CHARACTERIZATION OF PSL(3,Q) BY NSE

S. ASGARY and N. AHANJIDEH

Communicated by Alexandru Buium

Let G be a group and $\pi_e(G)$ be the set of element orders of G. Suppose that $k \in \pi_e(G)$ and m_k is the number of elements of order k in G. Set $\mathrm{nse}(G) := \{m_k : k \in \pi_e(G)\}$. Let $M = PSL(3, q)$, where q is a prime power and $r = (q^2 + q + 1)/(3, q - 1)$ is a prime number and G be a finite group such that $r \mid |G|$, $r^2 \nmid |G|$ and $|G|_3 = |M|_3$. In this paper, we prove that $G \cong M$ if and only if $\mathrm{nse}(G) = \mathrm{nse}(M)$.

AMS 2010 Subject Classification: 20D05, 20D20.

Key words: set of the number of elements of the same order, prime graph, the classification theorem of finite simple groups.

1. INTRODUCTION

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. We denote by $\pi_e(G)$ the set of element orders of G. Set $m_k = m_k(G) := |\{g \in G : \text{the order of } g \text{ is } k\}|$ and $\mathrm{nse}(G) := \{m_k(G) : k \in \pi_e(G)\}$.

Throughout this paper, we denote by ϕ the Euler totient function. If G is a finite group and r is a prime, then we denote by $S_r(G)$ a Sylow r-subgroup of G, by $\mathrm{Syl}_r(G)$ the set of Sylow r-subgroups of G and $n_r(G)$ is the number of Sylow r-subgroups of G. $|cl_G(x)|$ denotes the size of the conjugacy class of G containing x. Let n be a positive integer and p be a prime number. Then $|n|_p$ denotes the p-part of n.

The prime graph $GK(G)$ of a finite group G is a graph whose vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge if and only if G contains an element of order pq (we write $p \sim q$). Let $t(G)$ be the number of connected components of $GK(G)$ and let $\pi_1, \pi_2, ..., \pi_{t(G)}$ be the connected components of $GK(G)$. If $2 \in \pi(G)$, then we always suppose that $2 \in \pi_1(G)$. $|G|$ can be expressed as a product of co-prime positive integers OC_i, $i = 1, 2, \ldots, t(G)$, where $\pi(OC_i) = \pi_i$. These OC_i's are called the order components of G and the set of order components of G will be denoted by $OC(G)$. Also if $2 \in \pi(G)$, we call $OC_2, ..., OC_{t(G)}$ the odd order components of G. The sets of order components of finite simple groups with disconnected prime graph can be obtained using [10] and [17].

Let $M_t(G) := \{g \in G : g^t = 1\}$. Then G and H are of the same order type if and only if $|M_t(G)| = |M_t(H)|$, $t = 1, 2, \ldots$. In 1987, J.G. Thompson put forward the following problem:

THOMPSONS PROBLEM. *Let* $T(G) = \{(k, m_k) : k \in \pi_e(G),\ m_k \in \mathrm{nse}(G)\}$, *where* m_k *is the number of elements of* G *of order* k. *Suppose that* H *is a group with* $T(G) = T(H)$. *If* G *is solvable, then is it true that* H *is also necessarily solvable?*

It is easy to see that if $T(G) = T(H)$, then $\mathrm{nse}(G) = \mathrm{nse}(H)$ and $|G| = |H|$. We say that the group G is characterizable by nse (and the order) if every group H with $\mathrm{nse}(G) = \mathrm{nse}(H)$ (and $|G| = |H|$) is isomorphic to G. Note that not all groups can be characterizable by nse. For instance, let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ and $H = \mathbb{Z}_2 \times \mathbb{Q}_8$. Then $|G| = |H|$ and $\mathrm{nse}(G) = \mathrm{nse}(H)$, while $G \ncong H$. In [12], it is shown that the simple groups $PSL(2, q)$, where $q \leq 13$, are characterizable by nse. In [9] and [15], it is proved that $PSL(2, p)$, where p is a prime, is characterizable by nse. Also, in [8], [2] and [3], the authors respectively showed that for the prime number p, $PGL(2, p)$, the alternating groups A_n, where $n \in \{p, p+1, p+2\}$ and, $C_n(2)$, ${}^2D_n(2)$ and ${}^2D_{n+1}(2)$, where $2^n + 1 = p$, are characterizable by nse under some extra conditions.

Throughout this paper, let q be a prime power such that $\frac{q^2+q+1}{(3,q-1)}$ is a prime, namely r and $M = PSL(3, q)$. In this paper, we are going to study the characterization of M by nse. In fact, we prove the following theorem:

MAIN THEOREM. *Let* G *be a finite group such that* $r \mid |G|$, $r^2 \nmid |G|$ *and* $|G|_3 = |M|_3$. *Then* $G \cong M$ *if and only if* $\mathrm{nse}(G) = \mathrm{nse}(M)$.

2. PRELIMINARIES

Definition 2.1 ([5]). Let a be a natural number and r be a prime such that $(a, r) = 1$. If n is the smallest natural number such that $r \mid (a^n - 1)$, then r is named a Zsigmondy prime of $a^n - 1$.

LEMMA 2.1 ([5]). *Let* a *and* n *be natural numbers, then there exists a Zsigmondy prime of* $a^n - 1$, *unless* $(a, n) = (2, 1)$, $(a, n) = (2, 6)$ *or* $n = 2$ *and* $a = 2^s - 1$ *for some natural number* s.

Remark 2.1. If l is a Zsigmondy prime of $a^n - 1$, then Fermat's little theorem shows that $n \mid l - 1$. Put

$$Z_n(a) = \{l : l \text{ is a Zsigmondy prime of } a^n - 1\}.$$

If $r \in Z_n(a)$ and $r \mid a^m - 1$, then we can see at once that $n \mid m$.

Lemma 2.2 ([4]). *Let G be a Frobenius group of even order with kernel K and complement H. Then $t(G) = 2$, the prime graph components of G are $\pi(H)$ and $\pi(K)$ and the following assertions hold:*
(1) *K is nilpotent;*
(2) $|K| \equiv 1 \pmod{|H|}$.

