COMMUTATORS HAVING IDEMPOTENT VALUES WITH AUTOMORPHISMS IN SEMI-PRIME RINGS

MOHAMMAD ASHRAF, MOHD ARIF RAZA and SAJAD AHMAD PARY

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In the present paper, it is shown that a semi-prime ring $\mathcal{R}$ of characteristic not 2 and 3 contains a non-zero central ideal of $\mathcal{R}$, if $\mathcal{R}$ admits an automorphism ζ such that $[s^\zeta, w]^m = [s^\zeta, w]$ for every $s, w \in \mathcal{R}$, where $1 < m \in \mathbb{Z}^+$. We shall also study the case when the underlying condition holds for the elements from a non-cental Lie ideal of a prime ring $\mathcal{R}$. The latter result is in the spirit of Herstein's theorem which deals with the commutator having idempotent values in rings.

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1. INTRODUCTION

In the mid forties of the twentieth century, after the development of the general structure theory for rings, a great deal of work was done that showed that under certain types of hypotheses, rings had to be commutative or almost commutative. A classical result of ring theory established by Jacobson states that if every element s of a ring $\mathcal{R}$ satisfies the condition $s^m = s$, where $1 < m \in \mathbb{Z}^+$, then $\mathcal{R}$ is commutative. This result generalizes the theorem of Wedderburn that every finite division ring is commutative and also the result that every Boolean ring is a commutative ring. Then much significant work in this area was done by I. N. Herstein alone by proving some classical commutativity theorems.

In [7], Herstein established that a ring $\mathcal{R}$ must be commutative if it satisfies $[s, w]^m = [s, w]$ for every $s, w \in \mathcal{R}$, where $1 < m \in \mathbb{Z}^+$. Inspired by the above mentioned result, a lot of work has been done to investigate the circumstances under which a ring becomes commutative, for instance, generalizing Herstein's identities, using restrictions on polynomials, introducing derivations, skew-derivations and generalized derivations on rings, looking for special properties for rings, etcetera. For ongoing contributions in this direction see [16] and references therein.

It is possible to constitute several problems by taking suitable conditions on the subset $\mathcal{K}(\mathcal{T},\mathcal{Y}) = \{[\mathcal{T}(s), s]^n : s \in \mathcal{Y}\}$, where $\mathcal{Y}$ is an appropriate subset of $\mathcal{R}$, while $\mathcal{T}$ is an additive map on $\mathcal{R}$ and $1 < m \in \mathbb{Z}^+$. In 2000, Carini and De Filippis [3] established that a prime ring $\mathcal{R}$ of characteristic not 2 is commutative if it satisfies $\mathcal{K}(\partial,\mathcal{Y}) = 0$, where ∂ is a non-zero derivation of $\mathcal{R}$ and $\mathcal{Y} = \mathcal{L}$. Recently, Wang [19] studied the similar identities for an automorphism ζ of a prime ring $\mathcal{R}$. In particular, he investigated the nature of an automorphism ζ satisfying $\mathcal{K}(\zeta,\mathcal{L}) = 0$. Many researchers have studied similar differential/functional identities on various ways (see [11, 12, 14, 15, 17–19] and references therein). In view of the above motivation it is genuine to discuss the case when $\mathcal{K}(\zeta,\mathcal{L}) \neq 0$, *i.e.*, commutator having idempotent values with an automorphism ζ of a ring $\mathcal{R}$. In this manuscript, we investigate the above mentioned problem and prove the following:

THEOREM 1.1. *Let $1 < m \in \mathbb{Z}^+$ and $\mathcal{R}$ be a semi-prime ring of characteristic not 2 and 3. If $\mathcal{R}$ admits an automorphism ζ such that $[s^\zeta, w]^m = [s^\zeta, w]$ for every $s, w \in \mathcal{R}$, then $\mathcal{R}$ contains a non-zero central ideal.*

