COHOMOLOGICALLY COMPLETE INTERSECTIONS WITH VANISHING OF BETTI NUMBERS

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Let I be an ideal of an n-dimensional local Gorenstein ring R. In this paper we will describe several necessary and sufficient conditions such that the ideal I becomes a cohomologically complete intersection. In fact, as a technical tool, it will be shown that the vanishing of the modules $H^i_I(R) = 0$ for all $i \neq c = \operatorname{grade}(I)$ is equivalent to the vanishing of the Betti numbers of $H^c_I(R)$. This gives a new characterization to check the cohomologically complete intersections property with the homological properties of the vanishing of Tor modules of $H^c_I(R)$.

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1. INTRODUCTION

For a commutative Noetherian local ring $(R, \mathfrak{m}, k)$ and an ideal $I \subseteq R$ we denote $H^i_I(R)$, $i \in \mathbb{Z}$, the local cohomology modules of R with respect to I. We refer to see [1] and [2] for the definition of local cohomology modules. It is one of the difficult questions to compute the cohomological dimension $\operatorname{cd}(I)$ of I with respect to R. Here $\operatorname{cd}(I) := \max\{i \in \mathbb{Z} : H^i_I(R) \neq 0\}$. Moreover it is well-known that $\operatorname{grade}(I) \leq \operatorname{cd}(I)$.

The ideal I is called a cohomologically complete intersection if $H^i_I(R) = 0$ for all $i \neq c = \operatorname{grade}(I)$. Note that cohomologically complete intersections property helps us to decide whether an ideal is set-theoretically complete. As a first step M. Hellus and P. Schenzel (see [5, Theorem 0.1]) have shown that if I is a cohomologically complete intersection in $V(I) \setminus \{\mathfrak{m}\}$ over an n-dimensional local Gorenstein ring R then $H^i_I(R) = 0$ for all $i \neq c = \operatorname{grade}(I)$ if and only if $\dim_k(\operatorname{Ext}^i_R(k, H^c_I(R))) = \delta_{n,i}$. This was the first time that the cohomologically complete intersections property of I is completely encoded in homological properties of the module $H^c_I(R)$. Moreover the above characterization of cohomologically complete intersections looks like a Gorenstein property.

After that several authors have studied this cohomologically complete intersections property. For instance the author and M. Zargar (see [9, Theorem 1.1] and [13, Theorem 1.1]) have generalized this result to a maximal Cohen-Macaulay module of finite injective dimension over a finite dimensional local ring. For an extension to an arbitrary finitely generated R-module we refer to [6, Theorem 4.4].

Here we are succeeded to prove a dual statement:

THEOREM 1.1. *Let R be a local Gorenstein ring of dimension n and I an ideal with $c = \operatorname{grade}(I)$. Then for all $\mathfrak{p} \in V(I)$ the following conditions are equivalent:*

(a) $H^i_I(R) = 0$ *for all $i \neq c$, that is I is a cohomologically complete intersection.*

(b) *The natural homomorphism*

$$\operatorname{Tor}_c^{R_\mathfrak{p}}(E_{R_\mathfrak{p}}(k(\mathfrak{p})), H^c_{IR_\mathfrak{p}}(R_\mathfrak{p}))) \to E_{R_\mathfrak{p}}(k(\mathfrak{p}))$$

is an isomorphism and $\operatorname{Tor}_i^{R_\mathfrak{p}}(E_{R_\mathfrak{p}}(k(\mathfrak{p})), H^c_{IR_\mathfrak{p}}(R_\mathfrak{p})) = 0$ *for all $i \neq c$.*

(c) *The natural homomorphism*

$$E_{R_\mathfrak{p}}(k(\mathfrak{p})) \to H^c_{\mathfrak{p}R_\mathfrak{p}}(\operatorname{Hom}_{R_\mathfrak{p}}(H^c_{IR_\mathfrak{p}}(R_\mathfrak{p}), E_{R_\mathfrak{p}}(k(\mathfrak{p}))))$$

is an isomorphism and $H^i_{\mathfrak{p}R_\mathfrak{p}}(\operatorname{Hom}_{R_\mathfrak{p}}(H^c_{IR_\mathfrak{p}}(R_\mathfrak{p}), E_{R_\mathfrak{p}}(k(\mathfrak{p})))) = 0$ *for all $i \neq c$.*

(d) *The natural homomorphism*

$$\operatorname{Tor}_c^{R_\mathfrak{p}}(k(\mathfrak{p}), H^c_{IR_\mathfrak{p}}(R_\mathfrak{p})) \to k(\mathfrak{p})$$

is an isomorphism and $\operatorname{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}), H^c_{IR_\mathfrak{p}}(R_\mathfrak{p})) = 0$ *for all $i \neq c$.*

In the above Theorem 1.1 $k(\mathfrak{p})$ denotes the residue field of the local Gorenstein ring $R_\mathfrak{p}$ with the injective hull $E_{R_\mathfrak{p}}(k(\mathfrak{p}))$. Moreover the existence of the above natural homomorphisms is shown in Theorem 2.3. The new point of view here is a new characterization of an ideal I to be a cohomologically complete intersection is equivalent to the following property of Betti numbers of $H^c_{IR_\mathfrak{p}}(R_\mathfrak{p})$

$$\dim_{k(\mathfrak{p})}(\operatorname{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}), H^c_{IR_\mathfrak{p}}(R_\mathfrak{p}))) = \delta_{c,i}$$

for all $\mathfrak{p} \in V(I)$.