Lemma 2.3. *If x is an element of $M - \{1\}$, then either $|cl_M(x)|_r = |M|_r$ or $|cl_M(x)|_r < |M|_r$ and*

$$|cl_M(x)| = \frac{|M|(q-1)d}{|GL(1,q^3)|} = \frac{|GL(3,q)|}{|GL(1,q^3)|}.$$

Proof. It follows from [1, Corollary 2.8]. □

Lemma 2.4 ([6]). *Let t be a positive integer dividing $|G|$. Then $t \mid |M_t(G)|$.*

From Lemma 2.4, it may be concluded that:

Corollary 2.1. *For a finite group G:*
(i) *if $n \mid |G|$, then $n | \sum_{s|n} m_s$;*
(ii) *if $n \in \pi_e(G)$, then $m_n = \phi(n)k$, where k is the number of cyclic subgroups of order n in G. In particular, $\phi(n) \mid m_n$.*
(iii) *if $R \in \mathrm{Syl}_r(G)$ is cyclic of prime order r, then $m_r = n_r(G)(r-1)$;*
(iv) *if $P \in \mathrm{Syl}_p(G)$ is cyclic of prime order p and $r \in \pi(G) - \{p\}$, then $m_{rp} = n_p(G)(p-1)(r-1)k$, where k is the number of cyclic subgroups of order r in $C_G(P)$.*

Lemma 2.5 ([11]). *If $n \geq 6$ is a natural number, then there are at least $s(n)$ prime numbers p_i such that $(n+1)/2 < p_i < n$. Here*

$$\begin{aligned} s(n) &= 1, \ \ for\ 6 \leq n \leq 13;\\ s(n) &= 2, \ \ for\ 14 \leq n \leq 17;\\ s(n) &= 3, \ \ for\ 18 \leq n \leq 37;\\ s(n) &= 4, \ \ for\ 38 \leq n \leq 41;\\ s(n) &= 5, \ \ for\ 42 \leq n \leq 47;\\ s(n) &= 6, \ \ for\ n \geq 48. \end{aligned}$$

3. MAIN RESULTS

In this section, let G be a finite group such that $r \mid |G|$, $r^2 \nmid |G|$ and $\mathrm{nse}(G) = \mathrm{nse}(M)$. In the following, we are going to bring some useful lemmas which will be used during the proof of the main theorem:

LEMMA 3.1. $m_r(M) = \frac{1}{3}(r-1)q^3(q^2-1)(q-1)$.

Proof. By [7], $|N_M(S_r(M))| = 3.\frac{q^3-1}{(q-1)(3,q-1)}$ and hence,

$$n_r(M) = \frac{|M|}{|N_M(S_r(M))|} = \frac{1}{3}q^3(q^2-1)(q-1).$$

Now, since $S_r(M)$ is cyclic, Corollary 2.1(iii) implies that $m_r(M) = n_r(M)(r-1) = \frac{1}{3}(r-1)q^3(q^2-1)(q-1)$, as desired. □

LEMMA 3.2.

(i) *For $u \in \pi_e(M)$, either $r \mid m_u(M)$ or $u = r$;*

(ii) *For every $u \in \pi_e(G)$, $r \nmid m_u(G)$ if and only if $m_u(G) = m_r(M)$;*

(iii) $m_r(G) = m_r(M)$;

(iv) $m_2(G) = m_2(M)$.

Proof. (i) Obviously, $m_u(M) = \sum_{O(x_k)=u} |cl_M(x_k)|$, where $x_k s$ are selected from distinct conjugacy classes of M. Thus Lemma 2.3 completes the proof of (i).

(ii) Since $m_u(G) \in \text{nse}(G) = \text{nse}(M)$, (i) completes the proof.

(iii) By Corollary 2.1(i), we have $r \mid (1 + m_r(G))$ and hence, $r \nmid m_r(G)$. Thus $m_r(G) = m_r(M)$, by (ii).

(iv) By Corollary 2.1(ii), for every $u \in \pi_e(M)$, $\phi(u) \mid m_u(M)$. Thus if $u > 2$, then m_u is even. On the other hand, $2 \mid (1 + m_2(M))$ and hence, $m_2(M)$ is odd. Applying the same reasoning shows that the only odd number in nse(G) is $m_2(G)$ and hence, $m_2(G) = m_2(M)$, as wanted. □

LEMMA 3.3. *For every $s \in \pi(G) - \{r\}, sr \notin \pi_e(G)$.*

Proof. Suppose on the contrary, $sr \in \pi_e(G)$. Since $r^2 \nmid |G|$, we deduce that $S_r(G)$ is cyclic and hence, Corollary 2.1(iv) forces $m_{rs}(G) = (r-1)(s-1)n_r(G)k$, for some natural number k. Thus $m_{rs}(G) = m_r(G)(s-1)k$ and hence, one of the following holds:

(i) $r \mid m_{rs}(G)$. Then $r \mid (s-1)k$ and hence, $m_{rs}(G) > |M|$, along with Lemmas 3.1 and 3.2(iii). Thus $m_{rs}(G) \notin \text{nse}(M)$, which is a contradiction.

(ii) $r \nmid m_{rs}(G)$. Then Lemma 3.2(ii) shows that $m_{rs}(G) = m_r(G)$ and hence, by Corollary 2.1(iv), $s = 2$. On the other hand, Corollary 2.1 (i) and Lemma 3.2(iv) show that $2r \mid (1 + m_2(G) + m_r(G) + m_{2r}(G)) = 1 + m_2(M) + 2m_r(G)$. Now, since by Corollary 2.1 (i) and Lemma 3.2(i), $r \mid (1 + m_r(G))$ and $r \mid m_2(M)$, we deduce that $r \mid m_r(G)$, which is a contradiction. □

COROLLARY 3.1. *r is an odd order component of G.*

Proof. It follows from Lemma 3.3. □

LEMMA 3.4.
(i) $n_r(G) = n_r(M) = \frac{1}{3}q^3(q^2-1)(q-1)$;
(ii) $\frac{1}{3}|M| \mid |G|$ *and* $|G| \mid \frac{1}{3}r(r-1)q^3(q^2-1)(q-1)$.

Proof. Since $S_r(G)$ is cyclic, Corollary 2.1(iii) forces $m_r(G) = \phi(r)n_r(G)$. Thus by Lemma 3.2(iii), $\phi(r)n_r(M) = m_r(M) = m_r(G) = \phi(r)n_r(G)$ and hence (i) follows. Now let $s \in \pi(G) - \{r\}$. By Lemma 3.3, $S_s(G)$ acts fixed point freely on the set of elements of order r in G and hence, $|G|_s \mid m_r(G) = \phi(r)n_r(G)$. Also, $|G|_r = r$ and $n_r(G) = \frac{1}{3}q^3(q^2-1)(q-1)$. Thus $|M|/3 \mid |G|$ and $|G| \mid \frac{1}{3}r(r-1)q^3(q^2-1)(q-1)$. □

Proof of the main theorem. If $G \cong M$, then it is obvious that $\mathrm{nse}(G) = \mathrm{nse}(M)$. Now we assume that $\mathrm{nse}(G) = \mathrm{nse}(M)$. In the following, we show that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a non-abelian simple group.

