SOME RESULTS ON $q$-ANALOGUE OF THE BERNOULLI, EULER AND FIBONACCI MATRICES

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In this article, we study $q$-analogues of the Bernoulli, Euler and Fibonacci matrices. Some algebraic properties of these matrices are presented and proved. In particular, we show various factorizations of these matrices and their inverse matrices.

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1. INTRODUCTION

Matrix theory is recently used in the study of several combinatorial sequences. The matrix representation gives a powerful tool to obtain new or classical identities. In particular, Pascal type matrices [5, 7, 26, 36] have been used to obtain new interesting combinatorial identities involving sequences such as Fibonacci and Lucas sequence [22, 23, 32, 37], $k$-Fibonacci numbers [21], Catalan numbers [31], Stirling numbers and their generalizations [8, 11, 22, 24, 25, 27, 28], Bernoulli numbers [8, 35], among others. In the present paper, we are interested in lower-triangular matrices whose entries are $q$-Bernoulli numbers [1, 19], $q$-Euler numbers [20] and $q$-Fibonacci numbers [12]; for more details about these sequences see [13].

The $q$-Bernoulli polynomials, $B_{n,q}(x)$, are defined by the exponential generating function

$$F(x,t,q) := \frac{t e_q(x t)}{e_q(t) - 1} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q!} \quad (|t| < 2\pi),$$

where the $q$-exponential function $e_q(x)$ is defined by

$$e_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{((1-q)x; q)_{\infty}} \quad (0 < |q| < 1, \quad x < |1-q|^{-1}),$$

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with

\[(a; q)_{\infty} := \prod_{j=0}^{\infty} (1 - aq^j), \quad [n]_q = 1 + q + \cdots + q^{n-1} \quad \text{and} \quad [n]_q! = [1]_q [2]_q \cdots [n]_q.\]

In particular, if \(x = 0\) in \(B_{n,q}(x)\), we obtain the \(q\)-Bernoulli numbers \(B_{n,q}\) (cf. [13]). The \(q\)-exponential function satisfies the addition formula \(e_q(x + y) = e_q(x)e_q(y)\) provided that \(yx = qxy\) (\(q\)-commutative variables).

For a real or complex parameter \(\alpha\), the generalized \(q\)-Bernoulli polynomials \(B_{n,q}^{(\alpha)}(x)\) are defined by the generating function

\[(1) \quad G(x, t, \alpha, q) := \left( \frac{t}{e_q(t) - 1} \right)^\alpha e_q(xt) = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x) \frac{t^n}{[n]_q^n} \quad (|t| < 2\pi, \; \alpha \in \mathbb{C}).\]

Clearly, \(B_{n,q}^{(1)}(x) = B_{n,q}(x)\) and \(B_{n,q}^{(1)}(0) = B_{n,q}\) are the \(q\)-analogue of the Bernoulli polynomials and numbers, respectively.

The \(q\)-binomial coefficient is defined as

\[\left[ \begin{array}{c} n \\ k \end{array} \right]_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}},\]

and

\[(a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j).\]

Another way to write the \(q\)-binomial coefficient is

\[\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q ![n-k]_q!}.\]

From (1) we can derive the following equation

\[(2) \quad B_{n,q}^{(\alpha+\beta)}(x + y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q B_{k,q}^{(\alpha)}(x) B_{n-k,q}^{(\beta)}(y)\]

provided that \(x\) commutes with \(y\), i.e., if \(yx = qxy\). Indeed,

\[\sum_{i=0}^{\infty} B_{i,q}^{(\alpha+\beta)}(x + y) \frac{t^i}{[i]_q!} = \left( \frac{t}{e_q(t) - 1} \right)^{\alpha+\beta} e_q((x + y)t)\]

\[= \left( \frac{t}{e_q(t) - 1} \right)^\alpha e_q(xt) \left( \frac{t}{e_q(t-1)} \right)^\beta e_q(yt)\]

\[= \sum_{i=0}^{\infty} B_{i,q}^{(\alpha)}(x) \frac{t^i}{[i]_q!} \sum_{i=0}^{\infty} B_{i,q}^{(\beta)}(y) \frac{t^i}{[i]_q!}.\]
Some results on $q$-analogue of the Bernoulli, Euler and Fibonacci matrices

$$= \sum_{i=0}^{\infty} \left( \sum_{l=0}^{i} \binom{i}{l} q^{i-l} B_{i-l,q}^{(\beta)} (y) \right) \frac{t^{i}}{[i]q!}.$$ 

The result follows by comparing the coefficients.

In particular, upon setting $\beta = 0$ in identity (2) and interchanging $x$ and $y$, we obtain

$$B_{n,q}^{(\alpha)} (x + y) = \sum_{k=0}^{n} \binom{n}{k} q^{n-k} B_{k,q}^{(\alpha)} (y) x^{n-k}.$$ 

Specifically, if $\alpha = 1$

$$B_{n,q} (x + y) = \sum_{k=0}^{n} \binom{n}{k} q^{n-k} B_{k,q} (y) x^{n-k},$$

and if $y = 0$

$$B_{n,q} (x) = \sum_{k=0}^{n} \binom{n}{k} q^{n-k} B_{n-k,q} x^{k}.$$ 

The first few $q$-Bernoulli polynomials are

$$B_{0,q} (x) = 1, \quad B_{1,q} (x) = x - \frac{1}{q+1}, \quad B_{2,q} (x) = x^{2} - x + \frac{q^{2}}{(q+1)(q^{2}+q+1)},$$

$$B_{3,q} (x) = x^{3} - \frac{q^{2} + q + 1}{q+1} x^{2} + \frac{q^{2}}{q+1} x - \frac{(q-1)q^{3}}{(q+1)^{2}(q^{2}+1)},$$

$$B_{4,q} (x) = x^{4} - (q^{2} + 1) x^{3} + \frac{q^{2}(q^{2}+1)}{q+1} x^{2} + \frac{q^{3} - q^{4}}{q+1} x$$

$$+ \frac{q^{10} - q^{8} - 2q^{7} - q^{6} + q^{4}}{(q+1)^{2}(q^{2}+q+1)(q^{4}+q^{3}+q^{2}+q+1)}.$$ 

