# ON WORD STRUCTURE OF THE HECKE GROUP $H(\lambda_6)$ OVER REAL QUADRATIC FIELDS

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Let $n$ be a non-square positive integer and $Q^*(\sqrt{n})$ the set $\{(a+\sqrt{n})/c : a, (a^2-n)/c, c$ are relatively prime integers$\}$. Coset diagrams for orbits of the Hecke group $H(\lambda_6)$ acting on projective line over the set $Q^*(\sqrt{n})$ are known. If $\alpha$ is any real quadratic irrational number and $\alpha^{H(\lambda_6)}$ is an orbit of the group $H(\lambda_6)$ under $\alpha$ then $\alpha^{H(\lambda_6)} \subseteq Q^*(\sqrt{n})$. In this paper, we employ the coset diagrams to prove some of the results for action of $H(\lambda_6)$ on real quadratic fields, which are known to hold in case of well known modular group $H(\lambda_3)$. In fact, we investigate the question: when does an orbit of $H(\lambda_6)$ containing a circuit (closed path) of a given type exist? We also determine a condition for existence of both a real quadratic irrational number $\gamma$ $(= (a+\sqrt{n})/3c')$ and its algebraic conjugate $\bar{\gamma}$ $(= (a-\sqrt{n})/3c')$ in the orbit.

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*Key words:* coset diagrams, linear-fractional transformations, circuits, ambiguous numbers.

## 1. INTRODUCTION

The Hecke group $H(\lambda_q)$ is a finitely generated discrete subgroup of $PSL(2,R)$ generated by the transformations: $z \longmapsto -1/z$ and $z \longmapsto -1/(z+\lambda_q)$ of order 2 and $q$ respectively, where $\lambda_q = 2\cos(\pi/q)$, $q$ is an integer $> 2$. When $q=3$, the Hecke group $H(\lambda_3)$ is isomorphic to the modular group. When $q=6$, we have the group $H(\lambda_6) = H(\sqrt{3})$. If some results hold for one group belonging to a class of groups then it is an immediate question to ask whether the results hold for some other groups of the same class or for the whole class. In this paper, we investigate the group $H(\sqrt{3})$ acting on real quadratic fields for the results which are proved in [5] for the modular group. Motivation to this work comes from the known results which are proved in literature through use of coset diagrams for both modular group and the group $H(\sqrt{3})$.

Coset diagrams for the modular group are introduced in [2]. It is known that in case of action of the modular group on real quadratic fields a finite

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number of ambiguous numbers of the form $(a+\sqrt{n})/c$ exist for a fixed value of a non-square positive integer $n$; part of a coset diagram containing ambiguous numbers forms a single circuit and it is the only circuit in the orbit [4]. In [5], these circuits are further classified by finding a condition for existence of an orbit of the modular group containing a circuit of a given type. Moreover, necessary and sufficient conditions are also found for existence of two orbits of the modular group; one containing $(a+\sqrt{n})/c$ along with its conjugate $(a-\sqrt{n})/c$; the other containing $(a+\sqrt{n})/c$ along with $1/(a-\sqrt{n})/c$.

Let $H(\sqrt{3})$ be denoted by $H$ for remaining part of the paper. It is known that $H$ is generated by two linear-fractional transformations $x : z \to -1/3z$ and $y : z \to -1/3(z+1)$, satisfying the relations: $x^2 = y^6 = 1$ [6]. Suppose $\gamma$ denote a real quadratic irrational number $(a+\sqrt{n})/c$, where $n$ is a non-square positive integer and the integers $a$, $(a^2-n)/c$, $c$ are relatively prime. Let the algebraic conjugate $(a-\sqrt{n})/c$ of $\gamma$ be denoted $\bar{\gamma}$. If both $\gamma$ and $\bar{\gamma}$ have different signs then $\gamma$ is called an ambiguous number [6]. In [6], it is proved that a non-square positive integer $n$ does not change its value in orbit $\gamma^H$ of the group $H$ acting on the real quadratic fields; ambiguous numbers obtained are finite in number; part of a coset diagram containing such numbers forms a single closed path; and it is the only closed path in the orbit. In [1], it is shown that if $\gamma$ is of the form $(a+\sqrt{n})/3c'$, $c' \in Z$ then a closed path can be found in a coset diagram for $\gamma^H$. On the path all numbers are of the form $(a+\sqrt{n})/3c'$ and belong to $Q^*(\sqrt{n})$. We continue further by investigating word structure of the elements of $H$ generating the numbers $\gamma$. We discuss characteristics of the circuits containing these numbers. We find a condition for existence of an orbit of $H$ containing a circuit of a given type. In case of existence of the circuit, we determine a condition for existence of a real quadratic irrational number $\gamma$ along with its algebraic conjugate $\bar{\gamma}$ in the orbit.