2. PRELIMINARIES AND RESULTS

For the sake of completeness we shall begin with few preliminary notions which are required for the development of the proof of our main theorem. Some of these notions are classical and we present them briefly, $\mathcal{R}$ is a (semi-)prime ring and $\mathcal{Q} = \mathcal{Q}_{mr}(\mathcal{R})$ is the maximal right ring of quotients of $\mathcal{R}$. Also we know that any automorphism of $\mathcal{R}$ can be uniquely extended to an automorphism of $\mathcal{Q}$. An automorphism ζ of $\mathcal{R}$ is called $\mathcal{Q}$-inner if there exists an invertible element $g \in \mathcal{Q}$ such that $s^\zeta = gsg^{-1}$ for every $s \in \mathcal{R}$. Otherwise, ζ is called $\mathcal{Q}$-outer. To facilitate our discussion, we begin with the following known facts.

FACT 2.1 ([5, Theorem 1]). *Let $\mathcal{R}$ be a prime ring and $\mathcal{I}$ be a two sided ideal of $\mathcal{R}$. Then $\mathcal{I}, \mathcal{R}$ and $\mathcal{Q}$ satisfy the same generalized polynomial identities (GPIs) with automorphisms.*

FACT 2.2 ([9]). *Let $\mathcal{R}$ be a domain and ζ be an automorphism of $\mathcal{R}$ which is outer. If $\mathcal{R}$ satisfies a GPI $\Psi(s_i, s_i^\zeta)$, then $\mathcal{R}$ also satisfies the nontrivial GPI $\Psi(s_i, w_i)$, where s_i and w_i are distinct indeterminates.*

FACT 2.3 ([1, Lemma 7.1]). *Let $\mathcal{V}_\mathcal{D}$ be a vector space over a division ring $\mathcal{D}$ with $dim\mathcal{V}_\mathcal{D} \geq 2$ and $\mathcal{S} \in End(\mathcal{V})$. If s and $\mathcal{S}s$ are $\mathcal{D}$-dependent for every $s \in \mathcal{V}$, then there exists $\chi \in \mathcal{D}$ such that $\mathcal{S}s = \chi s$ for every $s \in \mathcal{V}$.*

Fact 2.4 ([5, Theorem 3]). *Suppose that $\mathcal{R}$ is a prime ring and $\mathcal{A}$ an independent subset of $\mathcal{G}$ modulo $\mathcal{H}_i$. Let $\phi = \Phi(s_i^{a_j}) = 0$ be a generalized identity with automorphisms of $\mathcal{R}$ reduced with respect to $\mathcal{A}$. If for every $s_i \in X$, $a_j \in \mathcal{G}$, the $s_i^{a_j}$-word degree of $\phi = \Phi(s_i^{a_j})$ is strictly less than $char(\mathcal{R})$ when $char(\mathcal{R}) \neq 0$, then $\Phi(z_{ij}) = 0$ is also a GPI of $\mathcal{R}$.*

Let $\mathcal{V}_{\mathcal{D}}$ be a right vector space over a division ring $\mathcal{D}$. We denote by $End(\mathcal{V}_{\mathcal{D}})$ the ring of $\mathcal{D}$-linear transformations on $\mathcal{V}_{\mathcal{D}}$. A map $\mathcal{S} : \mathcal{V} \to \mathcal{V}$ is called a semi-linear transformation if $\mathcal{S}$ is additive and there is an automorphism ζ of $\mathcal{D}$ such that $\mathcal{S}(v\xi) = (\mathcal{S}v)\zeta(\xi)$ for every $v \in \mathcal{V}$ and $\xi \in \mathcal{D}$. By a theorem of Jacobson [8, Isomorphism Theorem, p.79], $s^{\zeta} = \mathcal{S}s\mathcal{S}^{-1}$ for every $s \in End(\mathcal{V}_{\mathcal{D}})$, where ζ is an automorphism of $End(\mathcal{V}_{\mathcal{D}})$ and $\mathcal{S}$ is the invertible semi-linear transformation.

We start with the following propositions which are necessary for the establishment of our main theorem.