The $q$-Euler polynomials $E_{n,q} (x)$ are defined by means of the following generating function (cf. [20])

$$\sum_{n=0}^{\infty} E_{n,q} (x) \frac{t^{n}}{[n]q!} := \frac{2}{e_{q}(t) + 1} e_{q}(xt) \quad (|t| < \pi).$$

In the special case, $x = 0$, $E_{n,q} (0) := E_{n,q}$ are called the $q$-Euler numbers. The first few $q$-Euler numbers are

$$E_{0,q} = 1, \quad E_{1,q} = -\frac{1}{2}, \quad E_{2,q} = \frac{q-1}{4}, \quad E_{3,q} = -\frac{1}{8} (q+1) (q^{2} - 3q + 1),$$

$$E_{4,q} = \frac{1}{16} (q-1)(q+1) (q^{2} - 4q + 1) (q^{2} + q + 1), \ldots.$$ 

On the other hand, there exist several slightly different $q$-analogue of the Fibonacci sequence, among other references, see, [2–4, 9, 12, 29]. In particular,
we are interested in the polynomials introduced by Cigler [12]. The \( q \)-Fibonacci polynomials, \( F_{n,q}(x,s) \), are defined as follows

\[
F_{n,q}(x,s) := \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left\lfloor \frac{n-k-1}{2} \right\rfloor q^{(k+1)2} x^{n-1-2k} s^k.
\]

In particular, if we take \( x = s = 1 \), we obtain the \( q \)-Fibonacci numbers \( F_{n,q} \).

The first few \( q \)-Fibonacci number are

\[
\begin{align*}
F_{0,q} &= 0, & F_{1,q} &= 1, & F_{2,q} &= 1, & F_{3,q} &= q + 1, & F_{4,q} &= q^2 + q + 1, \\
F_{5,q} &= 2q^3 + q^2 + q + 1, & F_{6,q} &= q^5 + 2q^4 + 2q^3 + q^2 + q + 1, \\
F_{7,q} &= q^7 + 2q^6 + 3q^5 + 2q^4 + 2q^3 + q^2 + q + 1, \ldots
\end{align*}
\]

It is clear that if \( q = 1 \) we recover the well-known Fibonacci sequence.

In the present article, we study three families of matrices, the \( q \)-Bernoulli matrix, the \( q \)-Euler matrix and the \( q \)-Fibonacci matrix. Then we obtain several results that generalize the classical case \( (q = 1) \). The outline of this paper is as follows. In Section 2, we introduce the generalize \( q \)-Bernoulli matrix, then we derive some basic identities, in particular we find its inverse matrix. In Section 3, we find a factorization of the \( q \)-Bernoulli matrix in terms of the \( q \)-Pascal matrix. In Section 4, we show a relation between the \( q \)-Pascal matrix plus one and the \( q \)-Euler matrix. Finally, in Section 5 we introduce the \( q \)-Fibonacci matrix and we find its inverse matrix and some interesting factorizations. In particular, we show a relation between the \( q \)-Bernoulli polynomial matrix and the \( q \)-Fibonacci matrix.

2. THE \( q \)-ANALOGUE OF THE GENERALIZED BERNOULLI MATRIX

We define the \( q \)-analogue of the generalized \((n + 1) \times (n + 1)\) Bernoulli matrix \( \mathcal{B}_{n,q}^{(\alpha)}(x) := \mathcal{B}_q^{(\alpha)}(x) := [B_{i,j}^{(\alpha)}(x)](0 \leq i, j \leq n) \), where

\[
B_{i,j}^{(\alpha)}(x) = \begin{cases} 
\binom{i}{j} B_{i-j,q}^{(\alpha)}(x), & \text{if } i \geq j; \\
0, & \text{otherwise}.
\end{cases}
\]

The matrices \( \mathcal{B}_{n,q}^{(1)}(x) := \mathcal{B}_q^{(1)}(x) := \mathcal{B}_q(x) \) and \( \mathcal{B}_{n,q}^{(0)} := \mathcal{B}_q(0) := \mathcal{B}_q \) are called the \( q \)-Bernoulli polynomial matrix and the \( q \)-Bernoulli matrix, respectively. For more information about this matrix see [13, 16]. Note that this matrix is different from one recently studied by Tuglu and Kuş, [33].

In particular, if \( q = 1 \) we recover the generalized Bernoulli matrix [35].
Example 1. For $n = 3$, we have

$$\mathcal{B}_{3,q}(x)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x - \frac{1}{q+1} & 1 & 0 & 0 \\ x^2 - x + \frac{q^2}{q+1} & (q + 1)x - 1 & 1 & 0 \\ x^3 - \frac{(q^2+q+1)}{q+1}x^2 + \frac{q^2}{q+1}x - \frac{(q-1)q^3}{(q+1)^2(q^2+1)}(q^2+q+1)(x-x^2) + \frac{q^2}{q+1}(q^2+q+1)x - \frac{q^2+q+1}{q+1} & 1 \end{bmatrix}$$

and

$$\mathcal{B}_{3,q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{q+1} & 1 & 0 & 0 \\ \frac{q^2}{q^2+2q^2+2q+1} & -1 & 1 & 0 \\ \frac{q^2}{q+1} & \frac{q^2+q+1}{q+1} & 1 \end{bmatrix}.$$

Theorem 2. If $x$ commutes with $y$, then the following equalities hold for any $\alpha$ and $\beta$

$$\mathcal{B}_q^{(\alpha+\beta)}(x + y) = \mathcal{B}_q^{(\alpha)}(x)\mathcal{B}_q^{(\beta)}(y) = \mathcal{B}_q^{(\beta)}(y)\mathcal{B}_q^{(\alpha)}(x).$$

We need the following $q$-binomial identity.