Let $x \in G$ be an element of order r. Since r is the maximal prime divisor of $|G|$ and an odd order component of G, $C_G(x) =< x >$. Set $H = O_{r'}(G)$, the largest normal r'-subgroup of G. Since $< x >$ acts on H fixed point freely, H is a nilpotent group. Suppose that K be a normal subgroup of G such that K/H is a minimal normal subgroup of G/H. Then K/H is a direct product of copies of same simple group. Since $r \mid |K/H|$ and $r^2 \nmid |K/H|$, K/H is a simple group. On the other hand, since $< x >$ is a Sylow r-subgroup of K, $G = N_G(< x >)K$ by the Frattini argument and so $|G/K|$ divides $r-1$. Now, it's not too hard to prove that $|K/H| \neq r$. Therefore, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a non-abelian simple group.

In the following, assume that $d = (3, q-1)$ and $q' = p'^{\alpha}$, where p' is a prime and α is a positive integer. Also, for convenience let $f(q) = \frac{1}{3}r(r-1)q^3(q^2-1)(q-1)$. We are going to continue the prove of the main theorem in the following steps:

Step 1. K/H is not a Sporadic simple group.

Proof. Suppose that K/H is a Sporadic simple group. Thus $r = \frac{q^2+q+1}{d} \in \{5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 67, 71\}$. If $\frac{q^2+q+1}{d} \in \{5, 11\}$, then since q is a prime power, we get a contradiction. Assume that $\frac{q^2+q+1}{d} = 7$ and $d = 1$. Thus $q = 2$. But $|PSL(3,2)| = 2^3.3.7$ and $K/H \in \{M_{22}, J_1, J_2, HS\}$, so $5 \mid |K/H|$, which is a contradiction. If $d = 3$, then $q = 4$ and $|PSL(3,4)| = 2^6.3^2.5.7$. Also, $K/H \in \{M_{22}, J_1, J_2, HS\}$. Now, similar to the above we get a contradiction. The same argument rules out the other possibilities of r. □

Step 2. K/H cannot be an alternating group $\mathbb{A}_m$, where $m \geq 5$.

Proof. If $K/H \cong \mathbb{A}_m$, then since $\frac{q^2+q+1}{d} = r \in \pi(K/H)$, $r \leq m$. Also, since $q \geq 2$ is a prime power, $r \geq 7$. Thus by Lemma 2.5, there exists a prime number $u \in \pi(\mathbb{A}_m) \subseteq \pi(G)$ such that $(r+1)/2 < u < r$. Lemma 3.4(ii) forces $u \mid \frac{1}{3}r(r-1)q^3(q^2-1)(q-1)$. We can check at once that $u \nmid q$, $u \nmid q-1$ and $u \nmid r-1$. Thus $u \in Z_2(q)$. It follows that $u = r-2$, where $r = 7$ and $q = 4$. So $|M| = |PSL(3,4)| = 2^6.3^2.5.7$. Since $|\mathbb{A}_m|$ divides $|G|$, we get $m \in \{7,8\}$. Note that $|H|$ divides $|G|/|K/H|$ and $S_7(G)$ acts fixed point freely on H and hence, $7 \mid |H|-1$. Therefore, considering the orders of G and K/H shows that either $|H| = 8$ and $m = 7$ or $|H| = 1$. If $|H| = 8$ and $m = 7$, then we can assume that H is a 2-elementary abelian group and hence, $\mathbb{A}_7 \lesssim GL(3,2)$. Thus $|\mathbb{A}_7|$ divides $|GL(3,2)|$, which is a contradiction. If $|H| = 1$, then $G \cong \mathbb{A}_7, \mathbb{S}_7, \mathbb{A}_8$ or $\mathbb{S}_8$, which in two former cases $m_7(G) = m_7(\mathbb{A}_7) = 720 \neq 5760 = m_7(M)$, which is a contradiction. If $G \cong \mathbb{A}_8$, then $1344 \in \mathrm{nse}(G) - \mathrm{nse}(M)$ and if $G \cong \mathbb{S}_8$, then $763 \in \mathrm{nse}(G) - \mathrm{nse}(M)$, contradicting our assumptions. □

Step 3. $K/H = PSL(3,q)$.

Proof. By Steps 1 and 2, and the classification theorem of finite simple groups, K/H is a simple group of Lie type such that $t(K/H) \geq 2$ and $r \in OC(K/H)$. Thus K/H is isomorphic to one of the following groups:

Case 1. Let $t(K/H) = 2$. Then $OC_2(K/H) = r = \frac{q^2+q+1}{d}$. Thus we have:

1.1. If $K/H \cong C_n(q')$, where $n = 2^u \geq 2$, then $\frac{q'^n+1}{(2,q'-1)} = \frac{q^2+q+1}{d}$. If $(2,q'-1) = 2$, then $\frac{q'^n+1}{2} = r$, so $q'^n = \frac{2(q^2+q+1)-d}{d}$ and hence, $(q',r) = (q',q) = 1$. Also, since p' is odd, we have $p \mid q-1$ or $p' \mid q-1$ and hence, $(p'^{\alpha})^{n^2} = |K/H|_{p'} \leq |G|_{p'} \leq \frac{1}{(3,p')}(q^2-1)_{p'}(q-1)_{p'}(\frac{q^2+q+1-d}{d})_{p'} < (\frac{2(q^2+q+1)-d}{d})^2 < (p'^{\alpha})^{2n}$. Therefore, $n < 2$, which is a contradiction. If $(2,q'-1) = 1$ and $d = 1$, then we have $q'^n = q(q+1)$, which is impossible. If $(2,q'-1) = 1$ and $d = 3$, then $q' = 2^{\alpha}$ and $q'^2+1 = r$. Thus $2^{2\alpha}+1 = \frac{q^2+q+1}{3}$ and hence, $2^{2\alpha} = \frac{(q-1)(q+2)}{3}$. Since $3 \mid q-1$, $3 \mid q+2$ and hence, $3 \mid 2^{2\alpha}$, which is a contradiction. The same reasoning completes the proof in the case when either $K/H \cong B_n(q')$ or $K/H \cong {}^2D_n(q')$, where $n = 2^u \geq 4$.

1.2. If $K/H \cong C_s(3)$ or $B_s(3)$, where s is prime, then $\frac{3^s-1}{2} = \frac{q^2+q+1}{d}$. So $3^s = \frac{2}{d}(q^2+q+\frac{d+2}{2})$ and hence, either $(3,q) = 1$ and $3^{s+1} > q^2+q+\frac{d+2}{2}$ or $q = s = 3$. In the former case, since $3^{s^2} = |K/H|_3 \leq |G|_3 = (q^2-1)_3(q-1)_3 < (\frac{q^2+q+1-d}{d})^3 < 3^{3(s+1)}$, $s^2 < 3(s+1)$ and hence, $s = 3$. So $q(q+1) \in \{12, 38\}$. Obviously, $q(q+1) \neq 38$. If $q(q+1) = 12$, then $q = 3$, which is a contradiction. If $q = s = 3$, then $r = 13$ and $|M| = |PSL(3,3)| = 2^4.3^3.13$, so $5 \nmid |G|$. On the other hand, $5 \mid |C_3(3)| = |B_3(3)|$, which is a contradiction.

