G}_{n-1}\cdots\mathcal{G}_1\mathcal{I}_n^{-1}[x]\\ &= (\mathcal{I}_n[x]\mathcal{G}_n\mathcal{I}_n^{-1}[x])(\mathcal{I}_n[x]\mathcal{G}_{n-1}\mathcal{I}_n^{-1}[x])\cdots(\mathcal{I}_n[x]\mathcal{G}_1\mathcal{I}_n^{-1}[x])\\ &= \mathcal{G}_n[x]\mathcal{G}_{n-1}[x]\cdots\mathcal{G}_1[x].\end{aligned}$$

Moreover,

$$\mathcal{P}_n^{-1}[x] = \mathcal{F}_n[x]\mathcal{F}_{n-1}[x]\cdots\mathcal{F}_1[x] = \overline{\mathcal{P}}_n[x],$$

where $(\overline{\mathcal{P}}_n[x])_{i,j} := (\mathcal{P}_n)_{i,j}q^{\binom{i-j}{2}}(-x)^{i-j}$.

From above factorizations we have the following theorem.

Theorem 10. *For any x we have the following identities*

$$\mathscr{B}_q(x) = \mathcal{G}_n[x]\mathcal{G}_{n-1}[x]\cdots\mathcal{G}_1[x]\mathscr{B}_q,$$
$$\mathscr{B}_q(x)^{-1} = \mathscr{D}_q\mathcal{F}_n[x]\mathcal{F}_{n-1}[x]\cdots\mathcal{F}_1[x].$$

If $q = 1$ we obtain Corollary 3.3 of [35].

4. INVERSE MATRIX OF THE q-PASCAL MATRIX PLUS ONE

In this section, we give an explicit formula to the inverse matrix of the q-Pascal matrix plus one, and we show the relation between this inverse matrix and the q-Euler numbers. Yang and Liu [34] studied the case $q = 1$.

Lemma 11. *For any $n \geq 0$, we have*

$$\frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q E_{k,q}(x) + \frac{1}{2}E_{n,q}(x) = x^n,$$

where $E_{n,q}(x)$ are the q-Euler polynomials defined by

$$\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{[n]_q!} = \frac{2}{e_q(t)+1}e_q(xt), \quad (|t| < \pi).$$

Proof. From definition of q-Euler numbers we get

$$\left(\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{[n]_q!}\right)(e_q(t)+1) = \frac{2}{e_q(t)+1}e_q(xt)(e_q(t)+1) = 2e_q(xt).$$

Therefore,

$$\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q E_{i,q}(x) + E_{n,q}(x)\right)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}2x^n\frac{t^n}{[n]_q!}.$$

By comparing coefficients, we get the desired result. $\square$

Lemma 12. *If x commutes with y, then for any $n \geq 0$, we have*

$$E_{n,q}(x+y) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q E_{i,q}(x) y^{n-i}.$$

In particular, if $x = 0$ we have

$$E_{n,q}(y) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q E_{i,q} y^{n-i}.$$

Proof. From definition of q-Euler numbers and since x, y are q-commutative variables, we get

$$\sum_{n=0}^{\infty} E_{n,q}(x+y) \frac{t^n}{[n]_q!} = \frac{2}{e_q(t)+1} e_q((x+y)t) = \frac{2}{e_q(t)+1} e_q(xt) e_q(yt)$$
$$= \left(\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_q!} \right) e_q(yt) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q E_{i,q}(x) y^{n-i} \right) \frac{t^n}{[n]_q!}.$$

By comparing coefficients, we get the desired result. □

We define the q-analogue of the $(n+1) \times (n+1)$ Euler matrix $\mathscr{E}_{n,q} := [E_{i,j}] (0 \leq i, j \leq n)$, where

$$E_{i,j} = \begin{cases} \left[\begin{smallmatrix} i \\ j \end{smallmatrix} \right]_q E_{i-j,q}, & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 13. *For all positive integer n the following identity holds*

$$(\mathcal{P}_n + I_n)^{-1} = \frac{1}{2} \mathscr{E}_{n,q}.$$

Proof. From Lemma 11 we have the following matrix equation

$$\frac{1}{2} (\mathcal{P}_n + I_n) \mathbb{E}_{n,q}(x) = \mathbb{X}_n,$$

where $\mathbb{E}_{n,q}(x)$ and $\mathbb{X}_n$ are $n \times 1$ matrices defined by

$$\mathbb{E}_{n,q}(x) = [E_{0,q}(x), E_{1,q}(x), \ldots, E_{n-1,q}(x)]^T, \qquad \mathbb{X}_n = \left[1, x, \ldots, x^{n-1}\right]^T.$$