Lemma 3. The following identity holds for any positive integers $i, j, k$

$$\begin{bmatrix} i \\ k \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q = \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} i - j \\ k - j \end{bmatrix}_q.$$

Proof. From the definition of $q$-binomial coefficient we get

$$\begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} i - j \\ k - j \end{bmatrix}_q = \frac{[i]_q!}{[j]_q!} \frac{[i]_q! [i - j]_q! [k - j]_q! [i - j - k + j]_q!}{[i]_q! [i - j]_q! [k - j]_q! [i - j - k + j]_q!} = \frac{[i]_q! [k]_q!}{[k]_q! [i - j]_q! [k - j]_q! [i - k]_q! [j]_q!} = \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} i - j \\ k - j \end{bmatrix}_q. \quad \square$$

Proof of Theorem 2. From above lemma and Identity (2) we have

$$(\mathcal{B}_q^{(\alpha)}(x)\mathcal{B}_q^{(\beta)}(y))_{i,j} = \sum_{k=j}^{i} \begin{bmatrix} i \\ k \end{bmatrix}_q B_{i-k,q}^{(\alpha)}(x) \begin{bmatrix} k \\ j \end{bmatrix}_q B_{k-j,q}^{(\beta)}(y)$$

$$= \sum_{k=j}^{i} \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} i - j \\ k - j \end{bmatrix}_q B_{i-k,q}^{(\alpha)}(x)B_{k-j,q}^{(\beta)}(y) = \begin{bmatrix} i \\ j \end{bmatrix}_q \sum_{k=0}^{i-j} \begin{bmatrix} i - j \\ k \end{bmatrix}_q B_{i-j-k,q}^{(\alpha)}(x)B_{k,q}^{(\beta)}(y)$$

$$= \begin{bmatrix} i \\ j \end{bmatrix}_q B_{i-j,q}^{(\alpha+\beta)}(x + y) = (\mathcal{B}_q^{(\alpha+\beta)}(x + y))_{i,j}.$$

Then Equation (4) follows. \quad \square
If $q = 1$ we obtain Theorem 2.1 of [35].

**Corollary 4.** If the variables commute then the following equality holds for any integer $k \geq 2$

\[
\mathcal{B}_q^{(\alpha_1+\alpha_2+\cdots+\alpha_k)}(x_1 + x_2 + \cdots + x_k) = \mathcal{B}_q^{(\alpha_1)}(x_1)\mathcal{B}_q^{(\alpha_2)}(x_2) \cdots \mathcal{B}_q^{(\alpha_k)}(x_k).
\]

**Proof.** We proceed by induction on $k$. From Theorem 2 the equality clearly holds for $k = 2$. Now suppose that the result is true for all $i < k$. We prove it for $k$.

\[
\mathcal{B}_q^{(\alpha_1+\alpha_2+\cdots+\alpha_k)}(x_1 + x_2 + \cdots + (x_{k-1} + x_k)) = \mathcal{B}_q^{(\alpha_1)}(x_1)\mathcal{B}_q^{(\alpha_2)}(x_2) \cdots \mathcal{B}_q^{(\alpha_{k-1}+\alpha_k)}(x_{k-1} + x_k).
\]

By Theorem 2, we get

\[
\mathcal{B}_q^{(\alpha_1)}(x_1)\mathcal{B}_q^{(\alpha_2)}(x_2) \cdots \mathcal{B}_q^{(\alpha_{k-1}+\alpha_k)}(x_{k-1} + x_k) = \mathcal{B}_q^{(\alpha_1)}(x_1)\mathcal{B}_q^{(\alpha_2)}(x_2) \cdots \mathcal{B}_q^{(\alpha_{k-1})}(x_{k-1})\mathcal{B}_q^{(\alpha_k)}(x_k). \quad \square
\]

If we take $x_1 = x_2 = \cdots = x_k = x$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \alpha$, then $(\mathcal{B}_q^{(\alpha)}(x))^k = \mathcal{B}_q^{(\alpha_k)}(kx)$. In particular, if $\alpha = 1$, then $(\mathcal{B}_q^{(1)}(x))^k = \mathcal{B}_q^{(1)}(kx)$; if $x = 0$, $(\mathcal{B}_q^{(1)}(x))^k = \mathcal{B}_q^{(1)}$. Finally, if $\alpha = 1, x = 0$ we obtain $\mathcal{B}_q^k = \mathcal{B}_q^{(1)}, i.e.,$ a formula of the powers of the $q$-Bernoulli matrix.

Now, we study the inverse matrix of the $q$-Bernoulli matrix. Let $\mathcal{D}_q = [d_{i,j})(0 \leq i, j \leq n)$ be the $(n + 1) \times (n + 1)$ matrix defined by

\[
d_{i,j} = \begin{cases} 
\frac{1}{[i-j+1]_q} \left[ \begin{array}{c} i \\ j \\ q 
\end{array} \right], & \text{if } i \geq j; \\
0, & \text{otherwise.}
\end{cases}
\]

We need the following lemma [20].

**Lemma 5.** For any positive integer $n$

\[
\sum_{k=0}^{n} \frac{1}{[k+1]_q} \left[ \begin{array}{c} n \\ k \\ q 
\end{array} \right] B_{n-k,q} = [n]_q! \delta_{n,0},
\]

where $\delta_{n,m}$ is the Kronecker delta symbol.

