NONLINEAR DEGENERATED ELLIPTIC PROBLEMS WITH DUAL DATA AND NONSTANDARD GROWTH

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In this article, we prove the existence of solutions for the nonlinear $p(\cdot)$–degenerate problems involving nonlinear operators of the form

$$-\text{ div } a(x,u,\nabla u)+a_0(x,u,\nabla u)=f$$

where a and a_0 are Carathéodory functions satisfying some nonstandard growth and coercivity conditions. The second member f belongs to $W^{-1,p'(x)}(\Omega,\rho^{1-p'(x)})$ where $\rho(\cdot)$ is a weight function on Ω.

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1. INTRODUCTION

Spaces with variable exponent are relevant in the study of non-Newtonian fluids. The underlying integral energy appearing in the modeling of the so-called electrorheological fluids (see for instance [12, 13]) is $\displaystyle\int_\Omega |\nabla u|^{p(x)}\mathrm{d}x$ or $\displaystyle\int_\Omega \rho(x)|\nabla u|^{p(x)}\mathrm{d}x$. Accordingly, this naturally leads to study these fluids in the weighted variable exponent Sobolev space $W_0^{1,p(x)}(\Omega,\rho)$.

In the theory of image restoration, Chen, Levine and Stanich proposed in [9] the following model which minimizes the nonstandard growth functional

$$\min_{u\in L^2(\Omega)\cap BV(\Omega)}\int_\Omega \phi(x,\nabla u)\,\mathrm{d}x+\frac{\lambda}{2}(u-I)^2$$

where I is the observed noisy image, and

$$\phi(x,r)=\begin{cases}\dfrac{1}{p(x)}|r|^{p(x)} & |r|\le\beta,\\ r-\dfrac{\beta p(x)-\beta^{p(x)}}{p(x)} & |r|>\beta,\end{cases}$$

with $\beta > 0$ is fixed and $1 \leq p(x) \leq 2$ (see also [4]). Such energies occur also in elasticity [16].

Let Ω be a bounded open subset of $I\!R^N$, $N \geq 2$. In this paper, we study the problem of existence of solutions of the following nonlinear degenerated $p(x)$ elliptic problem

$$\text{(1)} \qquad \begin{cases} -\operatorname{div} a(x, u, \nabla u) + a_0(x, u, \nabla u) = f \quad in \quad \Omega \\ u = 0 \qquad on \quad \partial\Omega. \end{cases}$$

where $p(\cdot) : \overline{\Omega} \to I\!R^+$ is a continuous function satisfying

$$\text{(2)} \qquad 1 \leq p^- \leq p(x) \leq p^+ < +\infty,$$

and the log-Hölder continuity condition, *i.e.* there is a constant $C > 0$ such that for every $x, y \in \overline{\Omega}$, $x \neq y$ with $|x - y| \leq \dfrac{1}{2}$ one has

$$\text{(3)} \qquad |p(x) - p(y)| \leq \frac{C}{-\log|x - y|},$$

The operator $-\operatorname{div} a(x, u, \nabla u)$ is a Leray-Lions operator defined on $W_0^{1,p(x)}(\Omega, \rho)$, where ρ is a weight function, such that $f \in W^{-1,p'(x)}(\Omega, \rho^{1-p'(x)})$ satisfying some integrability conditions.

Our goal in this article is to prove, by using the theory of pseudo-monotone operators and compactness arguments, the existence of a least weak solution of problem (1). We then extend both a class of problems involving Leray-Lions type operators with variable exponents (see [2,15]) and a class of some degenerate problems involving special weights [2,3,6,11,15]. The principal prototype that we have in mind is the equation

$$\begin{cases} -\operatorname{div} (\rho(x)|\nabla u|^{p(x)-2}\nabla u) + \rho(x)g(u)|\nabla u|^{p(x)-1} = f \quad \text{in} \quad \Omega \\ u = 0 \qquad \text{on} \quad \partial\Omega. \end{cases}$$

This paper is divided into three sections, organized as follows: in Section 2, we introduce some basic properties of the space $W^{1,p(x)}(\Omega, \rho)$ and some useful lemmas. In Section 3, we prove the existence of solutions of our problem.

2. MATHEMATICAL PRELIMINARIES

2.1. WEIGHTED GENERALIZED LEBESGUE AND SOBOLEV SPACES

In what follows, we recall some definitions and basic properties of weighted Lebesgue and Sobolev spaces with variable exponents (more detailed description can be found in [1]). Let Ω an open bounded subset of $\mathbb{R}^N$ $(N \geq 1)$ and $p(\cdot)$ satisfying (2) and (3).

Definition 2.1. Let ρ be a function defined on Ω, ρ is called a weight function if it is a measurable and strictly positive *a.e.* in Ω.

Let us introduce the integrability conditions used on the framework of weighted variable Lebesgue and Sobolev spaces

$$(H_1) : \rho \in L^1_{loc}(\Omega)$$
$$(H_2) : \rho^{\frac{-1}{p(x)-1}} \in L^1_{loc}(\Omega)$$

We define the weighted variable exponent Lebesgue space by

$$L^{p(x)}(\Omega, \rho) = \big\{ u : \Omega \to \mathbb{R} \text{ measurable}, \int_\Omega |u(x)|^{p(x)} \rho(x) \mathrm{d}x < \infty \big\}.$$

The space $L^{p(x)}(\Omega, \rho)$ endowed withe the norm:

$$\|u\|_{p(x),\rho} = \inf \big\{ \lambda > 0, \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} \rho(x) \mathrm{d}x \leq 1 \big\}$$

is a uniformly convex Banach space, thus reflexive. We denote by $L^{p'(x)}(\Omega, \rho^*)$ the conjugate space of $L^{p(x)}(\Omega, \rho)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, and $\rho^*(x) = \rho(x)^{1-p'(x)}$.

