ON TANGENT GROUPS OF 2-STEP NILPOTENT PRE-LIE GROUPS

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We investigate several types of topological groups: 2-step nilpotent topological groups, groups with Lie algebra and pre-Lie groups. We investigate the mutual relations between these groups. We study tangent groups of topological groups with Lie algebra and characterize the 2-step nilpotent topological groups for which tangent groups are pre-Lie groups.

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1. INTRODUCTION

We present some elements of Lie theory for 2-step nilpotent topological groups, that allows us to construct the topological Lie algebra of such a group, and in particular we provide a detailed proof of Theorem 2.22 which says that every 2-step nilpotent topological group is a group with Lie algebra. That fact was discovered in [9]. We present some results on general topological groups useful in characterization of the Lie algebra of a 2-step nilpotent topological group. For clarity we recall below the definition of groups with Lie algebra and pre-Lie groups. We introduce the tangent group of a topological group with Lie algebra, denoted by $T(G)$. We show that the tangent group of a 2-step nilpotent topological group is in turn a 2-step nilpotent group, and finally as the main new result of the present paper, we characterize 2-step nilpotent topological groups for which the tangent group is a pre-Lie group (Theorem 3.4).

The main tool used in the present investigation is the differential calculus on topological groups which are not necessarily Lie groups. References for that research area and its related topics include [1–4, 6–10], and [5].

We conclude this introduction by the following definitions, which are basic for the present paper.

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We define $\Lambda(G)$ as the set of all continuous homomorphisms $\alpha : \mathbb{R} \to G$ from the additive group $(\mathbb{R}, +)$ to G. The elements of $\Lambda(G)$ are called continuous subgroups of G with one parameter. The set $\Lambda(G)$ is endowed with a natural topology as follows (see for instance [6]).

Let G be a topological group with the set of neighborhoods of $\mathbf{1} \in G$ denoted by $\mathcal{V}_G(\mathbf{1})$.

For arbitrary $n \in \mathbb{N}$ and $U \in \mathcal{V}_G(\mathbf{1})$ denote

$$W_{n,U} = \{(\gamma_1, \gamma_2) \in \Lambda(G) \times \Lambda(G) \mid (\forall t \in [-n, n]) \quad \gamma_2(t)\gamma_1(t)^{-1} \in U\}.$$

For every $\gamma_1 \in \Lambda(G)$ define $W_{n,U}(\gamma_1) = \{\gamma_2 \in \Lambda(G) \mid (\gamma_1, \gamma_2) \in W_{n,U}\}$. Then there exists a unique topology on $\Lambda(G)$ with the property that for each $\gamma \in \Lambda(G)$ the family $\{W_{n,U}(\gamma) \mid n \in \mathbb{N},\ U \in \mathcal{V}_G(\mathbf{1})\}$ is a fundamental system of neighborhoods of γ.

Definition 1.1 ([6,8]). We say that the topological group G is a *group with Lie algebra* if the topological space $\Lambda(G)$ has the structure of a topological Lie algebra over $\mathbb{R}$ whose algebraic operations satisfy the following conditions for all $t, s \in \mathbb{R}$ and $\lambda, \gamma \in \Lambda(G)$,

$$(t\lambda)(s) = \lambda(ts);$$
$$(\lambda + \gamma)(t) = \lim_{n\to\infty} (\lambda(\frac{t}{n})\gamma(\frac{t}{n}))^n;$$
$$[\lambda, \gamma](t^2) = \lim_{n\to\infty} (\lambda(\frac{t}{n})\gamma(\frac{t}{n})\lambda(-\frac{t}{n})\gamma(-\frac{t}{n}))^{n^2},$$

with uniform convergence on the compact subsets of $\mathbb{R}$.

The topological group G is a *pre-Lie group* if it is a group with Lie algebra and for every nonconstant $\gamma \in \Lambda(G)$ there exists a real-valued function f of class C^∞ on some neighborhood of $\mathbf{1} \in G$ such that $Df(\mathbf{1}; \gamma) \neq 0$.

2. LIE THEORY FOR 2-STEP NILPOTENT TOPOLOGICAL GROUPS

In this section, we provide a detailed proof of [9, Th.IV.1.24] (see Theorem 2.22 below)

Definition 2.1. Let G be a group. We denote

$$[G, G] = \{xyx^{-1}y^{-1}; x, y \in G\}$$

and

$$Z(G) = \{g \in G; xg = gx, (\forall)x \in G\}$$

which is called the *center* of G and is a commutative subgroup of G.

We say that G is a *2-step nilpotent* group if $[G, G] \subseteq Z(G)$.

We define the commutator map

$$c : G \times G \to Z(G), \quad c(x, y) := xyx^{-1}y^{-1}.$$

Everywhere in what follows we assume that G is 2-step nilpotent group, unless explicitly stated that G is an arbitrary group.

2.1. SOME BASIC PROPERTIES OF THE COMMUTATOR MAP

LEMMA 2.2. *The commutator map c is a bi-morphism, that is for all $x, y, a, b \in G$ we have:*
(a) $c(y, x) = (c(x, y))^{-1}$
(b) $c(x, ab) = c(x, b)c(x, a) = c(x, a)c(x, b)$
(c) $c(ab, x) = c(a, x)c(b, x)$.

Proof. The proof is based on direct calculations, thus:
a) $(c(x, y))^{-1} = (xyx^{-1}y^{-1})^{-1} = yxy^{-1}x^{-1} = c(y, x)$.
b) We have:

$$\begin{aligned} c(x, ab) = c(x, b)c(x, a) &\iff xabx^{-1}b^{-1}a^{-1} = xbx^{-1}b^{-1}xax^{-1}a^{-1} \\ &\iff abx^{-1}b^{-1} = (bx^{-1}b^{-1}x)ax^{-1} \\ &\iff abx^{-1}b^{-1} = abx^{-1}b^{-1}xx^{-1} \end{aligned}$$

which is true.
c) We have

$$\begin{aligned} c(ab, x) &= (c(x, ab))^{-1} \\ &= (c(x, a)c(x, b))^{-1} \\ &= (c(x, b))^{-1}(c(x, a))^{-1} \\ &= c(b, x)c(a, x) \\ &= c(a, x)c(b, x) \end{aligned}$$

and the proof ends. □

LEMMA 2.3. *Let G be a 2-step nilpotent group and $a, b \in G$. Then we have*
(a) $c(a^{-1}, b^{-1}) = c(a, b)$
(b) $c(a, b^{-1}) = c(b, a)$
(c) $c(a^{-1}, b) = c(b, a)$
(d) $c(a^m, b^n) = (c(a, b))^{mn}$ *for any m, n natural numbers.*
(e) $abba = baab = ab^2a = ba^2b$.