1.3. If $K/H \cong C_s(2)$, where s is prime, then $2^s - 1 = \frac{q^2+q+1}{d}$ and hence, $2^s = \frac{q^2+q+1+d}{d}$. Now, if $q \in \{2,4\}$, then $\frac{q^2+q+1}{d} = 7$ and hence, $s = 3$. In these cases, $|PSL(3,2)| = 2^3.3.7$ and $|PSL(3,4)| = 2^6.3^2.5.7$. So $2^9 \nmid |G|$. On the other hand, $2^9 \mid |C_3(2)|$, which is a contradiction. If $q \notin \{2,4\}$, then $(2,q) = 1$ and hence, $2^{s^2} = |K/H|_2 \leq |G|_2 \leq (\frac{q^2-1}{d})_2(q-1)_2(\frac{q^2+q+1-d}{d})_2 < (\frac{q^2+q+1+d}{d})^3 = 2^{3s}$. So $s^2 < 3s$, which implies that $s = 2$. This forces $\frac{q^2+q+1}{d} = 3$, which is impossible.

1.4. If $K/H \cong D_s(q')$, where $s \geq 5$ is prime and $q' = 2,3,5$, then $\frac{q'^s-1}{q'-1} = r$. Thus $q'^{s(s-1)}\prod_{i=1}^{s-1}(q'^{2i}-1) \mid \frac{1}{3}(r-1)q^3(q^2-1)(q-1)$. On the other hand, $r^5 = \frac{(q'^s-1)^5}{(q'-1)^5} < q'^{5s}$ and $q'^{s(s-1)}.q'^{\frac{s(s-1)}{2}} < q'^{s(s-1)}\prod_{i=1}^{s-1}(q'^{2i}-1) \leq \frac{1}{3}(r-1)q^3(q^2-1)(q-1) < r^5$, which implies that $q'^{s(s-1)+\frac{s(s-1)}{2}} < q'^{5s}$ and hence, $s < 5$, which is a contradiction.

1.5. If $K/H \cong {}^2D_n(3)$, where $9 \leq n = 2^m + 1$ and n is not prime, then $\frac{3^{n-1}+1}{2} = \frac{q^2+q+1}{d}$. Thus $(3,q) = 1$ and $3^{n-1} = \frac{2}{d}(q^2 + q + \frac{2-d}{2})$ and hence, $3^n > q^2 + q + \frac{2-d}{2}$. Since $3^{n(n-1)} = |K/H|_3 \leq |G|_3 = (q^2-1)_3(q-1)_3 < (q^2 + q + \frac{2-d}{2})^3 < 3^{3n}$, we obtain $n - 1 < 3$, which is impossible.

1.6. If $K/H \cong {}^2D_n(2)$, where $n = 2^m + 1 \geq 5$, then $2^{n-1} + 1 = \frac{q^2+q+1}{d}$. Thus $2^{n-1} = \frac{q^2+q+1-d}{d}$ and hence, $(2,q) = 1$. Therefore, $2^{n(n-1)} = |K/H|_2 \leq |G|_2 < (\frac{q^2+q+1-d}{d})^3 = 2^{3(n-1)}$, so $n < 3$, which is impossible.

1.7. If $K/H \cong D_{s+1}(q')$, where s is an odd prime and $q' = 2,3$, then $\frac{q'^s-1}{(2,q'-1)} = r$. Thus $q'^{s(s+1)}(q'^s+1)(q'^{s+1}-1)\prod_{i=1}^{s-1}(q'^{2i}-1) \mid \frac{1}{3}(r-1)q^3(q^2-1)(q-1)$. Also, $r^5 = \frac{(q'^s-1)^5}{(2,q'-1)^5} < q'^{5s}$ and $q'^{s(s+1)}q'^{s(s+1)/2} < q'^{s(s+1)}(q'^s+1)(q'^{s+1}-1)\prod_{i=1}^{s-1}(q'^{2i}-1) \leq \frac{1}{3}(r-1)q^3(q^2-1)(q-1) < r^5$, which implies that $q'^{3s(s+1)/2} < q'^{5s}$ and hence, $s < 3$, which is a contradiction.

1.8. If $K/H \cong {}^2D_s(3)$, where $5 < s \neq 2^m + 1$ and s is an odd prime, then $\frac{3^s+1}{4} = r$ and $3^{s(s-1)}\prod_{i=1}^{s-1}(3^{2i}-1) \mid \frac{1}{3}(r-1)q^3(q^2-1)(q-1)$. Thus $r^5 = \frac{(3^s+1)^5}{1024} < 3^{5s}$ and $3^{s(s-1)} < 3^{s(s-1)}\prod_{i=1}^{s-1}(3^{2i}-1) \leq \frac{1}{3}(r-1)q^3(q^2-1)(q-1) < r^5$, which implies that $s(s-1) < 5s$ and hence, $s - 1 < 5$, which is a contradiction.