2. PRELIMINARIES

In this section, we will recall a few preliminaries and auxiliary results. In this paper, we will denote by $(R, \mathfrak{m})$ a commutative Noetherian local ring of finite dimension n with unique maximal ideal $\mathfrak{m}$. Let $E = E_R(k)$ be the injective hull of the residue field $k = R/\mathfrak{m}$. We will denote by $D(\cdot) = \operatorname{Hom}_R(\cdot, E)$ the Matlis dual functor. Moreover, for the basic facts about commutative algebra and homological algebra we refer to [1,3,7] and [12].

Suppose that the homomorphism $X \to Y$ of complexes of R-modules induces an isomorphism in homologies. Then it is called a quasi-isomorphism. In this case we will write it as $X \xrightarrow{\sim} Y$.

In order to derive the natural homomorphisms of the next Theorem 2.3 we need the definition of the truncation complex. The truncation complex is firstly introduce in [11, Definition 4.1]. Let $E_R^{\cdot}(R)$ be a minimal injective resolution of a local Gorenstein ring R of $\dim(R) = n$. Let I be an ideal of R with $\operatorname{grade}(I) = c$. Then there is an exact sequence

$$0 \to H_I^c(R) \to \Gamma_I(E_R^{\cdot}(R))^c \to \Gamma_I(E_R^{\cdot}(R))^{c+1}.$$

Whence there is an embedding of complexes of R-modules $H_I^c(R)[-c] \to \Gamma_I(E_R^{\cdot}(R))$.

Definition 2.1. Let $C_R^{\cdot}(I)$ be the cokernel of the above embedding. This said to be the truncation complex of R with respect to I. Moreover there is a short exact sequence of complexes

$$0 \to H_I^c(R)[-c] \to \Gamma_I(E_R^{\cdot}(R)) \to C_R^{\cdot}(I) \to 0.$$

Note that one can easily see that $H^i(C_R^{\cdot}(I)) = 0$ for all $i \le c$ or $i > n$ and $H^i(C_R^{\cdot}(I)) \cong H_I^i(R)$ for all $c < i \le n$.

As a consequence of the truncation complex the following result was originally proved in [5, Lemma 2.2 and Corollary 2.3]. Note that a generalization of the next result to maximal Cohen-Macaulay modules was given in [9, Theorem 4.3 and Corollary 4.4]. Moreover recently it has also been extended to finitely generated R-modules (see [6, Lemma 4.2]). For the sake of completeness we add it here.

LEMMA 2.2. *Let $(R, \mathfrak{m})$ denote an n-dimensional Gorenstein ring. Let $I \subset R$ be an ideal with $c = \operatorname{grade} I$ and $d = n - c$. Then there are an exact sequence*

$$0 \to H_{\mathfrak{m}}^{n-1}(C_R^{\cdot}(I)) \to H_{\mathfrak{m}}^d(H_I^c(R)) \to E \to H_{\mathfrak{m}}^n(C_R^{\cdot}(I)) \to 0,$$

and isomorphisms $H_{\mathfrak{m}}^{i-c}(H_I^c(R)) \cong H_{\mathfrak{m}}^{i-1}(C_R^{\cdot}(I))$ for all $i \ne n, n+1$. Moreover if in addition $H_I^i(R) = 0$ for all $i \ne c$ then the map $H_{\mathfrak{m}}^d(H_I^c(R)) \to E$ is an isomorphism and $H_{\mathfrak{m}}^i(H_I^c(R)) = 0$ for all $i \ne d$.

Proof. For the proof see [5, Lemma 2.2 and Corollary 2.3]. □

In the following we will obtain some natural homomorphisms with the help of the short exact sequence of the truncation complex. These maps will be used further to investigate the property of cohomologically complete intersection ideals.

THEOREM 2.3. *Let $(R, \mathfrak{m})$ be a Gorenstein ring of dimension n and I an ideal with $c = \operatorname{grade}(I)$. Then we have the following results:*

(a) *There are an exact sequence*

$$0 \to \operatorname{Tor}^R_{-1}(E, C^{\cdot}_R(I)) \to \operatorname{Tor}^R_c(E, H^c_I(R)) \to E \to \operatorname{Tor}^R_0(E, C^{\cdot}_R(I)) \to 0$$

and isomorphisms $\operatorname{Tor}^R_{c-i}(E, H^c_I(R)) \cong \operatorname{Tor}^R_{-(i+1)}(E, C^{\cdot}_R(I))$ *for all* $i \neq 0, -1$.