The paper consists of two more sections. First section contains both the coset diagrams employed and terminology used. One example is also included in this section to explain the circuits and associated concepts. Second section discusses action of the group $H$ on $Q^*$ $(\sqrt{n})$ and the results.

## 2. DIAGRAMS AND TERMINOLOGY

Any 6-cycle of $y$ in a coset diagram is denoted by six vertices of a hexagon permuted anti-clockwise by $y$. Any two vertices interchanged by $x$ are joined by an edge. An alternating sequence $v_0$, $e_1$, $v_1$, $e_2$, . . ., $e_t$, $v_t$ of vertices and edges of a coset diagram is called a path in a coset diagram if $e_i$ joins $v_{i-1}$ and $v_i$ for each $i = 1, 2, 3, . . . ., t$, where $e_i \neq e_j$ (for $i \neq j$). By a circuit, we mean a closed path of edges and hexagons. Let $n_1$, $n_2$, . . ., $n_{2t}$ be a sequence of positive integers. By a circuit of type $(n_1, n_2, \ldots, n_{2t})$ we mean the circuit such that $n_1$ number of hexagons have four vertices outside the circuit; $n_2$ number of hexagons have four vertices inside the circuit; and this pattern is maintained till last positive integer $n_{2t}$ in the sequence. This circuit induces an element $g = (yx)^{n_1}(y^{-1}x)^{n_2}...(y^{-1}x)^{n_{2t}}$ of $H$, which fixes a particular vertex of a hexagon lying on the circuit. As an example a circuit is shown in Fig. 1. This circuit induces an element $g = (yx)^3(y^{-1}x)^4(yx)^2(y^{-1}x)^2(yx)^3(y^{-1}x)$ of $H$, which fixes the vertex $v_0$ as shown in the diagram. This circuit is of the type (3, 4, 2, 2, 3, 1).