From Lemma 12 we have the following matrix equation

$$\mathbb{E}_{n,q}(x) = \mathscr{E}_{n,q} \mathbb{X}_n.$$

Therefore

$$\frac{1}{2} (\mathcal{P}_n + I_n) \mathscr{E}_{n,q} = I_n. \quad \square$$

If $q = 1$ we obtain the main result in [34].

5. SOME RELATIONS BETWEEN q-PASCAL MATRIX AND q-FIBONACCI MATRIX

In this section, we give some relations between q-Pascal matrix and q-Fibonacci matrix. Moreover, we obtain the inverse matrix of the q-Fibonacci matrix.

We define the q-analogue of the $(n+1)\times(n+1)$ Fibonacci matrix $\mathscr{F}_{n,q} := \mathscr{F}_q := [f_{i,j}](0 \le i,j \le n)$, where

$$f_{i,j} = \begin{cases} F_{i-j,q}, & \text{if } i \ge j; \\ 0, & \text{otherwise.} \end{cases}$$

A generalization of the q-Fibonacci matrix and its inverse were recently studied in [30].

Example 14. The q-Fibonacci matrix for $n = 4$ is

$$\mathscr{F}_{4,q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ q+1 & 1 & 1 & 0 & 0 \\ q^2+q+1 & q+1 & 1 & 1 & 0 \\ 2q^3+q^2+q+1 & q^2+q+1 & q+1 & 1 & 1 \end{bmatrix}.$$

We need the following auxiliary sequence and matrix. Let N_n be the sequence defined by $N_0 = 1$, $N_1 = 1$ and

$$N_n = N_{n-1} + \sum_{k=1}^{n-1} (-1)^{n-k} F_{n-k+2,q} N_{k-1}, \ n \ge 2.$$

The first few terms of N_n are

$$1, \ 1, \ -q, \ q^2-q, \ 2q^2-2q^3, \ q^5+2q^4-4q^3+q^2, \ -q^7-2q^6-q^5+7q^4-3q^3, \ldots.$$

The $(n+1)\times(n+1)$ lower Hessenberg matrix $\mathscr{C}_n := [c_{i,j}](0 \le i,j \le n)$ is defined by

$$c_{i,j} = \begin{cases} F_{i-j+2,q}, & i-j+2 \ge 1; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 15 ([6]). *Let A_n be an $n\times n$ lower Hessenberg matrix for all $n \ge 1$ and define* $\det(A_0) = 1$. *Then,* $\det(A_1) = a_{11}$ *and for $n \ge 2$*

$$\det(A_n) = a_{n,n}\det(A_{n-1}) + \sum_{r=1}^{n-1} [(-1)^{n-r} a_{n,r} (\prod_{j=r}^{n-1} a_{j,j+1}) \det(A_{r-1})].$$

Lemma 16. *For $n \ge 1$, $\det(\mathscr{C}_n) = N_n$.*

Proof. We proceed by induction on n. The result clearly holds for $n = 1$. Now suppose that the result is true for all positive integers less than or equal to n. We prove it for $n+1$. In fact, by using Lemma 15 we have

$$\begin{aligned}
\det(\mathscr{C}_{n+1}) &= c_{n+1,n+1}\det(\mathscr{C}_n) + \sum_{i=1}^{n}\left[(-1)^{n+1-i}a_{n+1,i}\prod_{j=i}^{n}a_{j,j+1}\det(\mathscr{C}_{i-1})\right] \\
&= \det(\mathscr{C}_n) + \sum_{i=1}^{n}\left[(-1)^{n+1-i}F_{n-i+3,q}\det(\mathscr{C}_{i-1})\right] \\
&= N_n + \sum_{i=1}^{n}\left[(-1)^{n+1-i}F_{n-i+3,q}N_{i-1}\right] = N_{n+1}. \quad \square
\end{aligned}$$

We need the following Theorem of Chen and Yu [10].