(b) *There are an exact sequence*

$$0 \to H^0_{\mathfrak{m}}(D(C^{\cdot}_R(I))) \to E \to H^c_{\mathfrak{m}}(D(H^c_I(R))) \to H^1_{\mathfrak{m}}(D(C^{\cdot}_R(I))) \to 0$$

and isomorphisms $H^{c+i}_{\mathfrak{m}}(D(H^c_I(R))) \cong H^{i+1}_{\mathfrak{m}}(D(C^{\cdot}_R(I)))$ *for all* $i \neq 0, -1$.

(c) *There are an exact sequence*

$$0 \to \operatorname{Tor}^R_{-1}(k, C^{\cdot}_R(I)) \to \operatorname{Tor}^R_c(k, H^c_I(R)) \to k \to \operatorname{Tor}^R_0(k, C^{\cdot}_R(I)) \to 0$$

and isomorphisms $\operatorname{Tor}^R_{c-i}(k, H^c_I(R)) \cong \operatorname{Tor}^R_{-(i+1)}(k, C^{\cdot}_R(I))$ *for all* $i \neq 0, -1$.

Proof. Let $F^R_{\cdot}$ be a free resolution of E. Then the short exact sequence of the truncation complex induces the following short exact sequence of complexes of R-modules

$$(2.1) \quad 0 \to (F^R_{\cdot} \otimes_R H^c_I(R))[-c] \to F^R_{\cdot} \otimes_R \Gamma_I(E^{\cdot}_R(R)) \to F^R_{\cdot} \otimes_R C^{\cdot}_R(I) \to 0$$

Now let $\underline{y} = y_1, \ldots, y_r \in I$ such that $\operatorname{Rad} I = \operatorname{Rad}(\underline{y})R$ and $\check{C}_{\underline{y}}$ denote the Čech complex with respect to $\underline{y}$. Since $E^{\cdot}_R(R)$ is a complex of injective R-modules and $F^R_{\cdot}$ is a right bounded complex of flat R-modules. Then by [10, Theorem 3.2] the middle complex is a quasi-isomorphic to $F^R_{\cdot} \otimes_R \check{C}_{\underline{y}} \otimes_R E^{\cdot}_R(R)$.

Moreover $\check{C}_{\underline{y}}$ and $F^R_{\cdot}$ are complexes of flat R-modules so there are the following quasi-isomorphisms

$$F^R_{\cdot} \otimes_R \check{C}_{\underline{y}} \xrightarrow{\sim} F^R_{\cdot} \otimes_R \check{C}_{\underline{y}} \otimes_R E^{\cdot}_R(R), \text{ and}$$
$$F^R_{\cdot} \otimes_R \check{C}_{\underline{y}} \xrightarrow{\sim} E \otimes_R \check{C}_{\underline{y}}.$$

But $\operatorname{Supp}_R(E) = V(\mathfrak{m})$ it follows that $E \otimes_R \check{C}_{\underline{y}} \cong E$. Therefore we get the homologies $H^i(F^R_{\cdot} \otimes_R \Gamma_I(E^{\cdot}_R(R))) = 0$ for all $i \neq 0$ and $H^i(F^R_{\cdot} \otimes_R \Gamma_I(E^{\cdot}_R(R))) = E$ for $i = 0$. Then the statement (a) can be deduced from the homology sequence of the above exact sequence 2.1.

Now apply the functor $D(\cdot) = \operatorname{Hom}_R(\cdot, E)$ to the short exact sequence of the truncation complex then it induces the following short exact sequence

$$0 \to D(C^{\cdot}_R(I)) \to D(\Gamma_I(E^{\cdot}_R(R))) \to D(H^c_I(R))[c] \to 0.$$

Let $F_\cdot(R/\mathfrak{m}^s)$ be a free resolution of $R/\mathfrak{m}^s$ for each $s \in \mathbb{N}$. Apply $\operatorname{Hom}_R(F_\cdot(R/\mathfrak{m}^s), \cdot)$ to the last sequence. Then we get the following exact sequence of complexes:

$$0 \to \operatorname{Hom}_R(F_\cdot(R/\mathfrak{m}^s), D(C_R^\cdot(I))) \to \operatorname{Hom}_R(F_\cdot(R/\mathfrak{m}^s), D(\Gamma_I(E_R^\cdot(R)))) \to \operatorname{Hom}_R(F_\cdot(R/\mathfrak{m}^s), D(H_I^c(R)))[c] \to 0.$$