Proof. a) $c(a^{-1}, b^{-1}) = c(a, b) \iff a^{-1}b^{-1}ab = aba^{-1}b^{-1} \iff$ $(ba)^{-1}ab = ab(ba)^{-1} \iff abba = baab \iff ab(bab^{-1}a^{-1})a^{-1}b^{-1} = \mathbf{1}$ $\iff aba^{-1}b^{-1}bab^{-1}a^{-1} = \mathbf{1} \iff c(a, b)c(b, a) = \mathbf{1}$ which is true and solve point a).

b) $c(a, b^{-1}) = c(b, a) \iff c(a, b^{-1})c(a, b) = \mathbf{1} \iff c(a, \mathbf{1}) = \mathbf{1}$ which is true.

c) $c(a^{-1}, b) = c((a^{-1})^{-1}, b^{-1}) = c(a, b^{-1}) = c(b, a)$.

d) $c(a^m, b^n) = c(a^m, b)c(a^m, b) \dots c(a^m, b) = (c(a^m, b))^n =$ $(c(a, b)c(a, b) \dots c(a, b))^n = (c(a, b)^m)^n = (c(a, b))^{mn}$. $\square$

And finally, assertion e) is obvious from proof of point a).

2.2. LIE BRACKETS FOR CONTINUOUS SUBGROUPS WITH ONE PARAMETER

LEMMA 2.4. *Let G be a 2-step nilpotent topological group and $\alpha, \beta : \mathbb{R} \to G$ two continuos morphisms of groups (from $\Lambda(G)$). Then we have*

$$c(\alpha(1), \beta(t^2)) = c(\alpha(t), \beta(t)), (\forall)t \in \mathbb{R}$$

Proof. The above relation is obvious for $t = 0$.

Let $t = \frac{m}{n}$ with m, n nonzero natural numbers. If we denote $\alpha(\frac{1}{n}) = a, \beta(\frac{m}{n^2}) = b$ then the formula $c(\alpha(1), \beta(t^2)) = c(\alpha(t), \beta(t))$ is equivalent to $c(a^n, b^m) = c(a^m, b^n) \iff (c(a, b))^{nm} = (c(a, b))^{mn}$ which holds true. Therefore we have shown that $c(\alpha(1), \beta(t^2)) = c(\alpha(t), \beta(t))$ for any t positive rational number.

Let $t \in \mathbb{R}, t > 0$. There exists a sequence t_n of positive rational numbers such that $\lim\limits_{n\to\infty} t_n = t$. Since α, β are continuous we obtain

$$c(\alpha(1), \beta(t^2)) = \lim_{n\to\infty} c(\alpha(1), \beta(t_n^2)) = \lim_{n\to\infty} c(\alpha(t_n), \beta(t_n)) = c(\alpha(t), \beta(t)).$$

If $t \in \mathbb{R}, t < 0$ then we have

$$c(\alpha(1), \beta(t^2)) = c(\alpha(-t), \beta(-t)) = c((\alpha(-t))^{-1}, (\beta(-t))^{-1}) = c(\alpha(t), \beta(t))$$

and the proof ends. $\square$

LEMMA 2.5. *Let G be a 2-step nilpotent group, $a \in G$ and $\beta : \mathbb{R} \to G$ a group morphism. We define $\lambda : \mathbb{R} \to G$, $\lambda(t) := c(a, \beta(t))$. Then λ is again a group morphism.*

Proof. We have

$$\lambda(t)\lambda(s) = c(a, \beta(t))c(a, \beta(s)) = c(a, \beta(t)\beta(s)) = c(a, \beta(t + s)) = \lambda(t + s)$$

and this concludes the proof. $\square$

Lemma 2.6. *Let G be a 2-step nilpotent topological group and $\alpha, \beta \in \Lambda(G)$. Then we have*

$$[\alpha, \beta](t) = c(\alpha(1), \beta(t)), (\forall) t \in \mathbb{R}$$

and $[\alpha, \beta] \in \Lambda(G)$

Proof. From

$$\begin{aligned}(\alpha(\frac{t}{n})\beta(\frac{t}{n})\alpha(-\frac{t}{n})\beta(-\frac{t}{n}))^{n^2} &= (c(\alpha(\frac{t}{n}), \beta(\frac{t}{n}))^{n^2} = c((\alpha(\frac{t}{n}))^n, (\beta(\frac{t}{n}))^n) \\ &= c(\alpha(\frac{t}{n}n), \beta(\frac{t}{n}n)) = c(\alpha(t), \beta(t)) \\ &= c(\alpha(1), \beta(t^2))\end{aligned}$$

we obtain

$$[\alpha, \beta](t^2) = c(\alpha(1), \beta(t^2)) \text{ and } [\alpha, \beta](t) = c(\alpha(1), \beta(t))$$

for all $t \in \mathbb{R}$, $t \geq 0$.

From

$$\begin{aligned}[\alpha, \beta](-t^2) &= ([\alpha, \beta](t^2))^{-1} = (c(\alpha(1), \beta(t^2)))^{-1} = c(\beta(t^2), \alpha(1)) \\ &= c(\alpha(1), (\beta(t^2))^{-1}) = c(\alpha(1), \beta(-t^2))\end{aligned}$$

we get $[\alpha, \beta](t) = c(\alpha(1), \beta(t))$ for all $t \in \mathbb{R}$, $t < 0$.

From the previous lemma it follows that $[\alpha, \beta] : \mathbb{R} \to G$ is a morphism.

Since G is a topological group we obtain that $[\alpha, \beta]$ is continuous so $[\alpha, \beta] \in \Lambda(G)$. □

2.3. SUM OF CONTINUOUS SUBGROUPS WITH A PARAMETER

Lemma 2.7. *Let G be a 2-step nilpotent group and $\alpha, \beta : \mathbb{R} \to G$ be two group morphisms. Then $(\alpha(s)\beta(s))^n = \alpha(ns)\beta(ns)c(\alpha(-s), \beta(\frac{n(n-1)}{2}s))$ for any $s \in \mathbb{R}$ and any natural number $n \geq 2$.*

Proof. We will prove the assertion by induction on $n \geq 2$.