**Theorem 6.** The inverse matrix of the $q$-Bernoulli matrix $\mathcal{B}_q$ is the matrix $\mathcal{D}_q$, i.e, $\mathcal{B}_q^{-1} = \mathcal{D}_q$. Furthermore, $(\mathcal{B}_q^{(k)})^{-1} = \mathcal{D}_q^k$.

**Proof.** From above lemma, we get

\[
(\mathcal{B}_q\mathcal{D}_q)_{i,j} = \sum_{k=j}^{i} \left[ \begin{array}{c} i \\ k \\ q 
\end{array} \right] B_{i-k,q} \cdot \frac{1}{[k-j+1]_q} \left[ \begin{array}{c} k \\ j \\ q 
\end{array} \right]
\]
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\[
B_{i-k,q} = \begin{bmatrix} i \atop j \end{bmatrix} \sum_{k=j}^{i} \frac{1}{[k-j+1]_q} \begin{bmatrix} i-j \atop k-j \end{bmatrix}_q
\]

This is 1 only when $i = j$, and 0 otherwise. Therefore $B_q D_q = I$, i.e., $B_q^{-1} = D_q$. Moreover, if $\alpha = 1$ and $x = 0$ we get

\[
(B_q^{(k)})^{-1} = (B_q^k)^{-1} = (B_q^{-1})^k = D_q^k. \quad \square
\]

If $q = 1$ we obtain Theorem 2.4 of [35].

3. THE $q$-ANALOOGUE OF THE GENERALIZED BERNOULLI MATRIX AND THE GENERALIZED $q$-PASCAL MATRIX

In this section, we show some relations between the $q$-analogues of the generalized Bernoulli matrix and the generalized Pascal matrix. The Pascal matrix is one of the most important matrices of mathematics. It arises in many different areas such as combinatorics, number theory, etc. Many kinds of generalizations of the Pascal matrix have been presented in the literature. In particular, we are interested in its $q$-analogue; see, e.g., [13–17,38].

The $(n+1) \times (n+1)$ $q$-Pascal matrix $P_n := P_{n,q} := [p_{i,j}]$ ($0 \leq i, j \leq n$) is defined with the $q$-binomial coefficients as follows

\[
p_{i,j} := \begin{cases} \begin{bmatrix} i \\ j \end{bmatrix}_q, & \text{if } i \geq j; \\ 0, & \text{otherwise}. \end{cases}
\]

For any nonnegative integers $n$ and $k$, the $(n+1) \times (n+1)$ matrices $I_n, S_n^{(k)}, D_n^{(k)}$ and $P_n^{(k)}$ are defined by

\[
(I_n)_{i,j} := \text{diag}(1, 1, \ldots, 1),
\]

\[
(S_n^{(k)})_{i,j} := \begin{cases} q^{(i-j)k}, & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases}
\]

\[
(D_n^{(k)})_{i,j} := \begin{cases} 1, & \text{if } i = j; \\ -q^k, & \text{if } j = i - 1; \\ 0, & \text{otherwise.} \end{cases}
\]

\[
(P_n^{(k)})_{i,j} := (P_n)_{i,j} q^{(i-j)k}.
\]
Clearly, $\mathcal{P}_n^{(0)} = \mathcal{P}_n$. Furthermore we need the $(n + 1) \times (n + 1)$ matrices

$$\overline{\mathcal{P}}_n^{(k)} := \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{P}_n^{(k)} \end{bmatrix}$$

$$\mathcal{F}_k := \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & D_k^{(n-k)} \end{bmatrix}, \quad (1 \leq k \leq n - 1) \text{ and } \mathcal{F}_n := D_n^{(0)},$$

$$\mathcal{G}_k := \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & S_k^{(n-k)} \end{bmatrix}, \quad (1 \leq k \leq n - 1) \text{ and } \mathcal{G}_n := S_n^{(0)}.$$

It is not difficult to see that $(D_n^{(k)})^{-1} = S_n^{(k)}$ and $\mathcal{F}_k^{-1} = \mathcal{G}_k$. Moreover, for any nonnegative integer $k$, we have the following factorization [38]

$$D_n^{(k)} \mathcal{P}_n^{(k)} = \overline{\mathcal{P}}_n^{(k+1)}.$$

By the above equation and the definition of the matrix $\mathcal{F}_k$, we get

$$\mathcal{F}_1 \mathcal{F}_2 \cdots \mathcal{F}_n \mathcal{P}_n = I_n \text{ or } \mathcal{P}_n = \mathcal{F}_n^{-1} \mathcal{F}_{n-1}^{-1} \cdots \mathcal{F}_1^{-1}.$$

Therefore, the $q$-Pascal matrix $\mathcal{P}_n$ can be factorized as follows

$$\mathcal{P}_n = \mathcal{G}_n \mathcal{G}_{n-1} \cdots \mathcal{G}_1.$$

Moreover, it is clear that the inverse of the $q$-Pascal matrix can be factorized as

$$\mathcal{P}_n^{-1} = \mathcal{F}_1 \mathcal{F}_2 \cdots \mathcal{F}_n = \overline{\mathcal{P}}_n,$$

where $(\overline{\mathcal{P}}_n)_{i,j} := (-1)^{i-j} (\mathcal{P}_n)_{i,j} q^{(i-j)\binom{i-j}{2}}$.

The $(n+1) \times (n+1)$ generalized $q$-Pascal matrix $\mathcal{P}_n[x] := [p_{i,j}] (0 \leq i, j \leq n)$, is defined by [38]

$$p_{i,j} = \begin{cases} \binom{i}{j}_q x^{i-j}, & \text{if } i \geq j; \\ 0, & \text{otherwise}. \end{cases}$$

**Example 7.** For $n = 4$, the generalized $q$-Pascal matrix is

$$\mathcal{P}_4[x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & (q+1)x & 1 & 0 \\ x^3 & (q^2+q+1)x^2 & (q^2+q+1)x & 1 \\ x^4 & (1+q)(1+q^2)x^3 & (1+q^2)(1+q+q^2)x^2 & (1+q)(1+q^2)x \\ x^5 & (1+q^3)(1+q^2)x^4 & (1+q^2)(1+q+q^2)x^3 & (1+q)(1+q^2)x^2 & 1 \end{bmatrix}$$

In [17], Ernst studied a generalization of this matrix.