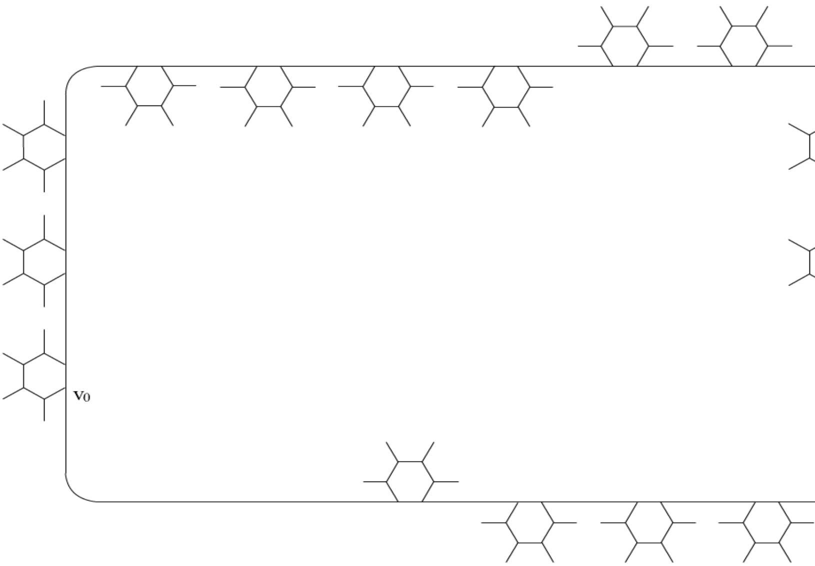


Fig. 1

## 3. ACTION OF $H$ ON $Q^*(\sqrt{n})$ AND RESULTS

We know that a finite number of ambiguous numbers of the form $(a + \sqrt{n})/c$ ($n$ is a non-square positive integer and $a$, $(a^2 - n)/c$, $c$ are relatively prime integers) exist for a fixed value of $n$ [6]. These numbers form a single circuit in a coset diagram for an orbit of $(a + \sqrt{n})/c$ and it is the only circuit containing the orbit. In [1], it has been proved further that in an orbit a circuit can be found in which all the ambiguous numbers are of the form $(a+\sqrt{n})/3c'$ (where $n$ is a non square positive integer and $a$, $(a^2-n)/3c'$ , $3c'$ are relatively prime integers). If $t$ denote the number of sets of hexagons on the circuit with four vertices outside the circuit and $\acute{t}$ the number of sets of hexagons on the circuit with four vertices inside, then $t = \acute{t}$. The total number of sets of hexagons in the circuit, then becomes $2t$. These sets of hexagons with four vertices outside/inside occur alternately in these circuits.

Next we prove our results which are analogous to theorems 2.1, 2.2, 2.3, 2.4, 2.5 proved in [5] for circuits in case of action of the modular group on real quadratic fields.

THEOREM 3.1. *Every finite order element of $H$, except the (group theoretic) conjugates of $x$, $y^{\pm 1}$, $y^2$, and $(yx)^n$, $n > 0$, has real quadratic irrational numbers as fixed points.*

*Proof.* Let $g : z \to (az + b)/(cz + d)$ belong to $H$ and $v_0$ a fixed point of $g$. Then,

$$cv_0^2 + (d - a)v_0 - b = 0. \tag{1}$$

It has real roots only when $(d - a)^2 + 4bc \geq 0$. Since $g$ belongs to $H$, therefore, $ad - bc = 1$ *or* 3. Let us discuss these two cases separately.

If the determinant $ad - bc = 3$, then $d^2 + a^2 - 2ad + 4(ad - 3) \geq 0$. It implies that $(a + d)^2 - 12 \geq 0$, where $a + d$ is trace of the matrix corresponding to $g$. Thus, for complex roots of (1), we have $(a + d)^2 < 12$ and the possible values of $a + d$ are 0, $\pm 1$, $\pm 2$, $\pm 3$.

If $a + d = 0$, then $g$ takes the form $g : z \to (az + b)/(cz - a)$ and $g^2 = 1$. Every element of order 2 is conjugate to $x$ [7]. So $g$ is conjugate to $x$. Thus, the fixed points of the conjugates of $x$ are some complex numbers. If $a + d = \pm 1$; since we can replace $a$, $b$, $c$, and $d$ by $-a$, $-b$, $-c$, and $-d$, in $g$; therefore, we can consider $a+ d = -1$. Now the possible values of order of g are 1, 2, 3, 6; therefore, $A^n = \lambda I$ only when $n = 1, 2, 3, 6$, where $A$ is a matrix corresponding to $g$. We know for a $2 \times 2$ matrix $A$:

$$A^2 = \lambda I \iff tr(A) = 0 \tag{2}$$
$$A^3 = \lambda I \iff tr^2(A) = \det(A) \tag{3}$$
$$A^6 = \lambda I \iff tr^2(A) = 3\det(A). \tag{4}$$

If $a+d = -1$, then $\det(A) = 1$. Thus, no matrix with trace $-1$ and determinant 3 having a finite order exists. The matrix corresponding to $g$ has infinite order.

If $a + d = \pm 2$, then again, none of the equations (2), (3), (4), holds for $A$ with determinant 3. It implies that such matrices correspond to an element of $H$ having the infinite order. Thus, $g$, having the infinite order, has complex numbers as fixed points.

If $a + d = \pm 3$, then (4) holds for $A$ with determinant 3. Thus, $g$ has order 6. It is a conjugate of $y$ because every element of $H$ of order 6 is a conjugate of $y$ [7]. It implies that fixed points of the conjugates of $y^{\pm 1}$are complex numbers. If $a + d = m \geq \sqrt{12}$, then $(a + d)^2 - 12 \geq 0$; the roots are real.

If $(a+d)^2-12$ is a perfect square, then we shall be dealing with a coset diagram for rational numbers. In this case, we know that $\infty$ is the only fixed point [6]. Thus, $(a+d)^2-12$ cannot be a perfect square, and the fixed points are real but irrational numbers.