In the case $n = 2$ we have

$$\begin{aligned}(\alpha(s)\beta(s))^2 &= \alpha(s)\beta(s)\alpha(s)\beta(s) = \alpha(s)\alpha(s)\alpha(-s)\beta(s)\alpha(s)\beta(-s)\beta(2s) \\ &= \alpha(2s)c(\alpha(-s), \beta(s))\beta(2s) = \alpha(2s)\beta(2s)c(\alpha(-s), \beta(s))\end{aligned}$$

Induction step n to $n+1$: We have

$$(\alpha(s)\beta(s))^{n+1} = (\alpha(s)\beta(s))^n\alpha(s)\beta(s)$$
$$= \alpha(ns)\beta(ns)\alpha(s)\beta(s)c(\alpha(-s),\beta(\frac{n(n-1)}{2}s))$$
$$= \alpha(ns)\alpha(s)\alpha(-s)\beta(ns)\alpha(s)\beta(-ns)\beta((n+1)s)c(\alpha(-s),\beta(\frac{n(n-1)}{2}s))$$
$$= \alpha((n+1)s)c(\alpha(-s),\beta(ns))c(\alpha(-s),\beta(\frac{n(n-1)}{2}s))\beta((n+1)s)$$
$$= \alpha((n+1)s)\beta((n+1)s)c(\alpha(-s),\beta(ns+\frac{n(n-1)}{2}s))$$
$$= \alpha((n+1)s)\beta((n+1)s)c(\alpha(-s),\beta(\frac{n(n+1)}{2}s))$$

and this concludes the proof. □

Lemma 2.8. *Let G be a 2-step nilpotent topological group and $\alpha,\beta \in \Lambda(G)$. Then we have*

$$c(\beta(s),\alpha(t)) = c(\beta(t),\alpha(s)), (\forall)s,t \in \mathbb{R}$$

Proof. The assertion is obvious for $s = 0$ or $t = 0$. If s,t are positive rational numbers, then $s = \frac{m}{n}, t = \frac{p}{q}$ where m,n,p,q are nonzero natural numbers.

The equality $c(\beta(s),\alpha(t)) = c(\beta(t),\alpha(s))$ is equivalent to

$$c(\beta(\frac{m}{n}),\alpha(\frac{p}{q})) = c(\beta(\frac{p}{q}),\alpha(\frac{m}{n}))$$
$$\iff (c(\beta(\frac{1}{n}),\alpha(\frac{1}{q})))^{mp} = c(\beta(\frac{1}{q}),\alpha(\frac{1}{n}))^{mp}$$
$$\iff (c(\beta(\frac{1}{nq}),\alpha(\frac{1}{q})))^{mpq} = c(\beta(\frac{1}{nq}),\alpha(\frac{1}{n}))^{mnp}$$
$$\iff (c(\beta(\frac{1}{nq}),\alpha(\frac{1}{nq})))^{mnpq} = c(\beta(\frac{1}{nq}),\alpha(\frac{1}{nq}))^{mnpq}$$

which holds true.

We thus showed that $c(\beta(s),\alpha(t)) = c(\beta(t),\alpha(s))$ is true for s,t positive rational numbers.

Let $s,t \in \mathbb{R}, s,t > 0$. There exists sequences s_n, t_n of positive rational numbers such that $\lim_{n\to\infty} t_n = t$, $\lim_{n\to\infty} s_n = s$. Since α,β are continuous we obtain

$$c(\beta(s),\alpha(t)) = \lim_{n\to\infty} c(\beta(s_n),\alpha(t_n)) = \lim_{n\to\infty} c(\beta(t_n),\alpha(s_n)) = c(\beta(t),\alpha(s))$$

There remain the cases $(s < 0, t > 0), (s < 0, t < 0), (s > 0, t < 0)$.

In the case $(s<0, t>0)$ we have

$$c(\beta(s),\alpha(t)) = c(\alpha(t),\beta(-s)) = c(\alpha(-s),\beta(t)) = c(\beta(t),\alpha(s)).$$

In the case $(s<0, t<0)$ we have

$$c(\beta(s),\alpha(t)) = c(\beta(-s),\alpha(-t)) = c(\beta(-t),\alpha(-s)) = c(\beta(t),\alpha(s)).$$

In the case $(s>0, t<0)$ we have

$$c(\beta(s),\alpha(t)) = c(\alpha(-t),\beta(s)) = c(\alpha(s),\beta(-t)) = c(\beta(t),\alpha(s)).$$

Therefore in all cases we obtain

$$c(\beta(s),\alpha(t)) = c(\beta(t),\alpha(s)), (\forall)s,t\in\mathbb{R}$$

and the proof ends. □

LEMMA 2.9. *Let G be a 2-step nilpotent topological group and $\alpha,\beta\in\Lambda(G)$. Then we have*

$$c(\beta(s),\alpha(t)) = c(\beta(1),\alpha(st)), (\forall)s,t\in\mathbb{R}$$

Proof. As in the proof of Lemma 2.8 it is sufficient to prove the required relation for s,t positive rational numbers, $s=\frac{m}{n}, t=\frac{p}{q}$ where m,n,p,q are nonzero natural numbers.

If we denote $\alpha(\frac{1}{nq})=a, \beta(\frac{1}{nq})=b$ it remains to prove that

$$c(a^{mq},b^{np}) = c(a^{nq},b^{mp}) \iff (c(a,b))^{mqnp} = (c(a,b))^{nqmp}$$

which is true and the proof ends. □

LEMMA 2.10. *Let G be a 2-step nilpotent group and $\alpha,\beta:\mathbb{R}\to G$ be two group morphisms. We define $\lambda:\mathbb{R}\to G, \lambda(t):=\alpha(t)\beta(t)c(\alpha(t),\beta(-\frac{t}{2}))$. Then λ is a group morphism.*

Proof. We must prove that for all $s,t\in\mathbb{R}$ one has $\lambda(s+t)=\lambda(s)\lambda(t)$.

