RADIAL POSITIVE SOLUTIONS OF SOME SEMILINEAR EQUATION INVOLVING THE DUNKL LAPLACIAN

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Let Δ_k be the Dunkl Laplacian on $\mathbb{R}^d$ associated with a reflection group W and a multiplicity function k. This paper deals with the existence of radial positive solutions of the semilinear equation

$$\Delta_k v = -v^\gamma \ \text{ on } \mathbb{R}^d.$$

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1. INTRODUCTION

We consider $\mathbb{R}^d$ with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and its associated norm $|\cdot|$. For $\alpha \in \mathbb{R}^d \setminus \{0\}$, we denote by σ_α the reflection with respect to the hyperplane orthogonal to α i.e.,

$$\sigma_\alpha x = x - 2\frac{\langle x, \alpha \rangle}{|\alpha|^2}\alpha.$$

A finite set $R \subset \mathbb{R}^d \setminus \{0\}$ is called a root system if $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha R \subset R$ for all $\alpha \in R$. A function $k : R \to \mathbb{R}$ is called multiplicity function if $k(\sigma_\alpha \beta) = k(\beta)$ for every $\alpha, \beta \in R$.

Throught this paper, we fix a root system R and a nonnegative multiplicity function k. The Dunkl Laplacian associated with R and k is given, for every C^2-function u by

$$(1) \qquad \Delta_k u(x) = \Delta u(x) + \sum_{\alpha \in R} k(\alpha) \left(\frac{\langle \nabla u(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{|\alpha|^2}{2} \frac{u(x) - u(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \right),$$

where Δ and ∇ denote respectively the classical Laplace operator and the gradient on $\mathbb{R}^d$.

The Dunkl Laplacian was introduced by C.F. Dunkl in [3]. The study of Dunkl operators is motivated by its interaction with various mathematics fields namely the analysis of certain exactly solvable models of mechanics [2, 5, 8], Fourier analysis and special function [6, 11, 12], algebra [7] and Feller processes with jumps [1, 4]. The main goal of this paper is to investigate the existence of radially symmetric function $v \in C^2(\mathbb{R}^d)$ such that

$$(2) \qquad \begin{cases} \Delta_k v = -v^\gamma & \text{on } \mathbb{R}^d \\ v > 0 & \text{on } \mathbb{R}^d \\ \lim_{|x|\to\infty} v(x) = 0, \end{cases}$$

where $\gamma > 1$. In the case where the multiplicity function k is identically vanishing, Δ_k is reduced to the classical Laplace operator Δ. In this case, a celebrated result of Pohozaev states that any solution of

$$\Delta v = -f(v)$$

on smooth bounded star-shaped domain D of $\mathbb{R}^d$ such that $u = 0$ on ∂D satisfies the so-called classical Pohozaev identity [10]. Using this result, it has been proved that the problem

$$\begin{cases} \Delta v = -v^\gamma & \text{on } \mathbb{R}^d \\ v > 0 & \text{on } \mathbb{R}^d \\ \lim_{|x|\to\infty} v(x) = 0 \end{cases}$$

admits a solution if and only if

$$\gamma \geq \frac{d+2}{d-2}.$$

We refer to [9] and the references therein for more details. Throughout this paper we denote

$$m = d + \sum_{\alpha \in R} k(\alpha)$$

and we assume that $m > 2$. Motivated by the existence result established for the classical Laplacian, our purpose consists in proving the following theorem:

Theorem 1. *Problem (2) admits a symmetric radial solution if and only if*

$$\gamma \geq \frac{m+2}{m-2}.$$

It is worth noting that, for nontrivial multiplicity function k, Pohozaev identity relative to Δ_k is not yet known. So, we are led to use a somewhat different method which is based upon tools from ordinary differential equations.