If determinant $ad-bc=1$, then (1) has real roots for $(a+d)^2-4 \geq 0$. It implies that $(a+d)^2 < 4$ for complex roots. In this case possible values of $a+d$ are 0, $\pm 1$.

If $a+d=0$, then $g$ takes the form $z \to (az+b)/(cz-a)$ and $g^2=1$. By the argument given earlier for the other case, again $g$ is a conjugate of $x$. Thus, the fixed points of the conjugates of $x$ are complex numbers.

When $a+d=\pm 1$, as discussed ealier in the proof it is sufficient to consider only $a+d=-1$. Consider $a+d=-1$, this implies that only possibility is (2); order of $g$ is 3; and g is a conjugate of $y^2$ [7]. This shows that fixed points of conjugates of $y^2$ are complex numbers.

If $a+d=\pm 2$, then the characteristic equation for $A$ is $A^2-2A+I=0$. By repeatedly multiplying this equation by $A$ and substituting $2A-I$ for $A^2$, we get $A^n-nA+(n-1)I=0$ for a positive integer $n$. Thus, $g$ in this case is a conjugate of $(yx)^n : z \longrightarrow z+n$ and $\infty$ is the only fixed point of it.

If $a+d=\pm 3$, then none of (2), (3), (4), holds for $A$ with determinant 1. Thus, we get real quadratic irrational numbers as fixed points except for the conjugates of $x$, $y^{\pm 1}$, $y^2$ and $(yx)^n$, $n>0$. $\square$

Let $\gamma$ be a real quadratic irrational number fixed by $g=(yx)^{n_1}(y^{-1}x)^{n_2} \ldots (y^{-1}x)^{n_{2t}}$, $n_i>0$ for all $i=1, 2, \ldots, 2t$, of $H$. If $B(g)$ denotes the matrix corresponding to $g$, then the size of its trace determines the size of the circuit $(n_1, n_2, \ldots, n_{2t})$ containing $\gamma$. This indeed means that there is a relationship between $n_1, n_2, \ldots, n_{2t}$ (sequence of positive integers) and the trace. In Theorem 3.2, we establish this relationship.

THEOREM 3.2. *Let $E=\{1, 2, \ldots, 2t\}$ be a cyclically ordered set of positive integers and the orbit of $\gamma$ contains a circuit of the type $(n_1, n_2, \ldots, n_{2t}), n_i>0$. Let $S$ be the collection of non-empty subsets of $E$ obtained by striking out any number of adjacent pairs of elements of $E$. Let $n_J=\prod_{i\in J} n_i$ for $J \in S$. Then, the trace of $B(g)$ is $2+\sum_{J\in S}\lambda_{i_J}n_J, \lambda_{i_J}=3^{k_J}$, $k_J$ is some positive integer.*

*Proof.* Consider an element $g=(yx)^{n_1}(y^{-1}x)^{n_2} \ldots (y^{-1}x)^{n_{2t}}$, $n_i>0$, of $H$, corresponding to the circuit of the type $(n_1, n_2, \ldots, n_{2t})$, such that a real quadratic irrational number $\gamma$ is fixed by $g$. Since $yx : z \longrightarrow z+1$ and $y^{-1}x : z \longrightarrow z/(3z+1)$ represent the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix},$$

therefore the matrix corresponding to $g$ has the form:

$$\begin{aligned} B(g) &= \begin{bmatrix} 1 & n_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3n_2 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & n_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3n_{i+1} & 1 \end{bmatrix} \\ & \cdots \begin{bmatrix} 1 & n_{2t-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3n_{2t} & 1 \end{bmatrix}. \end{aligned} \tag{5}$$

If we consider a matrix $A$ written in the form

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} \end{bmatrix} \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} \\ a_{21}^{(2)} & a_{22}^{(2)} \end{bmatrix} \cdots \begin{bmatrix} a_{11}^{(m)} & a_{12}^{(m)} \\ a_{21}^{(m)} & a_{22}^{(m)} \end{bmatrix},$$

the trace of $A$, of course, is of the type

$$\sum a_{\lambda m\lambda 1}^{(1)} a_{\lambda 1\lambda 2}^{(2)} \cdots a_{\lambda m-1\lambda m}^{(m)}. \tag{6}$$