The equality $\lambda(s+t)=\lambda(t)\lambda(s)$ is equivalent to

$$\alpha(t)\alpha(s)\beta(t)\beta(s)c(\alpha(t)\alpha(s),\beta(-\frac{t}{2})\beta(-\frac{s}{2}))$$
$$=\alpha(t)\beta(t)c(\alpha(t),\beta(-\frac{t}{2}))\alpha(s)\beta(s)c(\alpha(s),\beta(-\frac{s}{2}))$$

This equality is equivalent to:

$$\alpha(s)\beta(t)\beta(s)c(\alpha(t),\beta(-\frac{t}{2}))c(\alpha(t),\beta(-\frac{s}{2}))c(\alpha(s),\beta(-\frac{t}{2}))c(\alpha(s),\beta(-\frac{s}{2}))$$
$$=\beta(t)c(\alpha(t),\beta(-\frac{t}{2}))\alpha(s)\beta(s)c(\alpha(s),\beta(-\frac{s}{2}))$$

The equivalence of equalities can be extended thus:

$$\begin{aligned}
&\alpha(s)\beta(t)\beta(s)c(\alpha(t),\beta(-\frac{s}{2}))c(\alpha(s),\beta(-\frac{t}{2})) = \beta(t)\alpha(s)\beta(s)\\
&\iff \alpha(s)\beta(t)c(\alpha(t),\beta(-\frac{s}{2}))c(\alpha(s),\beta(-\frac{t}{2})) = \beta(t)\alpha(s)\\
&\iff c(\alpha(s),\beta(t))c(\alpha(t),\beta(-\frac{s}{2}))c(\alpha(s),\beta(-\frac{t}{2})) = \mathbf{1}\\
&\iff c(\alpha(s),\beta(\frac{t}{2}))c(\alpha(t),\beta(-\frac{s}{2})) = \mathbf{1}\\
&\iff c(\beta(\frac{s}{2}),\alpha(t)) = c(\beta(\frac{t}{2}),\alpha(s))\\
&\iff (c(\beta(\frac{s}{2}),\alpha(\frac{t}{2})))^2 = (c(\beta(\frac{t}{2}),\alpha(\frac{s}{2})))^2
\end{aligned}$$

which holds true and the proof ends. □

LEMMA 2.11. *Let G be a 2-step nilpotent topological group and $\alpha, \beta \in \Lambda(G)$. Then we have*

$$(\alpha+\beta)(t) = \alpha(t)\beta(t)c(\alpha(t),\beta(-\frac{t}{2})), (\forall)t \in \mathbb{R}$$

and $\alpha+\beta \in \Lambda(G)$.

Proof. We have

$$\begin{aligned}
c(\alpha(-\frac{t}{n}),\beta(\frac{n-1}{2}t)) &= (c(\alpha(-\frac{t}{2n}),\beta(\frac{(n-1)}{2}t))^2\\
&= (c(\alpha(-\frac{t}{2n}),\beta(\frac{n(n-1)}{2}\frac{t}{n}))^2\\
&= (c(\alpha(-\frac{t}{2n}),\beta(\frac{t}{n})))^{n(n-1)}\\
&= c(\alpha(-\frac{t}{2n}n),\beta(\frac{t}{n}(n-1)))\\
&= c(\alpha(-\frac{t}{2}),\beta(\frac{t(n-1)}{n}))
\end{aligned}$$

Further

$$\begin{aligned}
(\alpha+\beta)(t) &= \lim_{n\to\infty}(\alpha(\frac{t}{n})\beta(\frac{t}{n}))^n\\
&= \lim_{n\to\infty}\alpha(n\frac{t}{n})\beta(n\frac{t}{n})c(\alpha(-\frac{t}{n}),\beta(\frac{n-1}{2}t))\\
&= \lim_{n\to\infty}\alpha(t)\beta(t)c(\alpha(-\frac{t}{2}),\beta(\frac{t(n-1)}{n}))\\
&= \alpha(t)\beta(t)c(\alpha(-\frac{t}{2}),\beta(t))
\end{aligned}$$

$$
\begin{aligned}
&= \alpha(t)\beta(t)c(\beta(t),\alpha(\frac{t}{2}))\\
&= \alpha(t)\beta(t)(c(\beta(\frac{t}{2}),\alpha(\frac{t}{2})))^2\\
&= \alpha(t)\beta(t)c(\beta(\frac{t}{2}),\alpha(t))\\
&= \alpha(t)\beta(t)c(\alpha(-t),\beta(\frac{t}{2}))\\
&= \alpha(t)\beta(t)c(\alpha(t),\beta(-\frac{t}{2})).
\end{aligned}
$$

Let us mention that we used above the continuity of the morphisms α, β when we passed to the limit. From Lemma 2.10 it follows that $\alpha+\beta : \mathbb{R} \to G$ is a morphism. Since G is a topological group we obtain that $\alpha+\beta$ is continuous so $\alpha+\beta \in \Lambda(G)$ $\square$

2.4. SUBGROUPS WITH ONE PARAMETER IN GENERAL TOPOLOGICAL GROUPS

Lemmas (2.12–2.15) below are available for general topological groups, so we will not use the hypothesis that the group G is a 2-step nilpotent group.

LEMMA 2.12. *Let G be a topological group, $\alpha, \beta : \mathbb{R} \to G$ be two continuous morphisms of groups. We define $\lambda : \mathbb{R} \to G, \lambda(t) := \alpha(t)\beta(t)$. Then λ is a morphism if and only if α, β commute and in this case $\lambda = \alpha + \beta$.*

Proof. The map $\lambda : \mathbb{R} \to G$ is morphism if and only if we have

$$
\begin{aligned}
&(\forall s,t \in \mathbb{R})\ \lambda(t+s) = \lambda(t)\lambda(s)\\
&\iff (\forall s,t \in \mathbb{R})\ \alpha(t)\alpha(s)\beta(t)\beta(s) = \alpha(t)\beta(t)\alpha(s)\beta(s)\\
&\iff (\forall s,t \in \mathbb{R})\ \alpha(s)\beta(t) = \beta(t)\alpha(s) \iff \alpha,\beta \text{ commute.}
\end{aligned}
$$

We have

$$
\begin{aligned}
(\alpha+\beta)(t) &= \lim_{n\to\infty}(\alpha(\frac{t}{n})\beta(\frac{t}{n}))^n = \lim_{n\to\infty}(\alpha(\frac{t}{n}))^n(\beta(\frac{t}{n}))^n\\
&= \lim_{n\to\infty}\alpha(n\frac{t}{n})\beta(n\frac{t}{n}) = \alpha(t)\beta(t)
\end{aligned}
$$

so $\lambda = \alpha + \beta$. $\square$

In the following lemmas, for all $\alpha \in \Lambda(G)$ and $t \in \mathbb{R}$ we use the notation $t\alpha$ for the element of $\Lambda(G)$ defined by $(t\alpha)(s) = \alpha(ts)$ for all $s \in \mathbb{R}$.