Now we look at the cohomologies of the complex in the middle. By Hom-Tensor Duality it is isomorphic to $D(F_\cdot(R/\mathfrak{m}^s) \otimes_R \Gamma_I(E_R^\cdot(R)))$ (see [3, Proposition 5.15]). Since the Matlis dual functor $D(\cdot)$ is exact and cohomologies commutes with exact functor. So there is an isomorphism

$$H^i(D(F_\cdot(R/\mathfrak{m}^s) \otimes_R \Gamma_I(E_R^\cdot(R)))) \cong D(H^{-i}(F_\cdot(R/\mathfrak{m}^s) \otimes_R \Gamma_I(E_R^\cdot(R))))$$

for all $i \in \mathbb{Z}$. Then by the proof of (a) we have $D(H^{-i}(F_\cdot(R/\mathfrak{m}^s) \otimes_R \Gamma_I(E_R^\cdot(R)))) = 0$ for each $i \neq 0$ and $D(H^{-i}(F_\cdot(R/\mathfrak{m}^s) \otimes_R \Gamma_I(E_R^\cdot(R)))) = D(R/\mathfrak{m}^s)$ for $i = 0$. Recall that support of $R/\mathfrak{m}^s$ is contained in $V(\mathfrak{m})$. Then by cohomology sequence there are an exact sequence

(2.2)

$$0 \to \operatorname{Hom}_R(R/\mathfrak{m}^s, D(C_R^\cdot(I))) \to D(R/\mathfrak{m}^s) \to \operatorname{Ext}_R^c(R/\mathfrak{m}^s, D(H_I^c(R))) \to \operatorname{Ext}_R^1(R/\mathfrak{m}^s, D(C_R^\cdot(I))) \to 0,$$

and isomorphisms $\operatorname{Ext}_R^{i+1}(R/\mathfrak{m}^s, D(C_R^\cdot(I))) \cong \operatorname{Ext}_R^{i+c}(R/\mathfrak{m}^s, D(H_I^c(R)))$ for all $i \neq 0, -1$ and $s \in \mathbb{N}$. Since the direct limit is an exact functor. So by passing to the direct limit of this gives rise to the following exact sequence

$$0 \to H_\mathfrak{m}^0(D(C_R^\cdot(I))) \to E \to H_\mathfrak{m}^c(D(H_I^c(R))) \to H_\mathfrak{m}^1(D(C_R^\cdot(I))) \to 0,$$

and isomorphisms

$$H_\mathfrak{m}^{c+i}(D(H_I^c(R))) \cong H_\mathfrak{m}^{i+1}(D(C_R^\cdot(I))).$$

for all $i \neq 0, -1$. This proves the statement in (b).

Now let $L_\cdot^R$ denote a free resolution of k. Then the short exact sequence of truncation complex induces the exact sequence

$$(2.3) \quad 0 \to (L_\cdot^R \otimes_R H_I^c(R))[-c] \to L_\cdot^R \otimes_R \Gamma_I(E_R^\cdot(R)) \to L_\cdot^R \otimes_R C_R^\cdot(I) \to 0.$$

Then by the proof of (a) the cohomologies of the middle complex of this last sequence are $H^i(L_\cdot^R \otimes_R \Gamma_I(E_R^\cdot(R))) = 0$ for all $i \neq 0$ and $H^i(L_\cdot^R \otimes_R \Gamma_I(E_R^\cdot(R))) = k$ for $i = 0$. This gives the statement in (c) by virtue of long exact sequence of cohomologies. This finishes the proof of the Theorem. $\square$

We close this section with the the following version of the Local Duality Lemma. The proof of it can be found in [2]. Note that in [4, Theorem 6.4.1], [6, Theorem 3.1], and [8, Lemma 2.4] there is a generalization of it to arbitrary cohomologically complete intersection ideals.

Lemma 2.4. *Let I be an ideal of a Gorenstein ring R with $\dim(R) = n$. Then for any R-module M and for all $i \in \mathbb{Z}$ we have:*

(1) $\operatorname{Tor}^R_{n-i}(M, E) \cong H^i_{\mathfrak{m}}(M)$.

(2) $D(H^i_{\mathfrak{m}}(M)) \cong \operatorname{Ext}^{n-i}_R(M, \hat{R})$. *Here $\hat{R}$ denotes the completion of R with respect to the maximal ideal.*

3. ON COHOMOLOGICALLY COMPLETE INTERSECTIONS

In this section, we will see when the natural homomorphisms, described in Theorem 2.3 above, are isomorphisms. In this regard the first result is obtained in the following Corollary provided that the ideal I is of cohomologically complete intersections. In fact the next result tells us that all the Betti numbers of the module $H^c_I(R)$ vanish except at degree c.

Corollary 3.1. *With the notation of Theorem 2.3 suppose in addition that $H^i_I(R) = 0$ for all $i \neq c$. Then the following are true:*

(a) *The natural homomorphism*

$$\operatorname{Tor}^R_c(E, H^c_I(R)) \to E$$

is an isomorphism and $\operatorname{Tor}^R_i(E, H^c_I(R)) = 0$ for all $i \neq c$

(b) *The natural homomorphism*

$$E \to H^c_{\mathfrak{m}}(D(H^c_I(R)))$$

is an isomorphism and $H^i_{\mathfrak{m}}(D(H^c_I(R))) = 0$ for all $i \neq c$.