Proposition 2.1. *Let $1 < m \in \mathbb{Z}^+$ and ζ be an automorphism of $End(\mathcal{V}_{\mathcal{D}})$ such that $[[s,w]^{\zeta},[z,t]]^m = [[s,w]^{\zeta},[z,t]]$, for every $s,w,z,t \in End(\mathcal{V}_{\mathcal{D}})$. If $dim(\mathcal{V}_{\mathcal{D}}) \geq 2$, then ζ is identity map of $End(\mathcal{V}_{\mathcal{D}})$.*

Proof. As remarked above, we have $s^{\zeta} = \mathcal{S}s\mathcal{S}^{-1}$ for every $s \in End(\mathcal{V}_{\mathcal{D}})$. Also, for every $v \in \mathcal{V}$, $\xi \in \mathcal{D}$, $\mathcal{S}(v\xi) = (\mathcal{S}v)\zeta(\xi)$. Using our hypotheses, we find that

$$\begin{aligned} 0 &= [[s,w]^{\zeta},[z,t]]^m - [[s,w]^{\zeta},[z,t]] \\ &= [\mathcal{S}[s,w]\mathcal{S}^{-1},[z,t]]^m - [\mathcal{S}[s,w]\mathcal{S}^{-1},[z,t]] \end{aligned}$$

for every $s,w,z,t \in End(\mathcal{V}_{\mathcal{D}})$. We divide our proof into the following cases:

First we assume that $\{v, \mathcal{S}v, \mathcal{S}^{-1}v\}$ is $\mathcal{D}$-independent. For this, let $s,w,z,t \in End(\mathcal{V}_{\mathcal{D}})$ such that

$$\begin{array}{lllll} sv = 0, & s\mathcal{S}^{-1}v = v, & s\mathcal{S}v = v, & zv = v, & z\mathcal{S}v = 0; \\ wv = v, & w\mathcal{S}^{-1}v = 0, & w\mathcal{S}v = 0, & tv = 0, & t\mathcal{S}v = 2v. \end{array}$$

We see that $[s,w]\mathcal{S}^{-1}v = -v$, $[z,t]\mathcal{S}v = 2v$, $[z,t]v = 0$ and hence, by the main assumption we get

$$0 = ([\mathcal{S}[s,w]\mathcal{S}^{-1},[z,t]]^m - [\mathcal{S}[s,w]\mathcal{S}^{-1},[z,t]])v = (2^m - 2)v \neq 0, \text{ a contradiction.}$$

Next we suppose that $\{v, \mathcal{S}v, \mathcal{S}^{-1}v\}$ is $\mathcal{D}$-dependent. Thus, for $\mu, \vartheta \in \mathcal{D}$ we can write $\mathcal{S}^{-1}v = v\mu + \mathcal{S}v\vartheta$. We have to prove that $\vartheta \neq 0$, otherwise if $\vartheta = 0$, then we get a contradiction as $\mathcal{S}^{-1}v = v\mu$ implies $v = \mathcal{S}^{-1}v\mu$. Now we take $s,w,z,t \in End(\mathcal{V}_{\mathcal{D}})$ such that

$$\begin{array}{llll} sv = 0, & s\mathcal{S}v = v, & zv = v, & z\mathcal{S}v = 0; \\ wv = v, & w\mathcal{S}v = 0, & tv = 0, & t\mathcal{S}v = 2v. \end{array}$$

We can easily see that $0 = ([\mathcal{S}[s,w]\mathcal{S}^{-1},[z,t]]^m - [\mathcal{S}[s,w]\mathcal{S}^{-1},[z,t]])v = (2^m\vartheta^m - 2\vartheta)v$, a contradiction.