**Theorem 8.** If $x$ commutes with $y$, then we have the following identity

$$\mathcal{B}_q(x+y) = \mathcal{P}_n[x] \mathcal{B}_q(y) = \mathcal{P}_n[y] \mathcal{B}_q(x).$$

In particular, $\mathcal{B}_q(x) = \mathcal{P}_n[x] \mathcal{B}_q$. 
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Proof. From Equation (3), we have

$$(\mathcal{P}_n[x] \mathcal{B}_q(y))_{i,j} = \sum_{k=j}^{i} \left[ \begin{array}{c} i \\ k \\ j \\ q \end{array} \right] x^{i-k} \left[ \begin{array}{c} k \\ j \\ q \end{array} \right] B_{k-j,q}(y)$$

$$= \left[ \begin{array}{c} i \\ j \\ q \end{array} \right] \sum_{k=j}^{i} \left[ \begin{array}{c} i-j \\ k-j \\ q \end{array} \right] B_{k-j,q}(y) x^{i-k} = \left[ \begin{array}{c} i \\ j \\ q \end{array} \right] \sum_{j=0}^{i-j} \left[ \begin{array}{c} i-j \\ k \end{array} \right] q B_{k,q}(y) x^{i-j-k}$$

$$= \left[ \begin{array}{c} i \\ j \\ q \end{array} \right] B_{i-j,q}(x+y) = (\mathcal{B}_q(x+y))_{i,j}. \quad \Box$$

Similarly, we have $\mathcal{B}_q(x+y) = \mathcal{P}_n[y] \mathcal{B}_q(x)$.

In particular, if $q = 1$ we obtain Theorem 3.1 of [35].

Example 9. If $n = 2$ we obtain the following factorization

$$\mathcal{B}_{2,q}(x) = \begin{bmatrix} 1 & 0 & 0 \\ x - \frac{1}{q+1} & 0 & 0 \\ x^2 - x + \frac{q^2}{q^2+2q+1} & (q+1)x - 1 & 1 \end{bmatrix} = \mathcal{P}_2[x] \mathcal{B}_{2,q}.$$ 

Let $\mathcal{S}_n^{(k)}[x]$ and $\mathcal{D}_n^{(k)}[x]$ be the $(n+1) \times (n+1)$ matrices defined by

$$(\mathcal{S}_n^{(k)}[x])_{i,j} = (\mathcal{S}_n^{(k)})_{i,j} x^{i-j} \quad \text{and} \quad (\mathcal{D}_n^{(k)}[x])_{i,j} = (\mathcal{D}_n^{(k)})_{i,j} x^{i-j}.$$ 

For $1 \leq k \leq n - 1$, we define the following matrices

$$\mathcal{F}_k[x] := \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & \mathcal{D}_{n-k}^{(k)}[x] \end{bmatrix}, \quad (1 \leq k \leq n - 1) \quad \text{and} \quad \mathcal{F}_n[x] := \mathcal{D}_n^{(0)}[x],$$

$$\mathcal{G}_k[x] := \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & \mathcal{S}_{n-k}^{(k)}[x] \end{bmatrix}, \quad (1 \leq k \leq n - 1) \quad \text{and} \quad \mathcal{G}_n[x] := \mathcal{S}_n^{(0)}[x].$$

Clearly, $\mathcal{F}_k[x] = \mathcal{G}_k^{-1}[x], 1 \leq k \leq n$. We need the $(n+1) \times (n+1)$ matrix

$$\mathcal{I}_n[x] := \text{diag}(1, x, \ldots, x^n).$$

By definition, we have [38]

$$\mathcal{S}_n^{(k)}[x] = \mathcal{I}_n[x] \mathcal{S}_n^{(k)} \mathcal{I}_n^{-1}[x],$$

$$\mathcal{G}_k[x] = \mathcal{I}_n[x] \mathcal{G}_k \mathcal{I}_n^{-1}[x],$$

$$\mathcal{P}_n[x] = \mathcal{I}_n[x] \mathcal{P}_n \mathcal{I}_n^{-1}[x].$$
Moreover, the generalized $q$-Pascal matrix $\mathcal{P}_n[x]$ can be factorized by the matrices $\mathcal{G}_k[x]$ as follows

\[
\mathcal{P}_n[x] = \mathcal{I}_n[x] \mathcal{G}_n \mathcal{G}_{n-1} \cdots \mathcal{G}_1 \mathcal{I}_n^{-1}[x]
\]

\[
= (\mathcal{I}_n[x] \mathcal{G}_n \mathcal{I}_n^{-1}[x]) (\mathcal{I}_n[x] \mathcal{G}_{n-1} \mathcal{I}_n^{-1}[x]) \cdots (\mathcal{I}_n[x] \mathcal{G}_1 \mathcal{I}_n^{-1}[x])
\]

\[
= \mathcal{G}_n[x] \mathcal{G}_{n-1}[x] \cdots \mathcal{G}_1[x].
\]

Moreover,

\[
\mathcal{P}_n^{-1}[x] = \mathcal{F}_n[x] \mathcal{F}_{n-1}[x] \cdots \mathcal{F}_1[x] = \overline{\mathcal{P}}_n[x],
\]

where $(\overline{\mathcal{P}}_n[x])_{i,j} := (\mathcal{P}_n)_{i,j} q^{(i-j)/2} (-x)^{i-j}$.