(c) *The natural homomorphism*

$$\operatorname{Tor}^R_c(k, H^c_I(R)) \to k$$

is an isomorphism and $\operatorname{Tor}^R_i(k, H^c_I(R)) = 0$ for all $i \neq c$. That is the Betti numbers of $H^c_I(R)$ satisfy

$$\dim_k(\operatorname{Tor}^R_i(k, H^c_I(R))) = \delta_{c,i}.$$

Proof. Since $H^i_I(R) = 0$ for all $i \neq c$ so the complex $C^{\cdot}_R(I)$ is an exact complex (by definition of the truncation complex). Let $F^R_{\cdot}$, $F_{\cdot}(R/\mathfrak{m}^s)$ and $L^{\cdot}_R$ be the free resolutions of E, $R/\mathfrak{m}^s$ and k respectively. Then it follows that all the following complexes are exact:

$$F^R_{\cdot} \otimes_R C^{\cdot}_R(I),$$

$$\operatorname{Hom}_R(F_{\cdot}(R/\mathfrak{m}^s), D(C^{\cdot}_R(I))), \text{ and}$$

$$L^{\cdot}_R \otimes_R C^{\cdot}_R(I).$$

Then from the long exact sequence of cohomologies of the sequences 2.1 and 2.3 we can easily obtain the statements in (a) and (c). Moreover the sequence 2.2 provides that the following natural homomorphism

$$D(R/\mathfrak{m}^s) \to \operatorname{Ext}^c_R(R/\mathfrak{m}^s, D(H^c_I(R)))$$

is an isomorphism and $\operatorname{Ext}^i_R(R/\mathfrak{m}^s, D(H^c_I(R))) = 0$ for all $i \neq c$. Note that the direct limit is an exact functor so by passing to the direct limit we get the statement in (b). Hence the proof of Corollary is complete. □

In order to prove Theorem 3.4 which is one of the main results of this section we need the following result. Note that the next Theorem provides us the equivalence of the natural homomorphisms of Theorem 2.3. Moreover this equivalence related to the vanishing of the Betti numbers of $H^c_I(R)$ for $\operatorname{grade}(I) = c$.

THEOREM 3.2. *Let I be an ideal of an n-dimensional Gorenstein ring R with* $\operatorname{grade}(I) = c$. *Then the following conditions are equivalent:*

(a) *The natural homomorphism*

$$E \to H^c_{\mathfrak{m}}(D(H^c_I(R)))$$

is an isomorphism and $H^i_{\mathfrak{m}}(D(H^c_I(R))) = 0$ for all $i \neq c$.

(b) *The natural homomorphism*

$$\operatorname{Tor}^R_c(k, H^c_I(R)) \to k$$

is an isomorphism and $\operatorname{Tor}^R_i(k, H^c_I(R)) = 0$ for all $i \neq c$.

(c) *The Betti numbers of $H^c_I(R)$ satisfy*

$$\dim_k(\operatorname{Tor}^R_i(k, H^c_I(R))) = \delta_{i,c}.$$

Proof. Note that the equivalence of (b) and (c) is obvious. Now we prove that (a) implies (b). Since $H^i_{\mathfrak{m}}(D(H^c_I(R))) = 0$ for all $i \neq c$. By Theorem 2.3 (b) it follows that $H^i_{\mathfrak{m}}(D(C^{\cdot}_R(I))) = 0$ for all $i \in \mathbb{Z}$. Let $\check{C}_{\underline{x}}$ be the Čech complex with respect to $\underline{x} = x_1, \ldots, x_s \in \mathfrak{m}$ such that $\operatorname{Rad}\mathfrak{m} = \operatorname{Rad}(\underline{x})R$. Then it implies that $\check{C}_{\underline{x}} \otimes_R D(C^{\cdot}_R(I))$ is an exact complex.

Suppose that $F^{\cdot}_R$ denotes a minimal injective resolution of $D(C^{\cdot}_R(I))$. Let us denote $X := \operatorname{Hom}_R(k, F^{\cdot}_R)$ then there is an isomorphism

$$\operatorname{Ext}^i_R(k, D(C^{\cdot}_R(I))) \cong H^i(X)$$

for all $i \in \mathbb{Z}$. We claim that the complex X is homologically trivial. To this end note that the support of each module of X is in $\{\mathfrak{m}\}$. It follows that there is an isomorphism of complexes

$$\check{C}_{\underline{x}} \otimes_R X \cong X.$$

So in order to prove the claim it will be enough to show that $H^i(\check{C}_{\underline{x}} \otimes_R X) = 0$ for all $i \in \mathbb{Z}$. Let $L^R_\cdot$ be a free resolution of k. Then by the above arguments the following complex is exact

$$Y := \operatorname{Hom}_R(L^R_\cdot, \check{C}_{\underline{x}} \otimes_R D(C^\cdot_R(I))).$$