1.9. If $K/H \cong G_2(q')$, where $2 < q' \equiv \epsilon \pmod 3$ and $\epsilon = \pm 1$, then $q^{2\prime} - \epsilon q' + 1 = \frac{q^2+q+1}{d}$. Thus we can check at once that $|K/H| > f(q)$, which is a contradiction. If $K/H \cong F_4(q')$, where q' is odd, then similar to the above, we get a contradiction.
1.10. If $K/H \cong {}^2F_4(2)'$, then $|K/H| = 2^{11}{\cdot}3^3{\cdot}5^2{\cdot}13$ and $\frac{q^2+q+1}{d} = 13$. If $d = 1$, then $q = 3$ and $|PSL(3,3)| = 2^4.3^3.13$, so $5 \nmid |G|$, which is a contradiction. If $d = 3$, then $q(q+1) = 38$, which is impossible.
1.11. If $K/H \cong PSU(4,2)$, then $|K/H| = 2^6 \cdot 3^4 \cdot 5$. Thus $\frac{q^2+q+1}{d} = 5$, which is impossible.
1.12. If $K/H \cong PSL(s,q')$, where $(s,q') \neq (3,2),\ (3,4)$ and s is an odd prime, then $r = \frac{q'^s-1}{(s,q'-1)(q'-1)}$ and $q'^{\frac{s(s-1)}{2}} \prod_{i=1}^{s-1}(q'^i - 1) \mid \frac{1}{3}(r-1)q^3(q^2-1)(q-1)$. On the other hand, $r^5 = \frac{(q'^s-1)^5}{(s,q'-1)^5(q'-1)^5} < q'^{5s}$ and $q'^{s(s-1)-s} < q'^{\frac{s(s-1)}{2}} \prod_{i=1}^{s-1}(q'^i - 1) \leq \frac{1}{3}(r-1)q^3(q^2-1)(q-1) < r^5$, which implies that $s(s-1) - s < 5s$. Hence $s = 3, 5$. If $s = 5$, then $\frac{q^2+q+1}{d} = \frac{q'^4+q'^3+q'^2+q'+1}{(5,q'-1)}$ and hence, $(q',q) = 1$. Also, $q'^{10}(q'-1)(q'^2-1)(q'^3-1)(q'^4-1) \mid \frac{1}{3}(r-1)q^3(q^2-1)(q-1)$. But $q'^{10} \nmid \frac{1}{3}(r-1)q^3(q^2-1)(q-1)$, which is a contradiction. If $s = 3$, then $\frac{q'^3-1}{(3,q'-1)(q'-1)} = \frac{q^2+q+1}{d}$ and hence, $\frac{q'^2+q'+1}{(3,q'-1)} = \frac{q^2+q+1}{d}$. Now, we divide the proof into the following subcases:
(i) Suppose that $(q',q) \neq 1$ and $q \mid q'$.
(1) Let $(3,q'-1) = 1$ and $d = 1$. Thus $q'^2 + q' + 1 = q^2 + q + 1$, so $q' = q$ and $K/H \cong PSL(3,q)$.
(2) If $(3,q'-1) = 3$ and $d = 3$, then $\frac{q'^2+q'+1}{3} = \frac{q^2+q+1}{3}$ and similar to the above, $K/H \cong PSL(3,q)$.
(3) If $(3,q'-1) = 1$ and $d = 3$, then $q'^2 + q' + 1 = \frac{q^2+q+1}{3}$ and hence, $q(q+1) = 3q'^2 + 3q' + 2$, which implies that $q \mid 3q'^2 + 3q' + 2$. Since $q \mid q'$, $q \mid 2$, thus $q = 2$ and hence, $q'(q'+1) = \frac{4}{3}$, which is a contradiction.
(4) If $(3,q'-1) = 3$ and $d = 1$, then $\frac{q'^2+q'+1}{3} = q^2 + q + 1$ and hence, $q(q+1) = \frac{(q'-1)(q'+2)}{3}$, which implies that $q \mid q'^2 + q' - 2$. Thus $q \mid (-2)$ and hence, $q = 2$. It follows that $18 = q'^2 + q' - 2$. Hence, $q' = 4$. But $(s,q') \neq (3,4)$ by assumption, which is a contradiction.
The same argument completes the proof when $q' \mid q$.
(ii) Assume that $(q',q) = 1$.
(1) If $(3,q'-1) = 1$ and $d = 1$, then $q'^2 + q' + 1 = q^2 + q + 1$ and hence, $q'(q'+1) = q(q+1)$. Since $(q',q) = 1$, we get a contradiction.

(2) If $(3, q'-1) = 3$ and $d = 3$, then $\frac{q'^2+q'+1}{3} = \frac{q^2+q+1}{3}$ and similar to the above, we get a contradiction.
(3) If $(3, q'-1) = 3$ and $d = 1$, then $\frac{q'^2+q'+1}{3} = q^2+q+1$ and hence, $\frac{q'^2+q'-2}{3} = q(q+1)$. Thus $\frac{(q'-1)(q'+2)}{3} = q(q+1)$, which implies that $3k(k+1) = q(q+1)$. Considering the different possibilities of q leads us to get a contradiction.
(4) If $(3, q'-1) = 1$ and $d = 3$, then similar to the above, we get a contradiction.
1.13. If $K/H \cong PSL(s+1, q')$, where $(q'-1) \mid (s+1)$ and s is an odd prime, then $r = \frac{q'^s-1}{q'-1}$ and $q'^{\frac{s(s+1)}{2}}(q'^{s+1}-1)\prod_{i=1}^{s-1}(q'^i-1) \mid \frac{1}{3}(r-1)q^3(q^2-1)(q-1)$. On the other hand, $r^5 = \frac{(q'^s-1)^5}{(q'-1)^5} < q'^{5s}$ and $q'^{s(s+1)-s} < q'^{\frac{s(s+1)}{2}}(q'^{s+1}-1)\prod_{i=1}^{s-1}(q'^i-1) \leq \frac{1}{3}(r-1)q^3(q^2-1)(q-1) < r^5 < q'^{5s}$, which implies that $s+1 < 6$. Hence $s = 3$, so $q'^2 + q' + 1 = \frac{q^2+q+1}{d}$. Since $(q'-1) \mid (s+1)$, $q' \in \{2, 3, 5\}$, which implies that $K/H \cong PSL(4,2)$, $K/H \cong PSL(4,3)$, $K/H \cong PSL(4,5)$. If $K/H \cong PSL(4,2)$, then since $PSL(4,2) \cong \mathbb{A}_8$, Step 2 leads us to get a contradiction. If $K/H \cong PSL(4,3)$, then $\frac{q^2+q+1}{d} = 13$ and hence $q = 3$. So $|K/H| \nmid |G|$, which is impossible. The same reasoning rules out the case when $K/H \cong PSL(4,5)$.
1.14. If $K/H \cong E_6(q')$, then $r = \frac{q'^6+q'^3+1}{(3,q'-1)}$ and $q'^{36}(q'^{12}-1)(q'^8-1)(q'^6-1)(q'^5-1)(q'^2-1) \mid \frac{1}{3}(r-1)q^3(q^2-1)(q-1)$. On the other hand, $r^5 = \frac{(q'^6+q'^3+1)^5}{(3,q'-1)^5} < q'^{45}$ and $q'^{36}(q'^{12}-1)(q'^8-1)(q'^6-1)(q'^5-1)(q'^2-1) \leq \frac{1}{3}(r-1)q^3(q^2-1)(q-1) < r^5 < q'^{45}$, which is a contradiction. The same reasoning rules out the case when $K/H \cong {}^2E_6(q')$, where $q' > 2$.
1.15. If $K/H \cong {}^3D_4(q')$, then $r = q'^4 - q'^2 + 1$ and $q'^{12}(q'^4+q'^2+1)(q'^6-1)(q'^2-1) \mid \frac{1}{3}(r-1)q^3(q^2-1)(q-1)$. But $q'^{12}(q'^4+q'^2+1)(q'^6-1)(q'^2-1) > \frac{1}{3}(r-1)q^3(q^2-1)(q-1)$, which is a contradiction.
1.16. If $K/H \cong PSU(s+1, q')$, where $(s, q') \neq (3,3), (5,2)$, $(q'+1) \mid (s+1)$ and s is an odd prime, then $r = \frac{q'^s+1}{q'+1}$ and $q'^{\frac{s(s+1)}{2}}(q'^{s+1}-1)\prod_{i=1}^{s-1}(q'^i-(-1)^i) \mid \frac{1}{3}(r-1)q^3(q^2-1)(q-1)$. Moreover, $r^5 = (\frac{q'^s+1}{q'+1})^5 = (q'^{s-1} - q'^{s-2} + ... + 1)^5 < q'^{5(s-1)}$ and $q'^{\frac{s(s+1)}{2}}.q'^{\frac{s(s-1)}{2}+s} < q'^{\frac{s(s+1)}{2}}(q'^{s+1}-1)\prod_{i=1}^{s-1}(q'^i-(-1)^i) \leq \frac{1}{3}(r-1)q^3(q^2-1)(q-1) < r^5$, which implies that $s^2 + s < 5(s-1)$. Thus $s < 2$, which is a contradiction.
1.17. If $K/H \cong PSU(s, q')$, where s is an odd prime, then $r = \frac{q'^s+1}{(q'+1)(s,q'+1)}$ and