Therefore v and $\mathcal{S}^{-1}v$ are $\mathcal{D}$-dependent for every $v \in \mathcal{V}$. By Fact 2.3, we have $\mathcal{S}^{-1}v = v\chi$, where $\chi \in \mathcal{D}$ and $v \in \mathcal{V}$. Thus for every $s \in End(\mathcal{V}_{\mathcal{D}})$, $\mathcal{S}^{-1}(sv) = sv\chi$ and hence $sv = \mathcal{S}(sv\chi) = \mathcal{S}(s(v\chi)) = \mathcal{S}s\mathcal{S}^{-1}(v) = s^{\zeta}v$ for every $s \in End(\mathcal{V}_{\mathcal{D}})$, $v \in \mathcal{V}$. Moreover, we find that $(s^{\zeta} - s)\mathcal{V} = (0)$ for every $s \in End(\mathcal{V}_{\mathcal{D}})$. Thus in all, $s^{\zeta} = s$ for every $s \in End(\mathcal{V}_{\mathcal{D}})$. This shows that ζ is the identity map of $End(\mathcal{V}_{\mathcal{D}})$. $\square$

PROPOSITION 2.2. *Let $1 < m \in \mathbb{Z}^+$ and $\mathcal{R}$ be a prime ring of characteristic not 2 and 3. If ζ is an outer automorphism of $\mathcal{R}$ such that $[[s,w]^{\zeta},[z,t]]^m = [[s,w]^{\zeta},[z,t]]$ for every $s,w,z,t \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Proof. Let ζ be the identity map on $\mathcal{R}$. Then $[[s,w],[z,t]]^m = [[s,w],[z,t]]$ for every $s,w,z,t \in \mathcal{R}$, *i.e.*, $\mathcal{R}$ is a polynomial identity (PI) ring. Thus, $\mathcal{R}$ and $\mathcal{M}_k(\mathcal{F})$ satisfy the same polynomial identities [10, Lemma 1], *i.e.*, for each $s,w,z,t \in \mathcal{M}_k(\mathcal{F})$, $[[s,w],[z,t]]^m = [[s,w],[z,t]]$. Let $k \geq 2$ and e_{ij} be the usual unit matrix. Then for $s = e_{12}, w = e_{21}, z = e_{11}$ and $t = e_{12}$, we get a contradiction $2e_{12} = 0$. Thus $k = 1$ and we get the required conclusion.

Next we suppose that ζ is a non-identity map. Therefore $[[s,w]^{\zeta},[z,t]]^m = [[s,w]^{\zeta},[z,t]]$ is a non-trivial GPI for $\mathcal{R}$ [4, Main Theorem]. Also, by Fact 2.1 , $\mathcal{Q}$ satisfies $[[s,w]^{\zeta},[z,t]]^m = [[s,w]^{\zeta},[z,t]]$. Moreover, $\mathcal{Q}$ is a primitive ring which is isomorphic to a dense ring of linear transformations of some vector space $\mathcal{V}$ over a division ring $\mathcal{D}$ [13, Theorem 3].

In case $\mathcal{Q}$ is a domain, $\mathcal{Q}$ satisfies the GPIs $[[y,x],[z,t]]^m = [[y,x],[z,t]]$ (Fact 2.2). By using the same argument as above, we find that $\mathcal{Q}$ is commutative and hence $\mathcal{R}$ is also commutative.

From now on, $\mathcal{Q}$ is not a domain. As mentioned above, we have $s^{\zeta} = \mathcal{S}s\mathcal{S}^{-1}$ for every $s \in \mathcal{Q}$. Therefore, we have $[\mathcal{S}[s,w]\mathcal{S}^{-1},[z,t]]^m = [\mathcal{S}[s,w]\mathcal{S}^{-1},[z,t]]$ for every $s,w,z,t \in \mathcal{Q}$. We realise that, if for any $v \in \mathcal{V}$ there exists $\vartheta \in \mathcal{D}$ such that $\mathcal{S}^{-1}v = v\vartheta$. In this case, by Proposition 2.1 we get a contradiction that ζ is the identity map. Therefore, there exists $v \in \mathcal{V}$ such that v and $\mathcal{S}^{-1}v$ are linearly $\mathcal{D}$-independent. In this case, first we take $dim\mathcal{V}_{\mathcal{D}} \geq 3$. Let $\{u, v, \mathcal{S}^{-1}v\}$ be linearly $\mathcal{D}$-independent for $u \in \mathcal{V}$. Therefore, by the density of $\mathcal{Q}$, there exist $s,w,z,t \in \mathcal{Q}$ such that

$$\begin{array}{llll} zv = v, & s\mathcal{S}^{-1}v = u, & yu = \mathcal{S}^{-1}u, & zu = 0; \\ tv = 0, & w\mathcal{S}^{-1}v = 0, & xv = 0, & tu = 2v. \end{array}$$

Therefore, we get

$$0 = ([\mathcal{S}[s,w]\mathcal{S}^{-1},[z,t]]^m - [\mathcal{S}[s,w]\mathcal{S}^{-1},[z,t]])v = (2^m - 2)v \neq 0,$$

again a contradiction.