In case of $B(g)$, the factor matrices are alternately

$$\begin{bmatrix} 1 & n_i \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 3n_j & 1 \end{bmatrix},$$

therefore in any term of (6) if some $a_{\lambda i\lambda j}^{(l)} = 0$, the entire term is zero. We can ignore any $a_{\lambda i\lambda j}^{(l)}$ which is 1. We consider only those terms which are neither 0 nor 1. Let us consider the following portion of three matrices from (5).

$$\begin{bmatrix} 1 & 0 \\ 3n_p & 1 \end{bmatrix} \begin{bmatrix} 1 & n_q \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3n_r & 1 \end{bmatrix}$$

Trace of $B(g)$ is sum of the certain products obtained by choosing one entry from each matrix in (5); therefore, we suppose that from the middle matrix of the three matrices, we choose $n_q$. In order to have a non-zero product, we choose $3n_r$ from the third matrix and 1 from the top left-hand corner of the first matrix. On the other hand, if from the first matrix, we choose $3n_p$ instead of 1, then from the third matrix, we need to choose 1 in the second row and the second column. In a similar way, consider the following portion of three matrices from (5):

$$\begin{bmatrix} 1 & n_p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3n_q & 1 \end{bmatrix} \begin{bmatrix} 1 & n_r \\ 0 & 1 \end{bmatrix},$$

and choose $3n_q$ from the middle matrix. We must then choose $n_p$ from the first matrix; 1 in the first row and first column of the third matrix or if we choose 1 in the second row and second column of the first matrix, then we need to choose $n_r$ from the third matrix. In fact, this means that we are striking out any number of adjacent pairs of elements of $E$. Thus, for $J \in S$, if we let

$n_J = \prod_{i\in J} n_i$, then the trace of $B(g)$ is $2 + \sum\limits_{J\in S} \lambda_{i_J} n_J, \lambda_{i_J} = 3^{k_J}$, $k_J$ is some positive integer. $\square$

For a given sequence of positive integers $n_1, n_2, \ldots, n_{2t}$, a circuit of the type $(n_1, n_2, \ldots, n_{2t^*}, n_1, n_2, \ldots, n_{2t^*}, \ldots, n_1, n_2, \ldots, n_{2t^*})$, where $t^*$ divides $t$, is said to have a period of length $2t^*$. In Theorem 3.3, we prove that there does not exist a circuit of this type in an orbit of $H$ acting on $Q^*(\sqrt{n})$.

Theorem 3.3. *For a given sequence of positive integers $n_1, n_2, \ldots, n_{2t}$, there does not exist a circuit, which has a period of length $2t^*$, where $t^*$ divides $t$, in an orbit of $H$.*

*Proof.* If there exists a circuit which has a period of length $2t^*$, then it will be of the type $(n_1, n_2, \ldots, n_{2t^*}, n_1, n_2, \ldots, n_{2t^*}, \ldots, n_1, n_2, \ldots, n_{2t^*})$ and will be as shown in Fig. 2.

It does not matter if we reverse the orientation of the hexagons on the circuit. If $v_0, v_1, \ldots, v_{t/t^*}$ are the vertices of the hexagons on the circuit as shown in Fig. 2 and $g = (yx)^{n_1}(y^{-1}x)^{n_2}\ldots(y^{-1}x)^{n_{2t^*}} \neq 1$, then $v_{i+1} = v_i g$, where $i = 0, 1, 2, \ldots, (t/t^* - 1)$ (indices modulo $t/t^*$). It implies that $v_0 \neq v_0 g$. Since for all $i$, $v_i = v_i(g)^{t/t^*}$ and $(g)^{t/t^*} \neq 1$; therefore, it is a contradiction to the fact that if $g \neq 1$ is an element of $H$, then $g$ has 1 or 2 fixed points, and these are the only fixed points of it unless $g^n = 1$ for some suitable $n$. Thus, no orbit $\gamma^H$ contains a circuit of the type $(n_1, n_2, \ldots, n_{2t^*}, n_1, n_2, \ldots, n_{2t^*}, \ldots, n_1, n_2, \ldots, n_{2t^*})$. $\square$