2. PROOF OF THEOREM 1

Let $v \in C^2(\mathbb{R}^d)$ be radially symmetric on $\mathbb{R}^d$. That is, there exists a C^2-function $\tilde{v} : [0, \infty[\to \mathbb{R}$ such that $v(x) = \tilde{v}(|x|)$ for every $x \in \mathbb{R}^d$. It is easy to verify that, for every $\alpha \in R$,

$$\langle \nabla v(x), \alpha \rangle = \frac{\tilde{v}'(|x|)}{|x|} \langle x, \alpha \rangle \quad \text{and} \quad v(\sigma_\alpha x) = v(x).$$

Thus, it follows from (1) that

$$\Delta_k v(x) = \tilde{v}''(|x|) + \frac{m-1}{|x|} \tilde{v}'(|x|),$$

and therefore, v is a solution of $\Delta_k v = -v^\gamma$ on $\mathbb{R}^d$ if and only if the function $\tilde{v}$ satisfies

$$\tilde{v}''(|x|) + \frac{m-1}{|x|} \tilde{v}'(|x|) = -\tilde{v}(|x|)^\gamma \quad \text{for all } x \in \mathbb{R}^d.$$

It is well known from the general theory of ordinary differential equation that, for every $a > 0$, there exists a unique positive C^2-function u on a maximal interval $[0, R_a[$ such that

$$\text{(3)} \qquad \begin{cases} u'' + \frac{m-1}{r} u' = -u^\gamma \text{ on } [0, R_a[\\ u(0) = a \\ u'(0) = 0. \end{cases}$$

The solution u is said global if $R_a = \infty$. We then deduce that Problem (2) admits a radial solution if and only if there exists $a > 0$ such that problem (3) has a global solution u satisfying

$$\lim_{r \to \infty} u(r) = 0.$$

In order to prove Theorem 1 we need the following lemma.

Lemma 2. *Let $a > 0$ and let u be a solution of (3) on a maximal interval $[0, R_a[$. Then u is nonincreasing on $[0, R_a[$ and*

$$\lim_{r \to R_a} u(r) = 0.$$

Furthermore, for every $r \in [0, R_a[$ the following holds:

(a)

$$\text{(4)} \qquad r^{m-1} u'(r) = -\int_0^r t^{m-1} u^\gamma(t) \mathrm{d}t.$$

(b) There exists $c > 0$ such that

$$\text{(5)} \qquad u(r) \leq c r^{-\frac{2}{\gamma - 1}}.$$

(c)

$$\frac{(m+2)-\gamma(m-2)}{(m-2)(\gamma+1)}\int_0^r t^{m-1}u^{\gamma+1}(t)\mathrm{d}t = \frac{2r^m u^{\gamma+1}(r)}{(m-2)(\gamma+1)} + \frac{r^m(u'(r))^2}{m-2} + r^{m-1}u'(r)u(r). \tag{6}$$

Proof. Writing the equation $u''+\frac{m-1}{r}u'=-u^\gamma$ in the form

$$(r^{m-1}u'(r))' = -r^{m-1}(u(r))^\gamma \tag{7}$$

and then integrating from 0 to r, we obtain (4) which implies that u is nonincreasing and then

$$r^{m-1}u'(r)\le -\frac{r^m}{m}u^\gamma(r).$$

Therefore, $u'(r)(u(r))^{-\gamma}\le -r/m$. Integrating this from 0 to r we obtain

$$\frac{1}{u^{\gamma-1}(r)}\ge\frac{1}{a^{\gamma-1}}+\frac{\gamma-1}{2m}r^2.$$

This yields the existence of $c>0$ such that (5) holds. The fact that $\lim_{r\to R_a}u(r)=0$ follows from the maximality condition if $R_a<\infty$, and from (5) in the case where $R_a=\infty$.

To get (6), on one hand we multiply (7) by $u(r)$ and then we integrate by parts the left hand side from 0 to r to obtain

$$r^{m-1}u(r)u'(r)-\int_0^r t^{m-1}(u'(t))^2\,\mathrm{d}t = -\int_0^r t^{m-1}(u(t))^{\gamma+1}\,\mathrm{d}t. \tag{8}$$