Moreover $L^R_\cdot$ is a right bounded complex of finitely generated free R-modules and $\check{C}_{\underline{x}}$ is a bounded complex of flat R-modules. So by [3, Proposition 5.14] Y is quasi-isomorphic to $\check{C}_{\underline{x}} \otimes_R \operatorname{Hom}_R(L^R_\cdot, D(C^\cdot_R(I)))$. So it is homologically trivial. Since $F^\cdot_R$ is a minimal injective resolution of $D(C^\cdot_R(I))$ so the following morphism of complexes

$$\check{C}_{\underline{x}} \otimes_R \operatorname{Hom}_R(L^R_\cdot, D(C^\cdot_R(I))) \to \check{C}_{\underline{x}} \otimes_R \operatorname{Hom}_R(L^R_\cdot, F^\cdot_R)$$

induces an isomorphism in cohomologies. Moreover we have

$$\check{C}_{\underline{x}} \otimes_R X \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R \operatorname{Hom}_R(L^R_\cdot, F^\cdot_R).$$

By the discussion above the complex on the right side is homologically trivial. It follows that the complex $\check{C}_{\underline{x}} \otimes_R X$ is homologically trivial. This proves the claim. Therefore $\operatorname{Ext}^i_R(k, D(C^\cdot_R(I))) = 0$ for all $i \in \mathbb{Z}$.

By Hom-Tensor Duality $D(\operatorname{Tor}^R_i(k, C^\cdot_R(I))) \cong \operatorname{Ext}^i_R(k, D(C^\cdot_R(I))) = 0$ for all $i \in \mathbb{Z}$. It implies that $\operatorname{Tor}^R_i(k, C^\cdot_R(I)) = 0$ for all $i \in \mathbb{Z}$. Then from Theorem 2.3(c) it follows that the natural homomorphism

$$\operatorname{Tor}^R_c(k, H^c_I(R)) \to k$$

is an isomorphism and $\operatorname{Tor}^R_i(k, H^c_I(R)) = 0$ for all $i \neq c$.

Conversely, note that there is a short exact sequence:

$$0 \to \mathfrak{m}^s/\mathfrak{m}^{s+1} \to R/\mathfrak{m}^{s+1} \to R/\mathfrak{m}^s \to 0.$$

for all $s \in \mathbb{N}$. Now apply the functor $\operatorname{Tor}^R_i(\cdot, C^\cdot_R(I))$ to this sequence then it induces the following exact sequence

$$\operatorname{Tor}^R_i(\mathfrak{m}^s/\mathfrak{m}^{s+1}, C^\cdot_R(I)) \to \operatorname{Tor}^R_i(R/\mathfrak{m}^{s+1}, C^\cdot_R(I)) \to \operatorname{Tor}^R_i(R/\mathfrak{m}^s, C^\cdot_R(I))$$

for all $i \in \mathbb{Z}$ and $s \in \mathbb{N}$. Then by induction on s and Theorem 2.3(c) this proves that $\operatorname{Tor}^R_i(R/\mathfrak{m}^s, C^\cdot_M(I))) = 0$ for all $i \in \mathbb{Z}$ and $s \in \mathbb{N}$. It follows that

$$\operatorname{Ext}^i_R(R/\mathfrak{m}^s, D(C^\cdot_R(I))) \cong D(\operatorname{Tor}^R_i(R/\mathfrak{m}^s, C^\cdot_R(I))) = 0$$

for all $i \in \mathbb{Z}$ and $s \in \mathbb{N}$. Recall that $\mathfrak{m}^s/\mathfrak{m}^{s+1}$ is a finite dimensional k-vector space. Then by the sequence 2.2 the natural homomorphism

$$D(R/\mathfrak{m}^s) \to \operatorname{Ext}^c_R(R/\mathfrak{m}^s, D(H^c_I(R)))$$

is an isomorphism and $\operatorname{Ext}^i_R(R/\mathfrak{m}^s, D(H^c_I(R))) = 0$ for all $i \neq c$. By passing to the direct limit of it we can easily obtain the statement in (a). This completes the proof of Theorem. □

The next Proposition is indeed in the proof of Theorem 3.4. So we will prove it firstly.

PROPOSITION 3.3. *Let R be an n-dimensional local Gorenstein ring. Let I be an ideal with* $\operatorname{grade}(I) = c$ *and* $d := n - c$. *If the natural homomorphism*

$$\operatorname{Tor}_c^R(k, H_I^c(R)) \to k$$

is an isomorphism and $\operatorname{Tor}_i^R(k, H_I^c(R)) = 0$ *for all* $i \neq c$. *Then the following conditions hold:*

(a) *The natural homomorphism*

$$H_{\mathfrak{m}}^d(H_I^c(R)) \to E$$

is an isomorphism and $H_{\mathfrak{m}}^i(H_I^c(R)) = 0$ *for all* $i \neq d$.

(b) *The natural homomorphism*

$$\operatorname{Tor}_c^R(E, H_I^c(R)) \to E$$

is an isomorphism and $\operatorname{Tor}_i^R(E, H_I^c(R)) = 0$ *for all* $i \neq c$.

Proof. Note that by Local Duality Lemma 2.4 (for $M = H_I^c(R)$) it will be enough to prove the statement in (b).