Finally, we consider the case when $dim\mathcal{V}_{\mathcal{D}} = 2$, *i.e.*, $\mathcal{Q} = \mathcal{M}_2(\mathcal{D})$. Thus, $[[s,w]^{\zeta},[z,t]]^m - [[s,w]^{\zeta},[z,t]] = 0$ for every $s,w,z,t \in \mathcal{Q}$. Since s^{ζ}-word degree is 2 and $char(\mathcal{R}) > 3$, by Fact 2.4, $[[y,x],[z,t]]^m - [[y,x],[z,t]] = 0$ for every

$s, w, t, z \in \mathcal{Q}$, which leads to the conclusion that $\mathcal{Q}$ is commutative (by using the same argument presented above), and hence $\mathcal{R}$ is commutative. This completes the proof. $\square$

THEOREM 2.1. *Let $1 < m \in \mathbb{Z}^+$ and $\mathcal{R}$ be a prime ring of characteristic different from 2 and 3 and $\mathcal{L}$ be a non-central Lie ideal of $\mathcal{R}$. If ζ is an automorphism of R such that $[u^\zeta, v]^m = [u^\zeta, v]$ for every $u, v \in \mathcal{L}$, then R is commutative.*

Proof. Since $\mathcal{L}$ is non-central Lie ideal of $\mathcal{R}$, there exists an ideal $\mathcal{I}$ of $\mathcal{R}$ such that $0 \neq [\mathcal{I}, \mathcal{R}] \subseteq \mathcal{L}$. Thus by the given hypotheses, $\mathcal{I}$ as well as $\mathcal{R}$ (Fact 2.1) satisfy $[[s, w]^\zeta, [z, t]]^m = [[s, w]^\zeta, [z, t]]$. Taking into consideration Proposition 2.2, we get the desire conclusion when ζ will be an outer automorphism of $\mathcal{R}$. Hence from now on, we assume that ζ is an inner automorphism, *i.e.*, $s^\zeta = psp^{-1}$ for every $s \in \mathcal{R}$. If $p \in \mathcal{C}$, then ζ is the identity map and we have nothing to prove. Assume that $p \notin \mathcal{C}$. Then

$$\Phi(r) = [p[s, w]p^{-1}, [z, t]]^m - [p[s, w]p^{-1}, [z, t]] = 0$$

is a non-trivial GPI of $\mathcal{R}$ and hence $\mathcal{Q}$ as well.

Assume that $\mathcal{F}$ is the algebraic closure of $\mathcal{C}$ when $\mathcal{C}$ is infinite and $\mathcal{F} = \mathcal{C}$ when $\mathcal{C}$ finite. Therefore $\mathcal{Q} \otimes_{\mathcal{C}} \mathcal{F}$ is a prime ring with extended centroid $\mathcal{F}$ [6, Theorem 3.5]. Clearly $\mathcal{Q} \cong \mathcal{Q} \otimes_{\mathcal{C}} \mathcal{C} \subseteq \mathcal{Q} \otimes_{\mathcal{C}} \mathcal{F}$. Thus we can have $\mathcal{Q}$ is a subring of $\mathcal{Q} \otimes_{\mathcal{C}} \mathcal{F}$ and in all $\Phi(r)$ is a non-trivial GPI of $\mathcal{Q} \otimes_{\mathcal{C}} \mathcal{F}$. If $\breve{\mathcal{Q}} = \mathcal{Q}_{mr}(\mathcal{Q} \otimes_{\mathcal{C}} \mathcal{F})$, then $\Phi(r)$ is also a non-trivial GPI on $\breve{\mathcal{Q}}$ ([2, Theorem 6.4.4]). Moreover, we get $\breve{\mathcal{Q}} \cong End(\mathcal{V}_{\mathcal{D}})$ (see Martindale's theorem [13]). As we have already mentioned above either $\mathcal{F}$ is algebraically closed or finite. Therefore $\mathcal{D} = \mathcal{F}$, when $\mathcal{D}$ is finite over $\mathcal{F}$. Thus in all $\breve{\mathcal{Q}} \cong End(\mathcal{V}_{\mathcal{F}})$. By Proposition 2.1, we get the required conclusion. $\square$