LEMMA 2.13. *Let G be a topological group, $\alpha : \mathbb{R} \to G$ be a continuous morphism of groups. Then*

$$\alpha + \alpha + \ldots + \alpha = n\alpha, (\forall)n \geq 2$$

Proof. We will do the proof by induction on $n \geq 2$.

In case $n = 2$ since α and α commute obtain for every $t \in \mathbb{R}$

$$(\alpha + \alpha)(t) = \alpha(t)\alpha(t) = \alpha(t + t) = \alpha(2t) = 2\alpha(t)$$

whence it follows that $\alpha + \alpha = 2\alpha$.

Transition from n to $n + 1$. Since $n\alpha$ and α commute obtain for every $t \in \mathbb{R}$

$$\begin{aligned}(\alpha + \alpha + \ldots + \alpha)(t) &= (n\alpha + \alpha)(t) = n\alpha(t)\alpha(t)\\ &= \alpha(nt)\alpha(t) = \alpha(nt + t) = \alpha((n + 1)t) = (n + 1)\alpha(t)\end{aligned}$$

whence it follows that $\alpha + \alpha + \ldots + \alpha = (n + 1)\alpha$ and proof ends. □

LEMMA 2.14. *Let G be a topological group, $a, b \in \mathbb{R}$ and $\alpha : \mathbb{R} \to G$ a morphism of groups. Then $(a + b)\alpha = a\alpha + b\alpha$ and $a(b\alpha) = (ab)\alpha$.*

Proof. Since $a\alpha, b\alpha$ commute we obtain for every $t \in \mathbb{R}$

$$\begin{aligned}(a\alpha + b\alpha)(t) &= a\alpha(t)b\alpha(t) = \alpha(at)\alpha(bt) = \alpha(at + bt) = \alpha((a + b)t)\\ &= (a + b)\alpha(t)\end{aligned}$$

so $(a + b)\alpha = a\alpha + b\alpha$.

We have $a(b\alpha)(t) = b\alpha(at) = \alpha(bat) = (ab)\alpha(t)$ so $a(b\alpha) = (ab)\alpha$. □

LEMMA 2.15. *Let G be a topological group, $a \in \mathbb{R}$ and $\alpha, \beta : \mathbb{R} \to G$ be two continuous morphisms of groups.*

Then $a(\alpha + \beta) = a\alpha + a\beta$ if $\alpha + \beta : \mathbb{R} \to G$ exist.

Proof. We have for all $t \in \mathbb{R}$,

$$\begin{aligned}(a(\alpha + \beta))(t) &= (\alpha + \beta)(at) = \lim_{n\to\infty} (\alpha(\frac{at}{n})\beta(\frac{at}{n}))^n = \lim_{n\to\infty} (a\alpha(\frac{t}{n})a\beta(\frac{t}{n}))^n\\ &= (a\alpha + a\beta)(t)\end{aligned}$$

so $a(\alpha + \beta) = a\alpha + a\beta$.

Let us point out that we used only existence in G of the limit which defines $\alpha + \beta$ and not the fact that $\alpha + \beta$ is a morphism or a continuous map. □

2.5. TOPOLOGICAL LIE ALGEBRA OF A 2-STEP NILPOTENT TOPOLOGICAL GROUP

In the following we will prove in detail Theorem 2.22, whose proof was sketched in [9, Th.IV.1.24]. On the way we also get other useful results (for example Proposition 2.20).

LEMMA 2.16. *Let G be a 2-step nilpotent topological group and $\alpha, \beta, \gamma \in \Lambda(G)$. Then $\alpha + \beta = \beta + \alpha$ and $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.*

Proof. We have $\alpha + \beta = \beta + \alpha \iff (\alpha + \beta)(t) = (\beta + \alpha)(t) \iff$

$$\alpha(t)\beta(t)c(\beta(\frac{t}{2}), \alpha(t)) = \beta(t)\alpha(t)c(\alpha(\frac{t}{2}), \beta(t))$$

We denote $\alpha(\frac{t}{2}) = a$, $\beta(\frac{t}{2}) = b$ and the previous relation is equivalent to $a^2b^2c(b, a^2) = b^2a^2c(a^2, b) \iff a^{-2}b^{-2}a^2b^2(c(b, a))^2 = (c(a, b))^2 \iff$

$$c(a^{-2}, b^{-2}) = (c(a, b))^2(c(a, b))^2 \iff c(a^2, b^2) = (c(a, b))^4$$

which holds true and we obtain $\alpha + \beta = \beta + \alpha$.

We have

$$\begin{aligned}((\alpha + \beta) + \gamma)(t) &= (\alpha + \beta)(t)\gamma(t)c(\gamma(\frac{t}{2}), (\alpha + \beta)(t)) \\ &= \alpha(t)\beta(t)\gamma(t)c(\beta(\frac{t}{2}), \alpha(t))c(\gamma(\frac{t}{2}), \alpha(t))c(\gamma(\frac{t}{2}), \beta(t))\end{aligned}$$

We have

$$\begin{aligned}(\alpha + (\beta + \gamma))(t) &= \alpha(t)(\beta + \gamma)(t)c((\beta + \gamma)(\frac{t}{2}), \alpha(t)) \\ &= \alpha(t)\beta(t)\gamma(t)c(\gamma(\frac{t}{2}), \beta(t))c(\beta(\frac{t}{2}), \alpha(t))c(\gamma(\frac{t}{2}), \alpha(t)) \\ &= ((\alpha + \beta) + \gamma)(t)\end{aligned}$$

so $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$. □

LEMMA 2.17. *Let G be a 2-step nilpotent topological group and $\alpha, \beta, \gamma \in \Lambda(G)$. Then we have*