${q'}^{\frac{s(s-1)}{2}}\prod_{i=1}^{s-1}({q'}^i-(-1)^i) \mid \frac{1}{3}(r-1)q^3(q^2-1)(q-1)$. Also, $r^5=\frac{({q'}^s+1)^5}{(q'+1)^5(s,q'+1)^5}<{q'}^{5(s-1)}$ and ${q'}^{\frac{s(s-1)}{2}}{q'}^{\frac{s(s-1)}{2}}<{q'}^{\frac{s(s-1)}{2}}\prod_{i=1}^{s-1}({q'}^i-(-1)^i)\leq\frac{1}{3}(r-1)q^3(q^2-1)(q-1)<r^5$. This implies that $s<5$. Thus $s=3$ and hence, $r=\frac{{q'}^2-q'+1}{(3,q'+1)}$. This shows that $q=q'-1$, $q(q+1)=3k(k+1)$ or $q'(q'-1)=3k(k+1)$. If $q'=q+1$, then we can see that $|K/H|\nmid|G|$, which is a contradiction. Considering the different possibilities of k in two latter cases shows that $(q,q')=(4,3)$, so $|G|_5=5$. Since $5\nmid|\mathrm{Aut}(PSU(3,3))|$, so $|H|_5=5$. But $S_7(G)$ acts fixed point freely on $S_5(H)$ and hence, $7\mid 5-1$, which is a contradiction.

Case 2. Let $t(K/H)=3$. Then $r\in\{OC_2(K/H),\ OC_3(K/H)\}$:

2.1. If $K/H\cong PSL(2,q')$, where $4\mid q'$, then the odd order components of K/H are $q'+1$ and $q'-1$. If $q'+1=r$, then $q'=r-1=\frac{(q^2+q+1)}{d}-1$ and hence, either $q'=q(q+1)$ or $q'=\frac{(q-1)(q+2)}{3}$, which are impossible. So let $q'-1=r$. Thus $q'=2^\alpha>4$, $q'=r+1=\frac{(q^2+q+1)}{d}+1$ and $n_r(K/H)=\frac{q'(q'+1)}{2}\mid n_r(G)=\frac{1}{3}q^3(q-1)(q^2-1)$. If $d=1$, then $q(q+1)=2(2^{\alpha-1}-1)$ and hence, either $q=2$ or $(q^3(q+1),|K/H|)\mid 6$. If $q=2$, then $q'=8$, so $n_r(K/H)\nmid n_r(G)$, which is a contradiction. In the latter case, $3q'(q'+1)\mid 2(q-1)^2$, which is impossible. Now let $d=3$. Then we can see at once that $(q+2)(q-1)=6(2^{\alpha-1}-1)$ and either $(q,q')=1$ or $q=4$ and $q'=8$. Thus if $(q,q')=1$, then the above statements show that $r+1=q'\mid 8|q+1|_2$, which is impossible. If $q=4$ and $q'=8$, then $|H|_5=5$ and hence, $7\mid 5-1$, which is a contradiction.

2.2. If $K/H\cong PSL(2,q')$, where $4\mid q'-1$, then $q'=r$ or $(q'+1)/2=r$. If $q'=r$, then $n_r(K/H)=(q'+1)\mid n_r(G)=\frac{1}{3}q^3(q-1)(q^2-1)$ and either $q'+1=q^2+q+2$ or $q'+1=\frac{(q^2+q+4)}{3}$. So we can see at once that $q'+1\mid 2$, which is a contradiction. If $\frac{(q'+1)}{2}=r$, then $n_r(K/H)=\frac{1}{2}q'(q'-1)\mid n_r(G)=\frac{1}{3}q^3(q-1)(q^2-1)$ and $q'=2q^2+2q+1$ or $q'=\frac{(2q^2+2q-1)}{3}$. This forces $q'=5$ and hence, $r=3$, which is impossible. The same reasoning completes the proof when $K/H\cong PSL(2,q')$ and $4\mid q'+1$.

2.3. If $K/H\cong PSU(6,2)$ or $K/H\cong PSL(3,2)$, then $|K/H|=2^{15}\cdot3^6\cdot5\cdot7\cdot11$ or $|K/H|=2^3\cdot3\cdot7$. It follows that $\frac{q^2+q+1}{d}\in\{7,11\}$. We can check that $\frac{q^2+q+1}{d}\neq 11$ and hence $\frac{q^2+q+1}{d}=7$. So $q\in\{2,4\}$. Thus $|PSU(6,2)|\nmid|G|$ and hence, $K/H\ncong PSU(6,2)$. Also, if $K/H\cong PSL(3,2)$ and $q=4$, then as mentioned in the previous cases $7\mid 5-1$, which is a contradiction. Therefore, $K/H\cong PSL(3,2)=M$, as desired.

2.4. If $K/H\cong {}^2D_s(3)$, where $s=2^t+1\geq 5$, then $\frac{3^s+1}{4}=\frac{q^2+q+1}{d}$ or $\frac{3^{s-1}+1}{2}=$

$\frac{q^2+q+1}{d}$. If $\frac{3^s+1}{4} = r$, then $3^{s(s-1)}\prod_{i=1}^{s-1}(3^{2i}-1) \mid \frac{1}{3}(r-1)q^3(q^2-1)(q-1)$. On the other hand, $r^5 = \frac{(3^s+1)^5}{1024} < 3^{5s}$ and $3^{2s(s-1)-s} < 3^{s(s-1)}\prod_{i=1}^{s-1}(3^{2i}-1) \leq \frac{1}{3}(r-1)q^3(q^2-1)(q-1) < r^5$. This implies that $2s(s-1) < 6s$ and hence, $s < 4$, which is a contradiction. The same reasoning rules out the other possibility.

2.5. If $K/H \cong {}^2D_{s+1}(2)$, where $s = 2^n - 1$ and $n \geq 2$, then $2^s + 1 = \frac{q^2+q+1}{d}$ or $2^{s+1} + 1 = \frac{q^2+q+1}{d}$. If $2^s + 1 = r$, then $2^s = q(q+1)$ or $2^s = \frac{(q-1)(q+2)}{3}$, which is impossible. The same reasoning rules out the other possibilities.