From above factorizations we have the following theorem.

**Theorem 10.** For any $x$ we have the following identities

\[
\mathcal{B}_q(x) = \mathcal{G}_n[x] \mathcal{G}_{n-1}[x] \cdots \mathcal{G}_1[x] \mathcal{B}_q,
\]

\[
\mathcal{B}_q(x)^{-1} = \mathcal{D}_q \mathcal{F}_n[x] \mathcal{F}_{n-1}[x] \cdots \mathcal{F}_1[x].
\]

If $q = 1$ we obtain Corollary 3.3 of [35].

### 4. Inverse Matrix of the $q$-Pascal Matrix Plus One

In this section, we give an explicit formula to the inverse matrix of the $q$-Pascal matrix plus one, and we show the relation between this inverse matrix and the $q$-Euler numbers. Yang and Liu [34] studied the case $q = 1$.

**Lemma 11.** For any $n \geq 0$, we have

\[
\frac{1}{2} \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q E_{k,q}(x) + \frac{1}{2} E_{n,q}(x) = x^n,
\]

where $E_{n,q}(x)$ are the $q$-Euler polynomials defined by

\[
\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{2}{e_q(t) + 1} e_q(xt), \quad (|t| < \pi).
\]

**Proof.** From definition of $q$-Euler numbers we get

\[
\left( \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_q!} \right) (e_q(t) + 1) = \frac{2}{e_q(t) + 1} e_q(xt) (e_q(t) + 1) = 2 e_q(xt).
\]

Therefore,

\[
\sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} \left[ \begin{array}{c} n \\ i \end{array} \right]_q E_{i,q}(x) + E_{n,q}(x) \right) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} 2^n x^n \frac{t^n}{[n]_q!}.
\]

By comparing coefficients, we get the desired result. $\square$
Lemma 12. If $x$ commutes with $y$, then for any $n \geq 0$, we have

$$E_{n,q}(x + y) = \sum_{i=0}^{n} \binom{n}{i}_q E_{i,q}(x) y^{n-i}. $$

In particular, if $x = 0$ we have

$$E_{n,q}(y) = \sum_{i=0}^{n} \binom{n}{i}_q E_{i,q} y^{n-i}. $$

Proof. From definition of $q$-Euler numbers and since $x, y$ are $q$-commutative variables, we get

$$\sum_{n=0}^{\infty} E_{n,q}(x + y) \frac{t^n}{[n]_q!} = \frac{2}{e_q(t) + 1} e_q((x + y)t) = \frac{2}{e_q(t) + 1} e_q(xt)e_q(yt)$$

$$= \left( \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_q!} \right) e_q(yt) = \sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} \binom{n}{i}_q E_{i,q}(x) y^{n-i} \right) \frac{t^n}{[n]_q!}. $$

By comparing coefficients, we get the desired result. □

We define the $q$-analogue of the $(n + 1) \times (n + 1)$ Euler matrix $E_{n,q} := [E_{i,j}] (0 \leq i, j \leq n)$, where

$$E_{i,j} = \begin{cases} \binom{i}{j}_q E_{i-j,q}, & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases} $$

Theorem 13. For all positive integer $n$ the following identity holds

$$(P_n + I_n)^{-1} = \frac{1}{2} E_{n,q}. $$

Proof. From Lemma 11 we have the following matrix equation

$$\frac{1}{2} (P_n + I_n) E_{n,q}(x) = X_n,$$

where $E_{n,q}(x)$ and $X_n$ are $n \times 1$ matrices defined by

$$E_{n,q}(x) = [E_{0,q}(x), E_{1,q}(x), \ldots, E_{n-1,q}(x)]^T, \quad X_n = [1, x, \ldots, x^{n-1}]^T. $$

From Lemma 12 we have the following matrix equation

$$E_{n,q}(x) = E_{n,q}(x).$$

Therefore

$$\frac{1}{2} (P_n + I_n) E_{n,q} = I_n. \quad \square$$

If $q = 1$ we obtain the main result in [34].
5. SOME RELATIONS BETWEEN $q$-PASCAL MATRIX AND $q$-FIBONACCI MATRIX

In this section, we give some relations between $q$-Pascal matrix and $q$-Fibonacci matrix. Moreover, we obtain the inverse matrix of the $q$-Fibonacci matrix.

We define the $q$-analogue of the $(n + 1) \times (n + 1)$ Fibonacci matrix $\mathcal{F}_{n,q} := \mathcal{F}_q := [f_{i,j}](0 \leq i, j \leq n)$, where

$$f_{i,j} = \begin{cases} F_{i-j,q}, & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases}$$

A generalization of the $q$-Fibonacci matrix and its inverse were recently studied in [30].

Example 14. The $q$-Fibonacci matrix for $n = 4$ is

$$\mathcal{F}_{4,q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ q+1 & 1 & 1 & 0 \\ 2q^2 + q^2 + q + 1 & q + 1 & 1 & 1 \end{bmatrix}.$$ 

We need the following auxiliary sequence and matrix. Let $N_n$ be the sequence defined by $N_0 = 1$, $N_1 = 1$ and

$$N_n = N_{n-1} + \sum_{k=1}^{n-1} (-1)^{n-k} F_{n-k+2,q} N_{k-1}, \quad n \geq 2.$$ 

The first few terms of $N_n$ are

$1, \; 1, \; -q, \; q^2-q, \; 2q^2-2q^3, \; q^5+2q^4-4q^3+q^2, \; -q^7-2q^6-q^5+7q^4-3q^3, \ldots$.