(a) $[\beta, \alpha] = -[\alpha, \beta]$
(b) $[[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] = 0 \in \Lambda(G)$
(c) $[\alpha + \beta, \gamma] = [\alpha, \gamma] + [\beta, \gamma]$
(d) $[a\alpha, \gamma] = a[\alpha, \gamma]$ *for any* $a \in \mathbb{R}$

Proof. For point a) we have

$$\begin{aligned}[\beta, \alpha] = -[\alpha, \beta] &\iff [\beta, \alpha](t) = (-[\alpha, \beta])(t) \\ &\iff c(\beta(1), \alpha(t)) = [\alpha, \beta](-t) = c(\alpha(1), \beta(-t)) \\ &\iff c(\beta(1), \alpha(t)) = c(\beta(t), \alpha(1))\end{aligned}$$

which holds true.

For b) it is sufficient to show that $[[\alpha, \beta], \gamma] = 0 \in \Lambda(G)$. We have

$$[[\alpha, \beta], \gamma](t) = c([\alpha, \beta](1), \gamma(t)) = c(c(\alpha(1), \beta(1)), \gamma(t)) = \mathbf{1}$$

for every $t \in \mathbb{R}$ hence we obtain $[[\alpha, \beta], \gamma] = 0 \in \Lambda(G)$.

For c) we have

$$\begin{aligned}[\alpha+\beta,\gamma](t) &= c((\alpha+\beta)(1),\gamma(t))\\ &= c(\alpha(1)\beta(1)c(\beta(\frac{1}{2}),\alpha(1)),\gamma(t))\\ &= c(\alpha(1)\beta(1),\gamma(t))\\ &= c(\alpha(1),\gamma(t))c(\beta(1),\gamma(t))\end{aligned}$$

We have

$$\begin{aligned}([\alpha,\gamma]+[\beta,\gamma])(t) &= [\alpha,\gamma](t)[\beta,\gamma](t)c([\beta,\gamma](\frac{t}{2}),[\alpha,\gamma](t))\\ &= c(\alpha(1),\gamma(t))c(\beta(1),\gamma(t))\\ &= [\alpha+\beta,\gamma](t)\end{aligned}$$

hence we obtain $[\alpha+\beta,\gamma]=[\alpha,\gamma]+[\beta,\gamma]$.

For d) we have $[a\alpha,\gamma](t)=c(a\alpha(1),\gamma(t))=c(\alpha(a),\gamma(t))$. But $a[\alpha,\gamma](t)=c(\alpha(1),\gamma(at))=c(\alpha(a),\gamma(t))=[a\alpha,\gamma](t)$ hence we obtain $[a\alpha,\gamma]=a[\alpha,\gamma]$. $\square$

Let X be a topological space and G be a topological group. On the set $C(X,G)$ of continuous maps from X to G we introduce the topology of uniform convergence on compact subsets of X in which the sets

$$\begin{aligned}E(\beta,K,V) &= \{\gamma\in C(X,G); (\beta(t))^{-1}\gamma(t)\in V, (\forall)t\in K\}\\ &= \{\gamma\in C(X,G); \gamma(t)\in\beta(t)V, (\forall)t\in K\}\end{aligned}$$

for every compact subsets $K\subseteq X$ and $V\in V_G(\mathbf{1})$ form a fundamental system of neighborhoods of β. On the set $\Lambda(G)$ we consider the topology induced on $C(\mathbb{R},G)$.

Lemma 2.18. *Let G be a topological group and $a\in\mathbb{R}$. Then the map*

$$f: C(\mathbb{R},G)\to C(\mathbb{R},G),\quad f(\beta):=a\beta$$

is continuous.

Proof. If $a=0$ is evident because f is constant. If $a\neq 0$ continuity result from $E(a\beta,aK,V)=E(\beta,K,V)$. $\square$

The following lemma and proposition are well-known but we prove them here for the sake of completeness

Lemma 2.19. *Let X,Y,T be topological spaces, $y_0\in Y, t_0\in T$ and K compact in X, $f:X\times Y\to T$ a continuous function for which $f(K\times\{y_0\})=\{t_0\}$. Then for every neighborhood V of t_0 there exists a neighborhood of U y_0 such that $f(K\times U)\subseteq V$.*

Proof. Let $x \in K$. Since f is continuous in (x, y_0) and $f(x, y_0) = t_0$ it follows that there exists D open neighborhood of (x, y_0) such that $f(D) \subseteq V$. We may assume $D = S_x \times U_x$ where S_x is an open neighborhood of x and U_x is an open neighborhood of y_0 and we have $f(S_x \times U_x) \subseteq V$. From $K \subseteq \cup_{t \in K} S_t$ using K compact we obtain a finite sub-covering, so there exists $n \geq 1$ and $x_1, \ldots, x_n \in K$ such that $K \subseteq S_{x_1} \cup S_{x_2} \cup \ldots \cup S_{x_n}$. We denote $U = U_{x_1} \cap \ldots \cap U_{x_n}$ and U is open in Y and $y_0 \in U$, so U is a neighborhood of y_0. We show that $f(K \times U) \subseteq V$. Let $x \in K$ and $y \in U$. There exists $j \in \{1, \ldots, n\}$ such that $x \in S_{x_j}$ and $y \in U_{x_j}$. From $f(x, y) \in f(S_{x_j} \times U_{x_j}) \subseteq V$ we obtain $f(K \times U) \subseteq V$ and the proof ends. □

Proposition 2.20. *Let X be a topological space and G be a topological group. Then the map $\phi : C(X, G) \times C(X, G) \to C(X, G), \phi(\alpha, \beta) := \alpha\beta$ is continuous, where $\alpha\beta \in C(X, G)$ is defined by $\alpha\beta(t) := \alpha(t)\beta(t)$ for all $t \in \mathbb{R}$.*

Proof. Let $\alpha, \beta \in C(X, G)$, K compact in X and $V \in V_G(\mathbf{1})$. We must find K_1, K_2 compacts in X and $V_1, V_2 \in V_G(\mathbf{1})$ such that

$$\phi(E(\alpha, K_1, V_1) \times E(\beta, K_2, V_2)) \subseteq E(\alpha\beta, K, V)$$

We take $K_1 = K_2 = K$. Let $\alpha_0 \in E(\alpha, K_1, V_1)$ and $\beta_0 \in E(\beta, K_2, V_2)$ and let $t \in K$. There exist $v_1 \in V_1$ and $v_2 \in V_2$ such that $\alpha_0(t) = \alpha(t)v_1$ and $\beta_0(t) = \beta(t)v_2$.