2.6. If $K/H \cong G_2(q')$, where $q' \equiv 0 \pmod 3$, then $q'^2 \pm q' + 1 = \frac{q^2+q+1}{d}$. We know that $|K/H| \mid |G|$ and $|G| \mid f(q)$, so $|K/H| \mid f(q)$. Since $|K/H| = q'^6(q'^2-1)(q'^6-1)$ and either $q'(q' \pm 1) = q(q+1)$ or $q'(q' \pm 1) = \frac{1}{3}(q-1)(q+2)$, we can check at once that $q'^6 \nmid |G|_{p'}$, which is a contradiction.

2.7. If $K/H \cong {}^2G_2(q')$, where $q' = 3^{2t+1} > 3$, then $q' - \sqrt{3q'} + 1 = \frac{q^2+q+1}{d}$ or $q' + \sqrt{3q'} + 1 = \frac{q^2+q+1}{d}$. Let $(3, q) = 1$. If $q' - \sqrt{3q'} + 1 = \frac{q^2+q+1}{d}$, then $q' > \frac{q^2+q+1-d}{d}$. Thus $(3^{2t+1})^3 = |K/H|_3 \leq |G|_3 < (\frac{q^2+q+1-d}{d})^3 < (3^{2t+1})^3$, which is a contradiction. Now let $q' + \sqrt{3q'} + 1 = \frac{q^2+q+1}{d}$. If $d = 1$, then $3^{t+1}(3^t + 1) = q(q+1)$. Now, since $(3, q) = 1$, $3 \nmid q$ and hence, $q \mid (3^t + 1)$ and $(q+1)_3 = 3^{t+1}$, which is impossible. If $d = 3$, then $q' + \sqrt{3q'} + 1 = \frac{q^2+q+1}{3}$ and hence, $3^{t+2}(3^t + 1) = (q-1)(q+2)$. Thus either $3^{t+1} \mid (q-1)$ and $(q+2) \mid 3(3^t + 1)$ or $3^{t+1} \mid (q+2)$ and $(q-1) \mid 3(3^t + 1)$. This forces $(q-1) = 3^{t+1}$ and $(q+2) = 3(3^t + 1)$. This guarantees that $|G|_3 \leq 3^{3t+2}$. On the other hand, $3^{3(2t+1)} = |K/H|_3 \leq |G|_3 \leq 3^{3t+2}$, which is a contradiction. Now assume that $(3, q) \neq 1$. So $d = 1$ and hence, $q' \pm \sqrt{3q'} + 1 = q^2 + q + 1$. This forces $q = 3^{t+1}$ and $q + 1 = 3^t \pm 1$, which is impossible.

2.8. If $K/H \cong F_4(q')$, where q' is even, then $q'^4 + 1 = \frac{q^2+q+1}{d}$ or $q'^4 - q'^2 + 1 = \frac{q^2+q+1}{d}$. Thus $r^6 = (q'^4 + 1)^6 < (q'^5)^6 = q'^{30}$ and $q'^{24}(q'^{12} - 1)(q'^8 - 1)(q'^6 - 1)(q'^2 - 1) \mid \frac{1}{3}r(r-1)q^3(q^2-1)(q-1)$. Thus $q'^{36} \leq \frac{1}{3}r(r-1)q^3(q^2-1)(q-1) < r^6 < q'^{30}$, which is a contradiction.

2.9. If $K/H \cong E_7(2)$, then $r \in \{73, 127\}$. Therefore, either $r = 73$ and $q = 8$ or $r = 127$ and $q = 19$. So either $|M| = |PSL(3,8)| = 2^9 \cdot 3^2 \cdot 7^2 \cdot 73$ or $|M| = |PSL(3,19)| = 2^4 \cdot 3^4 \cdot 5 \cdot 19^3 \cdot 127$. On the other hand, $13 \mid |E_7(2)|$, so $|K/H| \nmid |G|$, which is a contradiction.

2.10. If $K/H \cong E_7(3)$, then $r \in \{757, 1093\}$. One can check at once that $\frac{(q^2+q+1)}{d} \neq 1093$. If $\frac{q^2+q+1}{d} = 757$, then $d = 1$ and $q = 27$. On the other hand, $|PSL(3,27)| = 2^4 \cdot 3^9 \cdot 7 \cdot 13^2 \cdot 757$ and $5 \mid |E_7(3)|$, which is a contradiction.

2.11. If $K/H \cong {}^2F_4(q')$, where $q' = 2^{2t+1} \geq 2$, then $r = q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1$. In both cases, one can check at once that $|K/H| > |G|$, which is a contradiction.

Case 3. Let $t(K/H) \in \{4, 5\}$. Then

$$r \in \{OC_2(K/H),\ OC_3(K/H),\ OC_4(K/H),\ OC_5(K/H)\},$$

as follows:

3.1. If $K/H \cong {}^2E_6(2)$, then $\frac{q^2+q+1}{d} \in \{13, 17, 19\}$. Obviously, $\frac{q^2+q+1}{d} \neq 17$. If $\frac{q^2+q+1}{d} = 19$, then $d = 3$ and $q = 7$. Thus $|M| = |PSL(3,7)| = 2^5 \cdot 3^2 \cdot 7^3 \cdot 19$. On the other hand $11 \mid |{}^2E_6(2)|$, which is a contradiction. The same reasoning rules out the case when $r = 13$ and $q = 3$.
3.2. If $K/H \cong PSL(3,4)$, then $\frac{q^2+q+1}{d} \in \{5, 7, 9\}$. It is easy to check that $\frac{q^2+q+1}{d} \notin \{5, 9\}$. If $\frac{q^2+q+1}{d} = 7$, then either $q = 2$ or $q = 4$. In the former case, $|K/H| \nmid |G|$, which is a contradiction. So $q = 4$ and hence, $K/H \cong PSL(3,4) = M$, as desired.
3.3. If $K/H \cong {}^2B_2(q')$, where $q' = 2^{2t+1}$ and $t \geq 1$, then $r \in \{q' - 1, q' \pm \sqrt{2q'} + 1\}$. Let $q' - 1 = r$ and $d = 1$. Thus $2(2^{2t} - 1) = q(q + 1)$. If $|q|_2 = 2$, then $q + 1 = 3$ and hence, $t = 1$ and $M = PSL(3,2)$. Therefore, $5 \nmid |G|$ and $5 \mid |K/H|$, which is a contradiction. This forces $q \mid 2^t - 1$ or $q \mid 2^t + 1$ and hence, $q(q + 1) \leq 2(2^t + 1)(2^{t-1} + 1)$. Therefore, $t = 2$ and $q = 5$. Thus $|K/H| \nmid |G|$, which is a contradiction. If $q' - 1 = r$ and $d = 3$, then we can see that $2^2(3.2^{2t-1} - 1) = q(q + 1)$. If $|q|_2 = 2^2$, then $q + 1 = 5$ and $t = 1$, which is impossible as described above. Thus $|q + 1|_2 = 2^2$ and hence, $|q - 1|_2 = 2$. Also, $|r - 1|_2 = 2$. So $2^{2(2t+1)} \leq |K/H|_2 \leq |G|_2 \leq 2^5$, which is a contradiction.