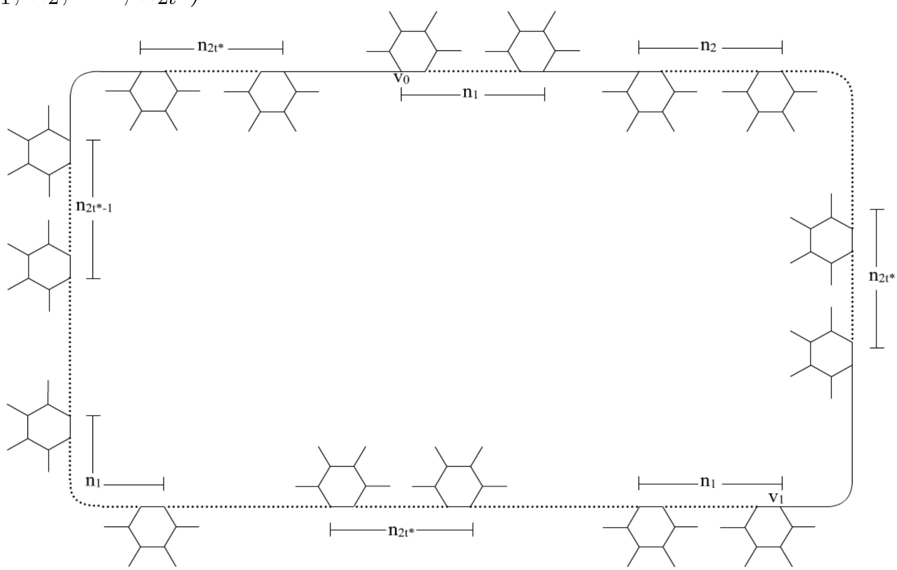


Fig. 2

THEOREM 3.4. *For a given sequence* $n_1, n_2, \ldots, n_{2t}$ *of positive integers there exists a real quadratic irrational number* $\gamma$ *such that a circuit in the orbit of* $\gamma$ *under* $H$ *has the type* $(n_1, n_2, \ldots, n_{2t})$ *if the circuit does not have a period of even length.*

*Proof.* In Theorem 3.3, it has been established that a given sequence has no circuit with repetitions. In order to prove this condition to be sufficient, we are to just show that a fixed point $k$ of $g = (yx)^{n_1}(y^{-1}x)^{n_2} \ldots (y^{-1}x)^{n_{2t}}$ is a real quadratic irrational number. Since $g$ fixes $k$, therefore, by Theorem 3.1 $g$ is not a conjugate of $x$, $y^{\pm 1}$, $y^2$, and $(yx)^n$, where $n > 0$. By using Theorem 3.2, the trace of matrix $B(g)$ is $r = 2 + \sum_{J\in S} \lambda_{i_J} n_J$, $\lambda_{i_J} = 3^{k_J}$ ($k_J$ is some positive integer, where $n_J = \prod_{i\in J} n_i$). It implies that $\sqrt{r^2 - 4}$ is not a complex number. Moreover, $r^2 - 4$ is not a perfect square. If it were a perfect square, then $k$ must be $\infty$ because of being fixed point of $g$ [6]. We know that $k$ is not $\infty$ and $det(B(g)) = 1$; therefore, $k$ is a real quadratic irrational number, and it belongs to an orbit $\gamma^H$. But in a coset diagram for the orbit, the ambiguous numbers form a set of circuits [6], and Theorem 3.3 implies that the orbit contains a circuit of the type $(n_1, n_2, \ldots, n_{2t})$. $\square$

In Theorem 3.5, we give the necessary and sufficient condition for a circuit to contain a real quadratic irrational number $\gamma$ along with its conjugate $\bar{\gamma}$.

THEOREM 3.5. *A circuit contains* $\gamma$ *with its conjugate* $\bar{\gamma}$ *if and only if the circuit is of the type* $(n_1, n_2, \ldots, n_{t-1}, n_t, n_t, \ldots, n_2, n_1)$.