Keeping in mind Theorem 2.1, we can write the following corollary

COROLLARY 2.1. *Let $1 < m \in \mathbb{Z}^+$ and $\mathcal{R}$ be a prime ring of characteristic not 2 and 3. If ζ is an automorphism of $\mathcal{R}$ such that $[s^\zeta, w]^m = [s^\zeta, w]$ for every $s, w \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Now we provide an example which shows that the same conclusion does not hold for semi-prime ring.

Example 2.1. Let $\mathcal{R} = \mathcal{M}_2(\mathcal{F}) \oplus \mathcal{M}_2(\mathcal{F})$ and $\mathcal{L} = \mathcal{M}_2(\mathcal{F}) \oplus 0$. We see that $\mathcal{L}$ is a non-zero Lie ideal of a semi-prime ring $\mathcal{R}$. Now we construct a mapping $\zeta : \mathcal{R} \to \mathcal{R}$ such that $(s_1, s_2)^\zeta = (s_2, s_1)$. It is easy to verify that that ζ satisfies the hypotheses of Theorem 2.1, *i.e.*, $[u^\zeta, v]^m = [u^\zeta, v]$ for every $u, v \in \mathcal{L}$, where $1 < m \in \mathbb{Z}^+$.

PROPOSITION 2.3. *Let $1 < m \in \mathbb{Z}^+$ and $\mathcal{R}$ be a prime ring of characteristic not 2 and 3 and let ζ be an epimorphism of $\mathcal{R}$ but not a monomorphism. If $\mathcal{R}$ satisfies $[s^\zeta, w]^m = [s^\zeta, w]$ for every $s, w \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Proof. By the given hypotheses, we have $[s^\zeta, w]^m = [s^\zeta, w]$ for every $s, w \in \mathcal{R}$. Let $\mathcal{X}$=Kerζ. Then $\mathcal{X}$ is a non-zero ideal of $\mathcal{R}$. For $s, w \in \mathcal{R}$ and $\kappa \in \mathcal{X}$ we have,

$$\begin{aligned} 0 &= [(s+\kappa)^\zeta, w+\kappa]^m - [(s+\kappa)^\zeta, w+\kappa] \\ &= [s^\zeta, w+\kappa]^m - [s^\zeta, w+\kappa]. \end{aligned}$$

It is well known that $\mathcal{X}$ and $\mathcal{R}$ satisfy the same GPIs [2, Theorem 6.4.4], and hence $[s^\zeta, w+\kappa]^m - [s^\zeta, w+\kappa] = 0$ for every $s, w, \kappa \in \mathcal{R}$. Replacing κ with $\kappa - w$ in the latter identity, we obtain $[s^\zeta, w]^m - [s^\zeta, w] = 0$ for every $s, w \in \mathcal{R}$. In the light of the above discussion and by Corollary 2.1, we get the desired result. □

Now we are in position to prove our main result.

Proof of Theorem 1.1. Let $\mathcal{W}$ be a prime ideal of $\mathcal{R}$, set $\widehat{\mathcal{R}} = \mathcal{R}/\mathcal{W}$ and write $\widehat{s} = s + \mathcal{W} \in \widehat{\mathcal{R}}$ for every $s \in \mathcal{R}$.