Now assume that $q' + \sqrt{2q'} + 1 = r$. If $d = 3$, then $\frac{q^2+q-2}{3} = 2^{t+1}(2^t + 1)$ and hence, $(q - 1)(q + 2) = 3.2^{t+1}(2^t + 1)$. Since $3 \mid q - 1$, $q - 1 = 3k$ for some positive integer k. Thus $3k(k + 1) = 2^{t+1}(2^t + 1)$ and hence, $k(k + 1) = 2^{t+1}(\frac{2^t+1}{3})$. Now, if $2^{t+1} \mid k$, then $k+1 \leq \frac{2^t+1}{3}$ and if $2^{t+1} \mid k+1$, then $k \leq \frac{2^t+1}{3}$, which are impossible.

If $d = 1$, then $q^2 + q + 1 = q' + \sqrt{2q'} + 1$ and hence, $q(q+1) = 2^{t+1}(2^t+1)$, which is impossible. The same reasoning rules out the case when $q' - \sqrt{2q'} + 1 = r$.
3.4. If $K/H \cong E_8(q')$, then $r \in \{\frac{q'^{10}+q'^5+1}{q'^2-q'+1} = q'^8 - q'^7 + q'^5 - q'^4 + q'^3 - q' + 1, \frac{q'^{10}-q'^5+1}{q'^2-q'+1} = q'^8 + q'^7 - q'^5 - q'^4 - q'^3 + q' + 1, \frac{q'^{10}+1}{q'^2+1} = q'^8 - q'^6 + q'^4 - q'^2 + 1, q'^8 - q'^4 + 1\}$. Thus $r < q'^9$. On the other hand, $r^5 < q'^{45}$ and $|G| \leq \frac{1}{3}(r - 1)q^3(q^2 - 1)(q - 1) < r^5$. Since $q'^{120} \mid |K/H|$ and $|K/H| \mid |G|$, we get a contradiction.

The above cases show that $K/H \cong PSL(3, q)$. $\square$

Step 4. $G \cong M$.

Proof. By the previous argument and step 3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong M = PSL(3,q)$. We claim that $H = 1$. Suppose on the contrary, $H \neq 1$, so $|H| \geq 2$. Let $t \in \pi(H)$. By Frattini's argument $N_G(S_t(H))H = G$, so $N_G(S_t(H))/N_H(S_t(H)) \cong G/H$. Since $r \mid |G/H|$, $r \mid |N_G(S_t(H))|$ and hence, $N_G(S_t(H))$ contains an element x of order r. Also, by Lemma 3.3, $rt \notin \pi_e(G)$, thus $< x >$ acts fixed point freely on $S_t(H) - \{1\}$, so $r \mid |S_t(H)| - 1$. On the other hand, $|S_t(H)| \mid |H|$ and since $|G| = |G/K||K/H||H|$ and $|K/H| = |PSL(3,q)|$, $|H| \leq r - 1$. This forces $r \leq r - 1$, which is impossible. Therefore $H = 1$ and hence, $K \cong M$.

We know that $G \leq AutM$. If $q = p^m$, then since by Lemma 3.3, $GK(G)$ is disconnected, by [14], $G/K \cong < \varphi > \times < \theta >$, where φ is the field automorphism of order 3^u and θ is the graph automorphism of order 2. Since $|G|_3 = |M|_3$, the order of φ is 1 and hence, $G = K$ or $G = K\cdot < \theta >$. If $G = K\cdot < \theta >$, then $m_2\big(K\cdot < \theta >\big) > m_2(K)$, but $nse(G)$ contains exactly one odd number $m_2(K)$, which is a contradiction. Therefore $G = K \cong M$. □

REFERENCES

[1] N. Ahanjideh, *Thompson's conjecture for some finite simple groups*. J. Algebra **344** (2011), 205–228.

[2] N. Ahanjideh and B. Asadian, *NSE characterization of some alternating groups*. J. Algebra Appl **14(2)** (2015).

[3] S. Asgary and N. Ahanjideh, *NSE characterization of some finite simple groups*. To appear in Sci. Ann. Comput. Sci., 2015.

[4] G.Y. Chen, *On Frobenius and 2-Frobenius group*. J. South China Normal Univ. Natur. Sci. Ed. **20(5)** (1995), 485–487.

[5] W. Feit, *On large Zsigmondy primes*. Proc. Amer. Math. Soc. **102(1)** (1988), 29–36.

[6] G. Frobenius, *Verallgemeinerung des Sylowschen Satze*. Berliner Sitz (1985), 981–993.

[7] B. Huppert, *Endliche Gruppen*. Springer-Verlag, 1967.

[8] A. Khalili Asboei, *A new characterization of* $PGL(2,p)$. J. Algebra Appl. **12(7)** (2013).

[9] A. Khalili Asboei and A. Iranmanesh, *A characterization of the linear groups* $L_2(p)$. Czechoslovak Math. J. **64(139)** (2014), 459–464.

[10] A.S. Kondratev, *Prime graph components of finite simple groups*. Math. USSR-Sb. **67(1)** (1990), 235–247.

[11] A.S. Kondratev and V.D. Mazurov, *Recognition of Alternating groups of prime degree from their element orders*. Sib. Math. J. **41(2)** (2000), 294–302.

[12] M. Khatami, B. Khosravi and Z. Akhlaghi, *A new characterization for some linear groups*. Monatsh. Math. **163** (2011), 39–50.

[13] P. Kleidman and M. Liebeck, *The subgroup structure of the finite classical groups*. London Math. Soc. Lecture Note Ser., 1990.

[14] M.S. Lucido, *Prime graph components of finite almost simple groups*, Rend. Semin. Mat. Univ. Padova **102** (1999), 1–22.

[15] C. Shao and Q. Jiang, *A new characterization of $PSL(2,p)$ by nse.* J. Algebra Appl. **13(4)** (2014).

[16] R. Shen, C.G. Shao, Q.H. Jiang, W.J. Shi and V. Mazurov, *A new characterization of A_5.* Monatsh. Math. **160** (2010), 337–341.

[17] J.S. Williams, *Prime graph components of finite groups.* J. Algebra **69** (1981), 487–513.

Received 19 January 2016

Shahrekord University
Department of Mathematics
Shahrekord, Iran
soleyman.asgary@stu.sku.ac.ir

Shahrekord University
Department of Mathematics
Shahrekord, Iran
ahanjideh.neda@sci.sku.ac.ir