We have $\alpha_0(t)\beta_0(t) = \alpha(t)v_1\beta(t)v_2 = \alpha(t)\beta(t)(\beta(t))^{-1}v_1\beta(t)v_2$.

We show that $(\beta(t))^{-1}v_1\beta(t)v_2 \in V$ for every $v_1 \in V_1$ and $v_2 \in V_2$ and any $t \in K$. Let $f : G \times G \times G \to G, f(x, y, z) := x^{-1}yxz$. Applying the previous lemma for the continuous function f and $X = G, Y = G \times G$, $y_0 = (\mathbf{1}, \mathbf{1}), t_0 = \mathbf{1}$ and the compact set $\beta(K) \subseteq G$ we obtain that there exist U an open neighborhood of $(\mathbf{1}, \mathbf{1})$ such that $f(\beta(K) \times U) \subseteq V$. Because U is open neighborhood of $(\mathbf{1}, \mathbf{1})$ in $G \times G$ we may assume $U = V_1 \times V_2$ with $V_1, V_2 \in V_G(\mathbf{1})$. From $f(\beta(K) \times V_1 \times V_2) \subseteq V$ it follows that $(\beta(t))^{-1}v_1\beta(t)v_2 \in V$ for every $v_1 \in V_1, v_2 \in V_2$ and any $t \in K$ and the proof ends. □

Proposition 2.21. *Let G be a 2-step nilpotent topological group. Then*

(1) *The map $\phi : \Lambda(G) \times \Lambda(G) \to \Lambda(G), \phi(\alpha, \beta) := \alpha + \beta$ is continuous.*

(2) *The map $\psi : \Lambda(G) \times \Lambda(G) \to \Lambda(G), \psi(\alpha, \beta) := [\alpha, \beta]$ is continuous.*

Proof. For (1) since G is a 2-step nilpotent topological group it follows that we have the relationship

$$(\alpha + \beta)(t) = \alpha(t)\beta(t)c(\alpha(t), \beta(-\frac{t}{2})) = \alpha(t)\beta(t)\alpha(t)\beta(-\frac{t}{2})\alpha(-t)\beta(\frac{t}{2})$$

so ϕ is a product of three functions from $\Lambda(G) \times \Lambda(G)$ to $C(\mathbb{R}, G)$ which are continuous by Proposition 2.20. Then ϕ is continuous. For (2) we use the equality $[\alpha,\beta](t) = c(\alpha(\mathbf{1}),\beta(t))$ and continuity of the map ψ follows with the same reasoning as for the assertion (1). □

From the preceding results on 2-step nilpotent topological groups we obtain the following theorem.

THEOREM 2.22 ([9, Th.IV.1.24]). *Every 2-step nilpotent topological group is a group with Lie algebra.*

3. TANGENT GROUP OF TOPOLOGICAL GROUP WITH LIE ALGEBRA

In this section, we introduce the following new notion, which generalizes the fact that the tangent bundle of any Lie group has a natural structure of Lie group.

Definition 3.1. Let G be a group with Lie algebra. The *tangent group* of G is $T(G)$, which is the set $G \ltimes \Lambda(G)$ endowed with the group operation $(x,\alpha)(y,\beta) = (xy, \alpha^y + \beta)$.

Here we define $\alpha^y \in \Lambda(G)$ by $\alpha^y(t) = y\alpha(t)y^{-1}$ for all $t \in \mathbb{R}, y \in G, \alpha \in \Lambda(G)$.

PROPOSITION 3.2. *Let G be a group with Lie algebra. Then $T(G)$ is a topological group.*

Proof. Associativity follows by the relationship

$$(\forall x \in G)(\forall \alpha,\beta \in \Lambda(G)) \quad (\alpha+\beta)^x = \alpha^x + \beta^x$$

which can be verified by direct calculation.

The unit element of $T(G)$ is $(\mathbf{1},0) \in T(G)$ and the inverse of (x,α) is $(x^{-1}, -\alpha^{x^{-1}})$. Continuity of operation on $T(G)$ results from the fact that G is a topological group, Lie algebra $\Lambda(G)$ is a topological algebra, and the action of G on $\Lambda(G)$ is continuous. □

PROPOSITION 3.3. *Let G be a 2-step nilpotent topological group. Then $T(G) = G \ltimes \Lambda(G)$ is a 2-step nilpotent group.*

Proof. Let $(g,\alpha),(h,\beta) \in T(G)$. We have

$$\begin{aligned}
c((g,\alpha),(h,\beta)) &= (g,\alpha)(h,\beta)(g,\alpha)^{-1}(h,\beta)^{-1}\\
&= (gh, \alpha^h+\beta)(g^{-1}h^{-1}, -\beta^{h^{-1}} - \alpha^{g^{-1}h^{-1}})\\
&= (c(g,h), (\alpha^h+\beta)^{g^{-1}h^{-1}} - \beta^{h^{-1}} - \alpha^{g^{-1}h^{-1}})\\
&= (c(g,h), \alpha^{hg^{-1}h^{-1}} + \beta^{g^{-1}h^{-1}} - \beta^{h^{-1}} - \alpha^{g^{-1}h^{-1}})\\
&= (c(g,h), (\alpha^{hg^{-1}h^{-1}} - \alpha^{g^{-1}h^{-1}}) + (\beta^{g^{-1}h^{-1}} - \beta^{h^{-1}})).
\end{aligned}$$

We now show that

$$Z(T(G)) = \{(x, \alpha) \in T(G); x \in Z(G), Im(\alpha) \subseteq Z(G)\}.$$

If $(x, \alpha) \in Z(T(G))$ then $(x, \alpha)(g, \lambda) = (g, \lambda)(x, \alpha)$ for every $(g, \lambda) \in T(G)$.