THEOREM 17. *Let $H = [h_{i,j}]$ be an $n \times n$ lower Hessenberg matrix,*

$$\tilde{H} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & & & \vdots \\ & H & & 0 \\ & & & 1 \end{bmatrix}$$

and $\tilde{H}^{-1} = \left[\begin{array}{c|c} [\alpha]_{n\times 1} & [L]_{n\times n} \\ \hline h & [\beta^T]_{1\times n} \end{array}\right]_{(n+1)\times(n+1)}$. *Then*

$$\det(H) = (-1)^n h \cdot \det(\tilde{H}), \tag{6}$$

$$L = H^{-1} + h^{-1}\alpha\beta^T, \tag{7}$$

$$H\alpha + he_n = 0, \tag{8}$$

where e_n is nth column of matrix $\mathcal{I}_n$.

LEMMA 18 ([18]). *Let e_1 be the first column of the identity matrix $\mathcal{I}_n$ and L, β, h be the matrices described in the Theorem 17. Then,*

$$\beta^T H + he_1 = 0. \tag{9}$$

THEOREM 19. *Let $\mathscr{F}_{n,q}$ be the $(n+1)\times(n+1)$ lower triangular q-Fibonacci matrix, then its inverse matrix is given by*

$$(\mathscr{F}_{n,q})^{-1} = [b_{i,j}] = \begin{cases} (-1)^{i-j}N_{i-j}, & i > j; \\ 1, & i = j; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. It is obvious that,

$$(\tilde{\mathscr{C}_n}) = \left[\begin{array}{ccc|c} 1 & 0 & \cdots & 0 \\ & & & \vdots \\ & \mathscr{C}_n & & 0 \\ & & & 1 \end{array}\right] = \mathscr{F}_{n+1,q}.$$

Let us construct the matrix $(\tilde{\mathscr{C}_n})^{-1} = \left[\begin{array}{c|c} [\alpha]_{n\times 1} & [L]_{n\times n} \\ \hline h & [\beta^T]_{1\times n} \end{array}\right]_{(n+1)\times(n+1)}$ and obtain the entries.

(1) By using (6) we obtain $h = \frac{(-1)^n \det(\mathscr{C}_n)}{\det(\tilde{\mathscr{C}_n})} = (-1)^n \det(\mathscr{C}_n)$,

(2) We obtain matrices $[\alpha]$ and $[\beta]$ by using (8) and (9):

$$[\alpha] = -(\mathscr{C}_n)^{-1}(-1)^n \det(\mathscr{C}_n) e_n = \begin{bmatrix} 1 \\ -\det \mathscr{C}_1 \\ \vdots \\ (-1)^{n-2} \det \mathscr{C}_{n-2} \\ (-1)^{n-1} \det \mathscr{C}_{n-1} \end{bmatrix}$$

and $[\beta^T] = -(-1)^n \det(\mathscr{C}_n) e_1 (\mathscr{C}_n)^{-1}$. Therefore $[\beta^T] = \begin{bmatrix} (-1)^{n-1} \det \mathscr{C}_{n-1} & (-1)^{n-2} \det \mathscr{C}_{n-2} & \cdots & -\det \mathscr{C}_1 & 1 \end{bmatrix}$.

(3) We obtain matrix $[L]$ by using (7) and Lemma 16:

$$\begin{aligned} [L] &= (\mathscr{C}_n)^{-1} + ((-1)^n \det \mathscr{C}_n)^{-1} \alpha \beta^T \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\det \mathscr{C}_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (-1)^{n-2} \det \mathscr{C}_{n-2} & \cdots & -\det \mathscr{C}_1 & 1 \end{bmatrix}. \end{aligned}$$

Consequently, combining these three steps and using Lemma 16, we obtain the required result. □

In particular, if $q = 1$ we obtain the inverse matrix of the Fibonacci matrix [23]. Lemma 16 and Theorem 19 were recently generalized in [30].