For this let $N_\alpha := D(R/\mathfrak{m}^\alpha)$ for each $\alpha \in \mathbb{N}$. Suppose that $F_.(N_\alpha)$ denote a minimal free resolution of N_α. Since the support of N_α is contained in $\{\mathfrak{m}\}$. Apply the functor $\cdot \otimes_R F_.(N_\alpha)$ to the short exact sequence of the truncation complex. By the proof of Theorem 2.3(a) this gives us the following natural homomorphisms

$$f_\alpha : \operatorname{Tor}_c^R(N_\alpha, H_I^c(R)) \to N_\alpha$$

for all $\alpha \in \mathbb{N}$. Now the short exact sequence $0 \to \mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1} \to R/\mathfrak{m}^{\alpha+1} \to R/\mathfrak{m}^\alpha \to 0$ induces the exact sequence

$$0 \to N_\alpha \to N_{\alpha+1} \to D(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}) \to 0$$

Moreover after the application of the functor $\operatorname{Tor}_i^R(\cdot, H_I^c(R))$ to this sequence we get the exact sequence

$$\operatorname{Tor}_i^R(N_\alpha, H_I^c(R)) \to \operatorname{Tor}_i^R(N_{\alpha+1}, H_I^c(R)) \to \operatorname{Tor}_i^R(D(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}), H_I^c(R))$$

for all $i \in \mathbb{Z}$. Since $N_1 = D(R/\mathfrak{m}) \cong k$. Then by induction on α and the vanishing of Tor modules this proves that $\operatorname{Tor}_i^R(N_\alpha, H_I^c(R)) = 0$ for all $i \neq c$ and $\alpha \in \mathbb{N}$.

Now we show that f_α is an isomorphism for all $\alpha \in \mathbb{N}$. Clearly f_1 is an isomorphism (by assumption). Then there is the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & \operatorname{Tor}_c^R(N_\alpha, H_I^c(M)) & \to & \operatorname{Tor}_c^R(N_{\alpha+1}, H_I^c(M)) & \to & \operatorname{Tor}_c^R(D(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}), H_I^c(M)) & \to & 0 \\ & & \downarrow f_\alpha & & \downarrow f_{\alpha+1} & & \downarrow f & & \\ 0 & \to & N_\alpha & \to & N_{\alpha+1} & \to & D(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}) & \to & 0 \end{array}$$

Note that the above row is exact because of $\operatorname{Tor}_i^R(N_\alpha, H_I^c(R)) = 0$ for all $i \neq c$ and $\alpha \in \mathbb{N}$. Moreover $\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}$ is a finite dimensional k-vector space. Then the natural homomorphism f is an isomorphism. Hence by induction and Snake lemma it implies that f_α is an isomorphism for all $\alpha \in \mathbb{N}$. Taking the direct limits of f_α induces the isomorphism

$$\operatorname{Tor}_c^R(E, H_I^c(R)) \to E.$$

Since the direct limit commutes with Tor functor and $\operatorname{Supp}_R(E) \subseteq V(\mathfrak{m})$. Moreover note that the vanishing of the above Tor modules implies that $\operatorname{Tor}_i^R(E, H_I^c(R)) = 0$ for all $i \neq c$. $\square$

Now we are able to prove our main result. Before proving it we will make some notation. Let R be a Gorenstein ring of dimension n and I an ideal with $c = \operatorname{grade}(I)$. We will denote by $k(\mathfrak{p})$ the residue field of $R_\mathfrak{p}$ with the injective hull $E_{R_\mathfrak{p}}(k(\mathfrak{p}))$ for $\mathfrak{p} \in V(I)$. Moreover we set $h(\mathfrak{p}) = \dim(R_\mathfrak{p}) - c$.

THEOREM 3.4. *Fix the previous notation. Then for all $\mathfrak{p} \in V(I)$ the following conditions are equivalent:*

(a) $H_I^i(R) = 0$ *for all $i \neq c$, that is I is a cohomologically complete intersection.*

(b) *The natural homomorphism*

$$H_{\mathfrak{p}R_\mathfrak{p}}^{h(\mathfrak{p})}(H_{IR_\mathfrak{p}}^c(R_\mathfrak{p})) \to E_{R_\mathfrak{p}}(k(\mathfrak{p}))$$

is an isomorphism and $H_{\mathfrak{p}R_\mathfrak{p}}^i(H_{IR_\mathfrak{p}}^c(R_\mathfrak{p})) = 0$ *for all $i \neq h(\mathfrak{p})$.*

(c) *The natural homomorphism*

$$\operatorname{Tor}_c^{R_\mathfrak{p}}(E_{R_\mathfrak{p}}(k(\mathfrak{p})), H_{IR_\mathfrak{p}}^c(R_\mathfrak{p}))) \to E_{R_\mathfrak{p}}(k(\mathfrak{p}))$$

is an isomorphism and $\operatorname{Tor}_i^{R_\mathfrak{p}}(E_{R_\mathfrak{p}}(k(\mathfrak{p})), H_{IR_\mathfrak{p}}^c(R_\mathfrak{p})) = 0$ *for all $i \neq c$.*