So $x \in Z(G)$ and $\alpha^g + \lambda = \lambda^x + \alpha, (\forall)g \in G, (\forall)\lambda \in \Lambda(G)$. From $x \in Z(G)$ we obtain that $\lambda^x = \lambda$ and $\alpha^g = \alpha, (\forall)g \in G$ hence it follows that $Im(\alpha) \subseteq Z(G)$ and we obtain relationship required.

Since G is 2-step nilpotent group we have that if $\alpha, \beta \in \Lambda(G)$ and $Im(\alpha), Im(\beta) \subseteq Z(G)$ then $Im(\alpha + \beta) \subseteq Z(G)$. To complete the proof we will show that if $x, y \in G$ and $\beta \in \Lambda(G)$ then $Im(\beta^x - \beta^y) \subseteq Z(G)$.

We have

$$\begin{aligned}(\beta^x - \beta^y)(t) &= \beta^x(t)\beta^y(-t)c(\beta^x(t), \beta^y(\frac{t}{2})) \\ &= x^{-1}\beta(t)xy^{-1}\beta(-t)yc(\beta^x(t), \beta^y(\frac{t}{2})) \\ &= c(x^{-1}, \beta(t))c(\beta(t), y^{-1})c(\beta^x(t), \beta^y(\frac{t}{2}))\end{aligned}$$

which belongs to $Z(G)$ because G is a 2-step nilpotent group.

It follows that $Im(\beta^x - \beta^y) \subseteq Z(G)$ and the proof ends. $\square$

Now we can obtain the main result of the present paper.

THEOREM 3.4. *Let G be a 2-step nilpotent topological group. Then $T(G) = G \ltimes \Lambda(G)$ is a pre-Lie group if and only if G is pre-Lie group and for every $\alpha \in \Lambda(G), \alpha \neq 0$ there exists a continuous linear functional $\psi : \Lambda(G) \to \mathbb{R}$ with $\psi(\alpha) \neq 0$.*

Proof. We first assume that $T(G) = G \ltimes \Lambda(G)$ is a pre-Lie group. Let $\gamma = (\gamma_1, \gamma_2) \in \Lambda(T(G))$ with $\gamma_1 \neq 0 \in \Lambda(G)$. Since G is pre-Lie group it follows that there exists $f : G \to \mathbb{R}$ of class C^∞ on a neighborhood U of $\mathbf{1} \in G$ such that $Df(\mathbf{1}; \gamma_1) \neq 0$.

We define $h : T(G) \to \mathbb{R}$ by $h(x, \lambda) := f(x)$ which is of class C^∞ on the neighborhood $U \times \Lambda(G)$ of $(\mathbf{1}, 0) \in T(G)$. We have

$$Dh((\mathbf{1}, 0); (\gamma_1, \gamma_2)) = Df(\mathbf{1}; \gamma_1) \neq 0.$$

Now let $\gamma = (\gamma_1, \gamma_2) \in \Lambda(T(G))$ with $\gamma_1 = 0 \in \Lambda(G)$ and $\gamma_2 : \mathbb{R} \to \Lambda(G)$ for which there exists $t_0 \in \mathbb{R}$ such that $\gamma_2(t_0) \neq 0 \in \Lambda(G)$. In addition γ_2 verifies $\gamma_2(t + s) = \gamma_2(t) + \gamma_2(s), (\forall)t, s \in \mathbb{R}$ so γ_2 is a continuous morphism of groups and we have $\gamma_2(t) = t\gamma_2(1), (\forall)t \in \mathbb{R}$.

From $\gamma_2(t_0) \neq 0 \in \Lambda(G)$ it follows that there exists $\psi : \Lambda(G) \to \mathbb{R}$ linear continuous such that $\psi(\gamma_2(t_0)) \neq 0$.

We now define $h : T(G) \to \mathbb{R}$ by $h(x, \lambda) := \psi(\lambda)$ which is of class C^∞ on $T(G)$. We have

$$\begin{aligned} Dh((\mathbf{1}, 0); (\gamma_1, \gamma_2)) &= \lim_{t \to 0} \frac{h(\mathbf{1}, \gamma_2(t)) - h(\mathbf{1}, 0)}{t} \\ &= \lim_{t \to 0} \frac{\psi(\gamma_2(t)) - \psi(0)}{t} \\ &= \lim_{t \to 0} \frac{t\psi(\gamma_2(\mathbf{1}))}{t} \\ &= \psi(\gamma_2(\mathbf{1})) = \frac{\psi(\gamma_2(t_0))}{t_0} \neq 0 \end{aligned}$$

so $T(G) = G \ltimes \Lambda(G)$ is pre-Lie group.

Conversely, we suppose that $T(G) = G \ltimes \Lambda(G)$ is a pre-Lie group and we prove that G is a pre-Lie group. Let $\alpha \in \Lambda(G), \alpha \neq 0$. Since $T(G) = G \ltimes \Lambda(G)$ is pre-Lie group for $(\alpha, 0) \in \Lambda(T(G))$ exists $h : T(G) \to \mathbb{R}$ of class C^∞ on a neighborhood $U \times V$ of $(\mathbf{1}, 0) \in T(G)$ and $Dh((\mathbf{1}, 0); (\alpha, 0)) \neq 0$.

We define $f : G \to \mathbb{R}$ by $f(x) := h(x, 0)$. Since $Df(\mathbf{1}; \alpha) = Dh((\mathbf{1}, 0); (\alpha, 0))$ it follows that $Df(\mathbf{1}; \alpha) \neq 0$ hence we obtain that G is a pre-Lie group.

Let $\alpha \in \Lambda(G), \alpha \neq 0$. We must find $\psi : \Lambda(G) \to \mathbb{R}$ linear and continuous such that $\psi(\alpha) \neq 0$. Since G is pre-Lie group it follows that exist $f : G \to \mathbb{R}$ of class C^∞ on a neighborhood U of $\mathbf{1} \in G$ such that $Df(\mathbf{1}, \alpha) \neq 0$. Let $\psi : \Lambda(G) \to \mathbb{R}$ given by $\psi(\lambda) := Df(\mathbf{1}, \lambda)$.

Since $\psi(\alpha) = Df(\mathbf{1}, \alpha) \neq 0$ and the derivative of f is linear and continuous it follows that ψ verifies the requirement and the proof ends. □

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