Example 20. The inverse matrix of $\mathscr{F}_{4,q}$ is

$$(\mathscr{F}_{4,q})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -q & -1 & 1 & 0 & 0 \\ q - q^2 & -q & -1 & 1 & 0 \\ 2q^2 - 2q^3 & q - q^2 & -q & -1 & 1 \end{bmatrix}.$$

The $(n+1)\times(n+1)$ lower triangular matrix $\mathcal{R}_{n,q} := [r_{i,j}]$, $(0 \le i,j \le n)$ is defined by

$$r_{i,j} = \begin{cases} \sum_{k=0}^{i-j}(-1)^k \begin{bmatrix} i \\ j+k \end{bmatrix}_q N_k, & \text{if } i \geqslant j; \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 21. *The following equality holds for any positive integer n*

$$\mathcal{P}_{n,q} = \mathcal{R}_{n,q}\mathscr{F}_{n,q}.$$

Proof. Note that it suffices to prove that $\mathcal{P}_{n,q}(\mathscr{F}_{n,q})^{-1} = \mathcal{R}_{n,q}$. For $i \ge j \ge 0$ we have

$$\sum_{k=0}^{n} p_{i,k}b_{k,j} = \sum_{k=0}^{n} \begin{bmatrix} i \\ k \end{bmatrix}_q (-1)^{k-j}N_{k-j} = \sum_{k=j}^{i} \begin{bmatrix} i \\ k \end{bmatrix}_q (-1)^{k-j}N_{k-j}$$
$$= \sum_{k=0}^{i-j}(-1)^k \begin{bmatrix} i \\ j+k \end{bmatrix}_q N_k = r_{i,j}$$

and it is obvious that $r_{i,j} = 0$ for $i - j > 0$, which implies that $\mathcal{P}_{n,q}(\mathcal{F}_{n,q})^{-1} = \mathcal{R}_{n,q}$, as desired. $\square$

In particular, if $q = 1$ we obtain Theorem 2.1 of [37].

Example 22.

$$\mathcal{P}_{4,q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & q+1 & 1 & 0 & 0 \\ 1 & q^2+q+1 & q^2+q+1 & 1 & 0 \\ 1 & q^3+q^2+q+1 & q^4+q^3+2q^2+q+1 & q^3+q^2+q+1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2q & q & 1 & 0 & 0 \\ -q^3-3q^2-q & -q & q^2+q & 1 & 0 \\ -2q^5-q^4-5q^3-q & -2q^4-q^3-3q^2 & q^4+q^2-q & q^3+q^2+q & 1 \end{bmatrix} \times$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ q+1 & 1 & 1 & 0 & 0 \\ q^2+q+1 & q+1 & 1 & 1 & 0 \\ 2q^3+q^2+q+1 & q^2+q+1 & q+1 & 1 & 1 \end{bmatrix} = \mathcal{R}_{4,q}\cdot\mathscr{F}_{4,q}.$$

The $(n+1)\times(n+1)$ lower triangular matrix $\mathcal{S}_{n,q} := [s_{i,j}]$, $(0 \leq i,j \leq n)$ is defined by

$$s_{i,j} = \begin{cases} \sum_{k=0}^{i-j}(-1)^{k+1}\begin{bmatrix} j+k \\ j \end{bmatrix}_q N_{i-j-k}, & \text{if } i \geqslant j \geqslant 1,\ i-j \text{ odd;} \\ \sum_{k=0}^{i-j}(-1)^{k}\begin{bmatrix} j+k \\ j \end{bmatrix}_q N_{i-j-k}, & \text{if } i \geqslant j \geqslant 1,\ i-j \text{ even;} \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 23. *The following equality holds for any positive integer n*

$$\mathcal{P}_{n,q} = \mathscr{F}_{n,q}\mathcal{S}_{n,q}.$$

Proof. The proof runs like in Theorem 21. $\square$

In particular, if $q=1$ we obtain the Theorem 2.1 of [22].

Example 24.

$$\mathcal{P}_{4,q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & q+1 & 1 & 0 & 0 \\ 1 & q^2+q+1 & q^2+q+1 & 1 & 0 \\ 1 & q^3+q^2+q+1 & q^4+q^3+2q^2+q+1 & q^3+q^2+q+1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ q+1 & 1 & 1 & 0 & 0 \\ q^2+q+1 & q+1 & 1 & 1 & 0 \\ 2q^3+q^2+q+1 & q^2+q+1 & q+1 & 1 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -q & q & 1 & 0 & 0 \\ -q^2 & -q+q^2 & q+q^2 & 1 & 0 \\ q^2-2q^3 & -2q^2+q^3 & -q+q^2+q^3+q^4 & q+q^2+q^3 & 1 \end{bmatrix} = \mathscr{F}_{4,q}\cdot\mathcal{S}_{4,q}.$$

Finally, we show a relation between the q-Fibonacci matrix and the q-Bernoulli matrix.