The $(n+1) \times (n+1)$ lower Hessenberg matrix $\mathcal{C}_n := [c_{i,j}](0 \leq i, j \leq n)$ is defined by

$$c_{i,j} = \begin{cases} F_{i-j+2,q}, & \text{if } i - j + 2 \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 15 ([6]). Let $A_n$ be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and define $\det(A_0) = 1$. Then, $\det(A_1) = a_{11}$ and for $n \geq 2$

$$\det(A_n) = a_{n,n} \det(A_{n-1}) + \sum_{r=1}^{n-1} [(-1)^{n-r} a_{n,r} \prod_{j=r}^{n-1} a_{j,j+1}] \det(A_{r-1}).$$

Lemma 16. For $n \geq 1$, $\det(\mathcal{C}_n) = N_n$. 

Proof. We proceed by induction on $n$. The result clearly holds for $n = 1$. Now suppose that the result is true for all positive integers less than or equal to $n$. We prove it for $n + 1$. In fact, by using Lemma 15 we have

\[
\det(C_{n+1}) = c_{n+1,n+1} \det(C_n) + \sum_{i=1}^{n} \left( -1 \right)^{n+1-i} a_{n+1,i} \prod_{j=i}^{n} a_{j,j+1} \det(C_{i-1})
\]

\[
= \det(C_n) + \sum_{i=1}^{n} \left( -1 \right)^{n+1-i} F_{n-i+3,q} \det(C_{i-1})
\]

\[
= N_n + \sum_{i=1}^{n} \left( -1 \right)^{n+1-i} F_{n-i+3,q} N_{i-1} = N_{n+1}. \quad \Box
\]

We need the following Theorem of Chen and Yu [10].

**Theorem 17.** Let $H = [h_{i,j}]$ be an $n \times n$ lower Hessenberg matrix,

\[
\tilde{H} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
& & & \\
& H & & 0 \\
& & 1 & 
\end{bmatrix}
\]

and $\tilde{H}^{-1} = \begin{bmatrix}
[\alpha]_{n \times 1} & [L]_{n \times n} \\
\hbar & [\beta^T]_{1 \times n}
\end{bmatrix}^{(n+1) \times (n+1)}$. Then

(6) $\det(H) = (-1)^n h \cdot \det(\tilde{H})$,

(7) $L = H^{-1} + h^{-1} \alpha \beta^T$,

(8) $H \alpha + h e_n = 0$,

where $e_n$ is $n$th column of matrix $I_n$.

**Lemma 18 ([18]).** Let $e_1$ be the first column of the identity matrix $I_n$ and $L$, $\beta$, $h$ be the matrices described in the Theorem 17. Then,

(9) $\beta^T H + h e_1 = 0$.

**Theorem 19.** Let $\mathcal{F}_{n,q}$ be the $(n+1) \times (n+1)$ lower triangular $q$-Fibonacci matrix, then its inverse matrix is given by

\[
(\mathcal{F}_{n,q})^{-1} = [b_{i,j}] = \begin{cases}
(-1)^{i-j} N_{i-j}, & i > j; \\
1, & i = j; \\
0, & \text{otherwise.}
\end{cases}
\]
Proof. It is obvious that,

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
\vdots & & & \\
\mathcal{C}_n & 0 & & 1
\end{pmatrix} = \mathcal{F}_{n+1,q}.
\]

Let us construct the matrix \((\tilde{\mathcal{C}}_n)^{-1} = \begin{bmatrix} [\alpha]_{n \times 1} & [L]_{n \times n} \end{bmatrix} \begin{bmatrix} [\beta^T]_{1 \times n} \end{bmatrix}_{(n+1) \times (n+1)}\) and obtain the entries.

1. By using (6) we obtain \(h = \frac{(-1)^n \det(\mathcal{C}_n)}{\det(\tilde{\mathcal{C}}_n)} = (-1)^n \det(\mathcal{C}_n),\)

2. We obtain matrices \([\alpha]\) and \([\beta]\) by using (8) and (9):

\[
[\alpha] = -(\mathcal{C}_n)^{-1}(-1)^n \det(\mathcal{C}_n) e_n =
\begin{bmatrix}
1 \\
-\det \mathcal{C}_1 \\
\vdots \\
(-1)^{n-2} \det \mathcal{C}_{n-2} \\
(-1)^{n-1} \det \mathcal{C}_{n-1}
\end{bmatrix}
\]

and \([\beta^T] = -(-1)^n \det(\mathcal{C}_n) e_1 (\mathcal{C}_n)^{-1}. \) Therefore

\[
[\beta^T] = \begin{bmatrix}
(-1)^{n-1} \det \mathcal{C}_{n-1} & (-1)^{n-2} \det \mathcal{C}_{n-2} & \cdots & -\det \mathcal{C}_1 & 1
\end{bmatrix}.
\]

3. We obtain matrix \([L]\) by using (7) and Lemma 16:

\[
[L] = (\mathcal{C}_n)^{-1} + ((-1)^n \det(\mathcal{C}_n)^{-1} \alpha \beta^T
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
-\det \mathcal{C}_1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 \\
(-1)^{n-2} \det \mathcal{C}_{n-2} & \cdots & -\det \mathcal{C}_1 & 1
\end{bmatrix}.
\]

Consequently, combining these three steps and using Lemma 16, we obtain the required result. \(\square\)

In particular, if \(q = 1\) we obtain the inverse matrix of the Fibonacci matrix [23]. Lemma 16 and Theorem 19 were recently generalized in [30].