On the other hand, multiplying (7) by $ru'(r)$ and then integrating by parts the right hand side from 0 to r, we obtain

$$\begin{aligned}\int_0^r tu'(t)(t^{m-1}u'(t))'\,\mathrm{d}t &= -\int_0^r t^m u'(t)u^\gamma(t)\,\mathrm{d}t\\ &= -\frac{r^m}{\gamma+1}(u(r))^{\gamma+1}+\frac{m}{\gamma+1}\int_0^r t^{m-1}(u(t))^{\gamma+1}\,\mathrm{d}t.\end{aligned} \tag{9}$$

But, a suitable integration by parts yields

$$\begin{aligned}\int_0^r tu'(t)(t^{m-1}u'(t))'\,\mathrm{d}t &= r^m(u'(r))^2-\int_0^r t^{m-1}(u'(t))^2\,\mathrm{d}t-\int_0^r t^m u''(t)u'(t)\,\mathrm{d}t\\ &= r^m(u'(r))^2-\int_0^r t^{m-1}(u'(t))^2\,\mathrm{d}t-\frac{1}{2}\int_0^r t^m\left((u'(t))^2\right)'\,\mathrm{d}t\\ &= \frac{r^m}{2}(u'(r))^2+\frac{m-2}{2}\int_0^r t^{m-1}(u'(t))^2\,\mathrm{d}t.\end{aligned} \tag{10}$$

Hence, combining (8), (9) and (10), we easily obtain (6). □

Now we turn to prove Theorem 1. As mentioned above, if there exists $a > 0$ such that Problem (3) admits a positive global solution u, then the positive function v defined for every $x \in \mathbb{R}^d$ by $v(x) = u(|x|)$ satisfies

$$\Delta_\kappa v = -v^\gamma$$

on the whole space $\mathbb{R}^d$. Moreover, by the above lemma

$$\lim_{|x|\to\infty} v(x) = \lim_{r\to+\infty} u(r) = 0.$$

Consequently, the theorem will be proved once we have shown that $\gamma \geq \frac{m+2}{m-2}$ if and only if Problem (3) admits a global solution for some $a > 0$.
To that end, assume that $\gamma \geq \frac{m+2}{m-2}$. Let $a > 0$ and let u be the solution of Problem (3) on the maximal interval $[0, R_a[$. Suppose that $R_a < \infty$. By the above lemma u is nonincreasing and $u(R_a) = 0$. Moreover, using (4), we get $|u'(r)| \leq ra^\gamma/m$ for every $r \in [0, R_a[$ and hence $\lim_{r\to R_a} u'(r)$ exists and is finite. Then, by letting r tend to R_a in (6) we obtain

$$\frac{(m+2) - \gamma(m-2)}{(m-2)(\gamma+1)} \int_0^{R_a} t^{m-1} u^{\gamma+1}(t)\,\mathrm{d}t = \lim_{r\to R_a} \frac{r^m}{m-2}(u'(r))^2 \geq 0.$$

But, since $(m+2) - \gamma(m-2) \leq 0$, this yields that $u = 0$ which is impossible. Hence $R_a = \infty$ and so u is a global solution of (3) as desired.
Conversely, let $a > 0$ and assume that Problem (3) admits a global solution u. Then (5) yields the existence of $c' > 0$ such that for every $r > 0$

$$\int_0^r t^{m-1}(u(t))^\gamma\,\mathrm{d}t \leq c'\, r^{m-\frac{2\gamma}{\gamma-1}},$$

and then it follows from (4) that

$$|u'(r)| \leq c' r^{-\frac{\gamma+1}{\gamma-1}}. \tag{11}$$

Now, we prove that $\gamma \geq \frac{m+2}{m-2}$ by contradiction. Suppose that $\gamma < \frac{m+2}{m-2}$. Then using the estimates (5) and (11) we derive

$$\lim_{r\to\infty} r^m u^{\gamma+1}(r) = \lim_{r\to\infty} r^m (u'(r))^2 = \lim_{r\to\infty} r^{m-1} u'(r) u(r) = 0.$$

Consequently, letting r tend to ∞ in (6), we immediately deduce that

$$\int_0^\infty t^{m-1} u^{\gamma+1}(t)\,\mathrm{d}t = 0$$

which implies that $u = 0$ contradicting $u(0) = a > 0$. Hence $\gamma \geq \frac{m+2}{m-2}$ as desired. $\square$

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