(d) *The natural homomorphism*

$$E_{R_\mathfrak{p}}(k(\mathfrak{p})) \to H_{\mathfrak{p}R_\mathfrak{p}}^c(\operatorname{Hom}_{R_\mathfrak{p}}(H_{IR_\mathfrak{p}}^c(R_\mathfrak{p}), E_{R_\mathfrak{p}}(k(\mathfrak{p}))$$

is an isomorphism and $H_{\mathfrak{p}R_\mathfrak{p}}^i(\operatorname{Hom}_{R_\mathfrak{p}}(H_{IR_\mathfrak{p}}^c(R_\mathfrak{p}), E_{R_\mathfrak{p}}(k(\mathfrak{p})))) = 0$ *for all $i \neq c$.*

(e) *The natural homomorphism*

$$\operatorname{Tor}_c^{R_\mathfrak{p}}(k(\mathfrak{p}), H_{IR_\mathfrak{p}}^c(R_\mathfrak{p})) \to k(\mathfrak{p})$$

is an isomorphism and $\operatorname{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}), H_{IR_\mathfrak{p}}^c(R_\mathfrak{p})) = 0$ *for all $i \neq c$.*

(f) *The Betti numbers of $H_{IR_\mathfrak{p}}^c(R_\mathfrak{p})$ satisfy*

$$\dim_{k(\mathfrak{p})}(\operatorname{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}), H_{IR_\mathfrak{p}}^c(R_\mathfrak{p}))) = \delta_{c,i}$$

Proof. We firstly prove that the statements in (a) and (b) are equivalent. Suppose that $H^i_I(R) = 0$ for all $i \neq c$. Then by [9, Proposition 2.7] it follows that $H^i_{\mathfrak{p}R_\mathfrak{p}}(H^c_{IR_\mathfrak{p}}(R_\mathfrak{p})) = 0$ for all $i \neq c = \operatorname{grade}(IR_\mathfrak{p})$ and $\mathfrak{p} \in V(I)$. So the result follows from Lemma 2.2.

Conversely, suppose that the statement in (b) is true. We use induction on $\dim_R(R/IR)$. Let $\dim_R(R/IR) = 0$ then it follows that $\operatorname{Rad}(IR) = \mathfrak{m}$. This proves the result since R is Gorenstein. Now let us assume that $\dim(R/IR) > 0$. Then it is easy to see that $\dim(R_\mathfrak{p}/IR_\mathfrak{p}) < \dim(R/IR)$ for all $\mathfrak{p} \in V(I) \backslash \{\mathfrak{m}\}$.

Moreover by the induction hypothesis for all $i \neq c$ and $\mathfrak{p} \in V(I) \setminus \{\mathfrak{m}\}$ we have

$$H^i_{IR_\mathfrak{p}}(R_\mathfrak{p}) = 0.$$

That is $\operatorname{Supp}(H^i_I(R)) \subseteq V(\mathfrak{m})$ for all $i \neq c$. Since our assumption is true for $\mathfrak{p} = \mathfrak{m}$. So by Lemma 2.2 it follows that $H^i_\mathfrak{m}(C^\cdot_R(I)) = 0$ for all $i \in \mathbb{Z}$.

Since $\operatorname{Supp}_R(H^i(C^\cdot_M(I))) \subseteq V(\mathfrak{m})$. So by [9, Lemma 2.5] in view of definition of the truncation complex we have

$$0 = H^i_\mathfrak{m}(C^\cdot_R(I)) \cong H^i(C^\cdot_R(I)) \cong H^i_I(R)$$

for all $c < i \leq n$. That is $H^i_I(R) = 0$ for all $i \neq c$. This completes the proof of the equivalence of (a) and (b).

Similarly we can prove that the statements in (b) and (c) are equivalent. Moreover by passing to the localization the statements in (c), (d) and (e) are equivalent (see Theorem 3.2 and Proposition 3.3). Finally the equivalence of the statements in (e) and (f) is obvious. This completes the equivalence of all the statements of Theorem. $\square$

Remark 3.5. Note that in Theorem 3.4 above one should have localization with respect to all $\mathfrak{p} \in V(I)$. To prove this let $R = k[|x_0, x_1, x_2, x_3, x_4|]$ denote the formal power series ring over any filed k. Suppose that $I = (x_0, x_1) \cap (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4)$. Then Hellus and Schenzel (see [5, Example 4.1]) proved that $\dim_k \operatorname{Ext}^i_R(k, H^2_I(R)) = \delta_{3,i}$ and $H^i_I(R) \neq 0$ for all $i \neq 2, 3$ with $\operatorname{grade}(I) = 2$.

Then by [9, Corollary 4.7] it follows that the natural homomorphism $H^d_\mathfrak{m}(H^c_I(R)) \to E$ is an isomorphism and $H^i_\mathfrak{m}(H^c_I(R)) = 0$ for all $i \neq 3$. This shows that the local conditions are necessary in order to prove Theorem 3.4.

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