Firstly we assume that $\mathcal{W}^\zeta \not\subseteq \mathcal{W}$. For $s, w \in \mathcal{R}$ and $\mathcal{X} \in \mathcal{W}$,

$$\widehat{0} = [\widehat{(s+\mathcal{X})^\zeta, w} + \mathcal{X}]^m - [\widehat{(s+\mathcal{X})^\zeta, w} + \mathcal{X}] = [\widehat{s^\zeta} + \widehat{\mathcal{X}^\zeta}, \widehat{w}]^m - [\widehat{s^\zeta} + \widehat{\mathcal{X}^\zeta}, \widehat{w}].$$

Thus $[\widehat{s^\zeta} + \widehat{z}, \widehat{w}]^m - [\widehat{s^\zeta} + \widehat{z}, \widehat{w}] = \widehat{0}$ for every $s, w \in \mathcal{R}$ and $z \in \mathcal{W}^\zeta$. Since $\mathcal{W}^\zeta \not\subseteq \mathcal{W}$, $\widehat{\mathcal{W}^\zeta} = (\mathcal{W}^\zeta + \mathcal{W})/\mathcal{W}$ is a non-zero ideal of the prime ring $\widehat{\mathcal{R}}$. By [2, Theorem 6.4.4], $[\widehat{s^\zeta} + \widehat{z}, \widehat{w}]^m - [\widehat{s^\zeta} + \widehat{z}, \widehat{w}] = \widehat{0}$ for every $s, w, z \in \mathcal{R}$. Replacing z by $z - s^\zeta$, we obtain $[\widehat{z}, \widehat{w}]^m - [\widehat{z}, \widehat{w}] = \widehat{0}$ for every $w, z \in \mathcal{R}$. This shows that $\widehat{\mathcal{R}}$ is commutative by Corollary 2.1. So $[\widehat{\mathcal{R}}, \widehat{\mathcal{R}}] = \widehat{0}$, and hence, $[\mathcal{R}, \mathcal{R}] \subseteq \mathcal{W}$.

Assume next that $\mathcal{W}^\zeta \subseteq \mathcal{W}$. Define $\widehat{\zeta} : \widehat{\mathcal{R}} \to \widehat{\mathcal{R}}$ by $\widehat{s}^{\widehat{\zeta}} = \widehat{s^\zeta}$. Then $\widehat{\zeta}$ is an epimorphism of $\widehat{\mathcal{R}}$. Then $\widehat{0} = \widehat{[s^\zeta, w]^m} - \widehat{[s^\zeta, w]} = [\widehat{s^\zeta}, \widehat{w}]^m - [\widehat{s^\zeta}, \widehat{w}]$ for all $s, w \in \mathcal{R}$. By Corollary 2.1 and Proposition 2.3, $\widehat{\mathcal{R}}$ is commutative, that is, $[\mathcal{R}, \mathcal{R}] \subseteq \mathcal{W}$.

Keeping in mind the argument presented as above, for any prime ideal $\mathcal{W}$ of $\mathcal{R}$, either $\mathcal{R}^\zeta \subseteq \mathcal{W}$ or $[\mathcal{R}, \mathcal{R}] \subseteq \mathcal{W}$. Thus we can write $\mathcal{R}^\zeta[\mathcal{R}, \mathcal{R}] \subseteq \bigcap_i \mathcal{W}_i = (0)$ and $[\mathcal{R}^\zeta, \mathcal{R}] \subseteq \bigcap_i \mathcal{W}_i = (0)$, where $\mathcal{W}_i$ are all prime ideals of $\mathcal{R}$. In particular, since $\mathcal{R}^\zeta \neq 0$, the non-zero ideal generated by $\mathcal{R}^\zeta$ is central in $\mathcal{R}$, we get the desired result.

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Aligarh Muslim University,
Department of Mathematics,
Aligarh-202002 India
mashraf80@hotmail.com

King Abdulaziz University,
Faculty of Science & Arts-Rabigh,
Department of Mathematics,
KSA
arifraza03@gmail.com

Aligarh Muslim University,
Department of Mathematics,
Aligarh-202002 India
paryamu@gmail.com