The $(n+1)\times(n+1)$ lower triangular matrix $\mathcal{T}_{n,q} := [t_{i,j}]$, $0 \leq i,j \leq n$ is defined by

$$t_{i,j} = \begin{cases} \begin{bmatrix} i \\ j \end{bmatrix}_q B_{i-j,q}(x) + \sum_{k=j}^{i-1}(-1)^{i-k}\begin{bmatrix} k \\ j \end{bmatrix}_q N_{i-k}B_{k-j,q}(x), & \text{if } i \geqslant j \geqslant 0 \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 25. *The following equality holds for any positive integer n*

$$\mathscr{B}_{n,q}(x) = \mathscr{F}_{n,q}\mathcal{T}_{n,q}.$$

Proof. The proof runs like in Theorem 21. $\square$

In particular, if $q = 1$ we obtain the Theorem 4.1 of [35].

Example 26. If $x = 0$ and $n = 3$ we get

$$\mathscr{B}_{3,q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{q+1} & 1 & 0 & 0 \\ \frac{q^2}{q^3+2q^2+2q+1} & -1 & 1 & 0 \\ -\frac{(q-1)q^3}{(q+1)^2(q^2+1)} & \frac{q^2}{q+1} & -\frac{q^2+q+1}{q+1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ q+1 & 1 & 1 & 0 \\ q^2+q+1 & q+1 & 1 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{q+2}{q+1} & 1 & 0 & 0 \\ -\frac{q^4+2q^3-1}{(q+1)(q^2+q+1)} & -2 & 1 & 0 \\ \frac{q\left(-q^7-2q^6-2q^5+3q^3+5q^2+3q+2\right)}{(q+1)^2(q^2+1)(q^2+q+1)} & \frac{1}{q+1} & -\frac{q^2+2q+2}{q+1} & 1 \end{bmatrix} = \mathscr{F}_{3,q} \cdot \mathcal{T}_{3,q}.$$

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REFERENCES

[1] W.A. Al-Salam, *q-Bernoulli numbers and polynomials.* Math. Nachr. **17** (1959), 239–260.

[2] G.E. Andrews, *Fibonacci numbers and the Rogers-Ramanujan identities.* Fibonacci Quart. **42** (2004), *1*, 3–19.

[3] H. Belbachir and A. Benmezai, *An alternative approach to Cigler's q-Lucas polynomials.* Appl. Math. Comput. **226** (2014), 691–698.

[4] H. Belbachir and A. Benmezai, *A q analogue for bi^{s}nomial coefficients and generalized Fibonacci sequence.* C. R. Math. Acad. Sci. Paris I **352** (2014), *3*, 167–171.

[5] R. Brawer and M. Pirovino, *The linear algebra of the Pascal matrix.* Linear Algebra Appl. **174** (1992), 13–23.

[6] N.D. Cahill, J.R. D'Errico, D.A. Narayan and J.Y. Narayan, *Fibonacci determinants.* College Math. J. **33** (2002), 221–225.

[7] G.S. Call and D.J. Velleman, *Pascal's matrices.* Amer. Math. Monthly **100** (1993), *4*, 372–376.

[8] M. Can and M. Cihat Dağli, *Extended Bernoulli and Stirling matrices and related combinatorial identities.* Linear Algebra Appl. **444** (2014), 114–131.

[9] L. Carlitz, *Fibonacci notes. III. q-Fibonacci numbers.* Fibonacci Quart. **12** (1974), 317–322.

[10] Y-H. Chen and C-Y. Yu, *A new algorithm for computing the inverse and the determinant of a Hessenberg matrix.* Appl. Math. Comput. **218** (2011), 4433–4436.

[11] G.S. Cheon and J.S. Kim, *Factorial Stirling matrix and related combinatorial sequences.* Linear Algebra Appl. **357** (2002), 247–258.

[12] J. Cigler, *A new class of q-Fibonacci polynomials.* Electron. J. Combin. **10** (2003), # R19.

[13] T. Ernst, *A Comprehensive Treatment of q-Calculus*. Birkhäuser, Springer, 2012.