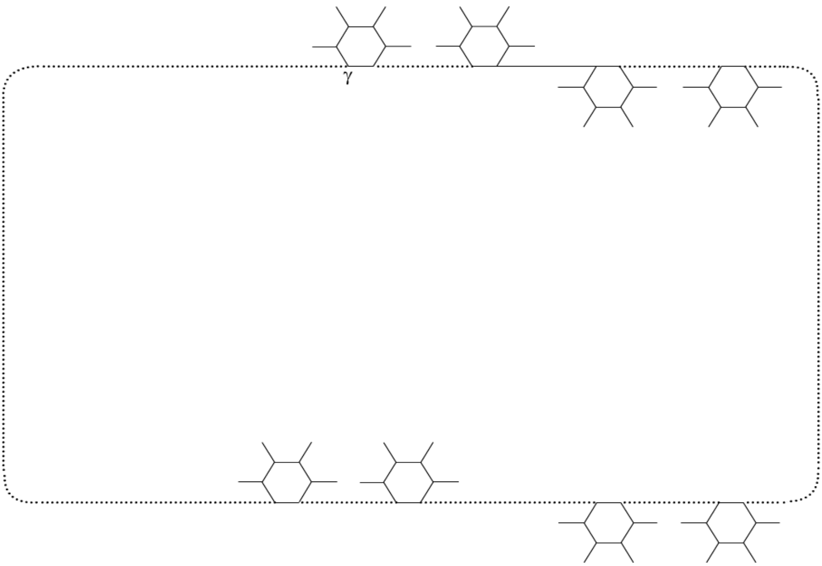


Fig. 3

*Proof.* We note that if $\gamma$ and $\bar{\gamma}$ are conjugates to each other, then $\gamma g$ and $\bar{\gamma} g$ are also conjugates to each other for every $g$ in $H$. This implies that it is enough to prove the result for any one element in the circuit to complete the proof. Suppose $\gamma$, $\bar{\gamma}$ belong to a circuit and $\gamma$ is fixed by $g = (yx)^{n_1}(y^{-1}x)^{n_2} \ldots (y^{-1}x)^{n_{2t}}$, where $n_i > 0$, $i = 1, 2, \ldots, 2t$. We index the vertices which belong to the circuit shown in Fig. 3 by the finite set $\{1, 2, \ldots, n\}$. If $\gamma$ occupies the vertices with odd labels, then no $\bar{\gamma}$ can occupy any of these vertices. If any $\bar{\gamma}$ occupies such a vertex, then $\bar{\gamma} = \gamma(yx)^{n_1}(y^{-1}x)^{n_2} \ldots (y^{-1}x)^{n_r}$ for some $r < t$. This implies that $\gamma$ and $\bar{\gamma}$ are fixed points of $g = (yx)^{n_1}(y^{-1}x)^{n_2} \ldots (y^{-1}x)^{n_{2t}}$ and $h = (yx)^{n_{r+1}}(y^{-1}x)^{n_{r+2}} \ldots (y^{-1}x)^{n_{2t}}(yx)^{n_1}(y^{-1}x)^{n_2} \ldots (y^{-1}x)^{n_r}$ respectively. However, $\gamma$ and $\bar{\gamma}$ being conjugate to each other are fixed by the same element of $H$. So $g$ must be equal to $h$. If it is the case, then $g = f^{\,m}$ for some $m > 1$; $\gamma$ will be a fixed point of $f$. By Theorem 3, this cannot happen except for $g^t$, $t \geq 1$ . It is a contradiction. Thus, no $\bar{\gamma}$ can occupy an odd labelled vertex. This means that all $\gamma$ occupy the vertices labelled with even numbers; so $\bar{\gamma}$ is fixed under the transformation $h = (yx)^{n_t}(y^{-1}x)^{n_{t-1}} \ldots (yx)^{n_1}(y^{-1}x)^{n_{2t}} \ldots (y^{-1}x)^{n_{t+1}}$. This shows that $\gamma$ corresponds to $(n_1, n_2, \ldots, n_{2t})$ and $\bar{\gamma}$ corresponds to $(n_t, n_{t-1}, \ldots, n_1, n_{2t}, n_{2t-1}, \ldots, n_{t+1})$ but with orientation of the hexagons in the reversed order. It means the type of the circuit corresponding to $\gamma$ is the same as the type of circuit corresponding to $\bar{\gamma}$, but with signs reversed and starting at a different point. Since the types must be the same, therefore, the circuit containing both $\gamma$ and $\bar{\gamma}$ must be of the type $(n_1, n_2, \ldots, n_{k-1}, n_k, n_k, \ldots, n_2, n_1)$. $\square$

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