Example 20. The inverse matrix of \(\mathcal{F}_{4,q}\) is

\[
(\mathcal{F}_{4,q})^{-1} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
-q & -1 & 1 & 0 & 0 \\
q - q^2 & -q & -1 & 1 & 0 \\
2q^2 - 2q^3 & q - q^2 & -q & -1 & 1
\end{bmatrix}.
\]
The \((n+1) \times (n+1)\) lower triangular matrix \(R_{n,q} := [r_{i,j}], (0 \leq i, j \leq n)\) is defined by
\[
r_{i,j} = \begin{cases} 
\sum_{k=0}^{i-j} (-1)^k \binom{i}{j+k} N_k, & \text{if } i \geq j; \\
0, & \text{otherwise.}
\end{cases}
\]

**Theorem 21.** The following equality holds for any positive integer \(n\)
\[
P_{n,q} = R_{n,q} F_{n,q}.
\]

**Proof.** Note that it suffices to prove that \(P_{n,q}(F_{n,q})^{-1} = R_{n,q}\). For \(i \geq j \geq 0\) we have
\[
\sum_{k=0}^{n} p_{i,k} b_{k,j} = \sum_{k=0}^{n} \binom{i}{k} (-1)^{k-j} N_{k-j} = \sum_{k=j}^{i} \binom{i}{k} (-1)^{k-j} N_{k-j} = \sum_{k=0}^{i-j} (-1)^k \binom{i}{j+k} N_k = r_{i,j}
\]
and it is obvious that \(r_{i,j} = 0\) for \(i - j > 0\), which implies that \(P_{n,q}(F_{n,q})^{-1} = R_{n,q}\), as desired. \(\square\)

In particular, if \(q = 1\) we obtain Theorem 2.1 of [37].

**Example 22.**
\[
P_{4,q} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & q+1 & 1 & 0 & 0 \\
1 & q^2 + q + 1 & q^2 + q + 1 & 1 & 0 \\
1 & q^3 + q^2 + q + 1 & q^4 + q^3 + 2q^2 + q + 1 & q^3 + q^2 + q + 1 & 1
\end{bmatrix}
\]
\[
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
-2q & q & 1 & 0 & 0 \\
-q^3 - 3q^2 - q & -q & q^2 + q & 1 & 0 \\
-2q^5 - q^4 - 5q^3 - q & -2q^4 - q^3 - 3q^2 & q^4 + q^2 - q & q^3 + q^2 + q & 1
\end{bmatrix}
\times
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
q+1 & 1 & 1 & 0 & 0 \\
q^2 + q + 1 & q+1 & 1 & 1 & 0 \\
2q^3 + q^2 + q + 1 & q^2 + q + 1 & q+1 & 1 & 1
\end{bmatrix}
= R_{4,q} \cdot F_{4,q}.
\]
The \((n + 1) \times (n + 1)\) lower triangular matrix \(S_{n,q} := [s_{i,j}]\), \(0 \leq i, j \leq n\) is defined by
\[
s_{i,j} = \begin{cases} 
\sum_{k=0}^{i-j}(-1)^{k+1} \binom{j+k}{j} q N_{i-j-k}, & \text{if } i \geq j \geq 1, \ i-j \text{ odd;} \\
\sum_{k=0}^{i-j}(-1)^{k} \binom{j+k}{j} q N_{i-j-k}, & \text{if } i \geq j \geq 1, \ i-j \text{ even;} \\
0, & \text{otherwise.}
\end{cases}
\]

**Theorem 23.** The following equality holds for any positive integer \(n\)
\[
\mathcal{P}_{n,q} = \mathcal{F}_{n,q} S_{n,q}.
\]

**Proof.** The proof runs like in Theorem 21. \(\square\)

In particular, if \(q = 1\) we obtain the Theorem 2.1 of [22].

**Example 24.**
\[
\mathcal{P}_{4,q} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & q + 1 & 1 & 0 & 0 \\
1 & q^2 + q + 1 & q^2 + q + 1 & 1 & 0 \\
1 & q^3 + q^2 + q + 1 & q^4 + q^3 + 2q^2 + q + 1 & q^3 + q^2 + q + 1 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
q + 1 & 1 & 1 & 0 & 0 \\
q^2 + q + 1 & q + 1 & 1 & 1 & 0 \\
2q^3 + q^2 + q + 1 & q^2 + q + 1 & q + 1 & 1 & 1
\end{bmatrix} \times
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
-q & q & 1 & 0 & 0 \\
-q^2 & -q + q^2 & q + q^2 & 1 & 0 \\
q^2 - 2q^3 & -2q^2 + q^3 & -q + q^2 + q^3 + q^4 & q + q^2 + q^3 & 1
\end{bmatrix} = \mathcal{F}_{4,q} \cdot S_{4,q}.
\]

Finally, we show a relation between the \(q\)-Fibonacci matrix and the \(q\)-Bernoulli matrix.

The \((n + 1) \times (n + 1)\) lower triangular matrix \(T_{n,q} := [t_{i,j}]\), \(0 \leq i, j \leq n\) is defined by
\[
t_{i,j} = \begin{cases} 
\binom{i}{j} q B_{i-j,q}(x) + \sum_{k=j}^{i-1}(-1)^{i-k} \binom{k}{j} q N_{i-k} B_{k-j,q}(x), & \text{if } i \geq j \geq 0 \\
0, & \text{otherwise.}
\end{cases}
\]

**Theorem 25.** The following equality holds for any positive integer \(n\)
\[
\mathcal{B}_{n,q}(x) = \mathcal{F}_{n,q} T_{n,q}.
\]
Proof. The proof runs like in Theorem 21. □

In particular, if $q = 1$ we obtain the Theorem 4.1 of [35].

Example 26. If $x = 0$ and $n = 3$ we get

$$B_{3,q} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-\frac{1}{q^2+1} & 1 & 0 & 0 \\
\frac{q^2+2q^2+2q+1}{(q+1)^2(q^2+1)} & -1 & 1 & 0 \\
\frac{q^2}{q+1} & \frac{q^2+q+1}{q+1} & \frac{q^2+q+1}{q+1}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
q+1 & 1 & 1 & 0 \\
q^2+q+1 & q+1 & 1 & 1
\end{bmatrix}
$$

$$= F_{3,q} \cdot \mathcal{T}_{3,q}.$$

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