[14] T. Ernst, *q-Leibniz functional matrices with applications to q-Pascal and q-Stirling matrices*. Adv. Stud. Contemp. Math. **22** (2012), 537–555.

[15] T. Ernst, *q-Pascal and q-Wronskian matrices with implications to q-Appell polynomials*. J. Discrete Math. (2013), Article ID450481, 1–10.

[16] T. Ernst, *q-Pascal and q-Bernoulli matrices and umbral approach*. Department of Mathematics Uppsala Universty D.M. Report 2008:23 (2008).

[17] T. Ernst, *Factorizations for q-Pascal matrices of two variables*. Spec. Matrices **3** (2015), 207–213.

[18] K. Kaygısız and A. Şahin, *A new method to compute the terms of generalized order-k Fibonacci numbers*. J. Number Theory **133** (2013), 3119–3126.

[19] D.S. Kim and T.K. Kim, *q-Bernoulli polynomials and q-umbral calculus*. Sci. China Math. **57** (2014), 1867–1874.

[20] D.S. Kim and T. Kim, *Some identities of q-Euler polynomials arising from q-umbral calculus*. J. Inequal. Appl. 2014: 1, (2014).

[21] G.Y. Lee and J.S. Kim, *The linear algebra of the k-Fibonacci matrix*. Linear Algebra Appl. **373** (2003), 75–87.

[22] G.Y. Lee, J.S. Kim and S.H. Cho, *Some combinatorial identities via Fibonacci numbers*. Discrete Appl. Math. **130** (2003), 527–534.

[23] G.Y. Lee, J.S. Kim and S.G. Lee, *Factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices*. Fibonacci Quart. **40** (2002), 203–211.

[24] T. Mansour, J.L. Ramírez and M. Shattuck, *A generalization of the r-Whitney numbers of the second kind*. J. Comb. **8**(2017), *1*, 29–55.

[25] I. Mező and J.L. Ramírez, *The linear algebra of the r-Whitney matrices*. Integral Transforms Spec. Funct. **26**(2015), *3*, 213–225.

[26] A.R. Moghaddamfar, S. Navid Salehy and S. Nima Salehy, *A matrix-theoretic perspective on some identities involving well-known sequences*. Bull. Malays. Math. Sci. Soc. (2015), 1–14.

[27] J.L. Ramírez and M. Shattuck, *Generalized r-Whitney numbers of the first kind*. Ann. Math. Inform. **46** (2016), 175–193.

[28] J.L. Ramírez and M. Shattuck, (p, q)*-Analogue of the r-Whitney-Lah numbers*. J. Integer Seq. **19** (2016), Article 16.5.6.

[29] J.L. Ramírez and V. Sirvent, *A q-analogue of the biperiodic Fibonacci sequence*. J. Integer Seq. **19** (2016), Article 16.4.6.

[30] A. Şahin, *On the q analogue of Fibonacci and Lucas matrices and Fibonacci polynomials*. Util. Math. **100** (2016), 113–125.

[31] S. Stanimirović, P. Stanimirović, M. Miladinović and A. Ilić, *Catalan matrix and related combinatorial identities*. Appl. Math. Comput. **215** (2009), 796–805.

[32] S. Stanimirović, J. Nikolov and I. Stanimirović, *A generalization of Fibonacci and Lucas matrices*. Discrete Appl. Math. **156** (2008), 2606–2619.

[33] N. Tuglu and S. Kuş, *q-Bernoulli matrices and their some properties*. Gazi Univ. J. of Science **28** (2015), 269–273.

[34] S.-L. Yang and Z.-K Liu, *Explicit inverse of the Pascal matrix plus one*. Int. J. Math. Math. Sci. Article ID 90901 (2006), 1–7.

[35] Z. Zhang and J. Wang, *Bernoulli matrix and its algebraic properties*. Discrete Appl. Math. **154** (2006), 1622–1632.

[36] Z. Zhang, *The linear algebra of the generalized Pascal matrix.* Linear Algebra Appl. **250** (1997), 51–60.

[37] Z. Zhang and X. Wang, *A factorization of the symmetric Pascal matrix involving the Fibonacci matrix.* Discrete Appl. Math. **155** (2007), 2371–2376.

[38] D. Zheng, *q-analogue of the Pascal matrix.* Ars Combin. **87** (2008), 321–336.

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