As in [7] we can prove the following proposition:

Proposition 2.1. (i) *For any* $u \in L^{p(x)}(\Omega, \rho)$ *and* $v \in L^{p'(x)}(\Omega, \rho^*)$, *we have*

$$|\int_\Omega uv \mathrm{d}x| \leq (\frac{1}{p_-} + \frac{1}{p'_-}) \|u\|_{p(x),\rho} \|v\|_{p'(x),\rho^*}.$$

(ii) *For all* p_1, p_2 *continuous on* $\overline{\Omega}$ *such that* $p_1(x) \leq p_2(x)$ *a.e* $x \in \overline{\Omega}$, *we have* $L^{p_2(x)}(\Omega, \rho) \hookrightarrow L^{p_1(x)}(\Omega, \rho)$ *and the embedding is continuous.*

Let us denote $I_\rho(u) = \displaystyle\int_\Omega |u|^{p(x)} \rho(x) \, \mathrm{d}x \quad \forall u \in L^{p(x)}(\Omega, \rho)$. By taking $I_\rho(u) = I(\rho^{\frac{1}{p(x)}} u)$, where $I(u) = \int_\Omega |u|^{p(x)} \mathrm{d}x$ and $\|\rho^{\frac{1}{p(x)}} u\|_{p(x)} = \|u\|_{p(x),\rho}$, we can prove the following result as a consequence of the corresponding one in [7].

Proposition 2.2. *For each* $u \in L^{p(x)}(\Omega, \rho)$,

(i) $\|u\|_{p(x),\rho} < 1$ *(resp.* $= 1$ *or* > 1*)* $\Leftrightarrow I_\rho(u) < 1$ *(resp.* $= 1$ *or* > 1*),*

(ii) $\|u\|_{p(x),\rho} > 1 \Rightarrow \|u\|^{p_-}_{p(x),\rho} \leq I_\rho(u) \leq \|u\|^{p_+}_{p(x),\rho}$
$\|u\|_{p(x),\rho} < 1 \Rightarrow \|u\|^{p_+}_{p(x),\rho} \leq I_\rho(u) \leq \|u\|^{p_-}_{p(x),\rho}$,

(iii) $\|u\|_{p(x),\rho} \to 0 \Leftrightarrow I_\rho(u) \to 0$ *and* $\|u\|_{p(x),\rho} \to \infty \Leftrightarrow I_\rho(u) \to \infty$.

We define the weighted variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega, \rho) = \{ u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega, \rho) \}.$$

with the norm $\|u\|_{1,p(x),\rho} = \|u\|_{p(x)} + \|\nabla u\|_{p(x),\rho}$. We denote by $W_0^{1,p(x)}(\Omega, \rho)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega, \rho)$.

We will use the following result of compact imbedding which can be proved in a similar manner to that of Theorem 4.8.2. in [5] (see also [8])

PROPOSITION 2.3. $W_0^{1,p(x)}(\Omega,\rho) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$.

The dual space of $W_0^{1,p(\cdot)}(\Omega,\rho)$, denoted $W^{-1,p'(\cdot)}(\Omega,\rho^*)$, is equipped with the norm

$$\|v\|_{W^{-1,p'(\cdot)}(\Omega,\rho^*)} = \inf\left\{\|v_0\|_{L^{p'(\cdot)}(\Omega,\rho^*)} + \sum_{i=1}^{N}\|v_i\|_{L^{p'(\cdot)}(\Omega,\rho^*)}\right\}$$

where the infinimum is taken on all possible decompositions $v = v_0 - \operatorname{div} F$.

Remark 2.1. We can see following [14, Theorem 3] that the the Poincaré inequality holds for the weighted Sobolev spaces $W_0^{1,p(\cdot)}(\Omega,\rho)$. In particular, this space has a norm $\|\cdot\|_{1,p(x)}$ given by

$$\|u\|_{1,p(x)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega,\rho)} \text{ for all } u \in W_0^{1,p(\cdot)}(\Omega,\rho),$$

which is equivalent to $\|\cdot\|_{1,p(x),\rho}$.

3. MAIN RESULT

3.1. BASIC ASSUMPTIONS AND TECHNICAL LEMMAS

Throughout the paper, let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$, $p(\cdot)$ satisfying (2) and (3) and ρ a weight function in Ω satisfying (H_1), (H_2). The function $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following conditions: For all $\xi, \eta \in \mathbb{R}^N$ and for almost every $x \in \Omega$,

$$|a(x,s,\xi)| \leq \beta\rho^{\frac{1}{p(x)}}(k(x) + |s|^{p(x)-1} + \rho^{\frac{1}{p'(x)}}|\xi|^{p(x)-1}), \tag{4}$$

$$[a(x,s,\xi) - a(x,s,\eta)](\xi - \eta) > 0 \quad \forall \xi \neq \eta, \tag{5}$$

$$a(x,s,\xi)\xi \geq \alpha\rho(x)|\xi|^{p(x)}, \tag{6}$$

where $k(x)$ is a positive function in $L^{p'(x)}(\Omega)$ and α and β are positive constants.

Let $a_0(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for *a.e.* $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$|a_0(x,s,\xi)| \leq \gamma(x) + g(s)\rho(x)|\xi|^{p(x)-1} \tag{7}$$

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$ is a continuous positive function that belongs to $L^1(\mathbb{R})$, while $\gamma(x)$ belongs to $L^1(\Omega)$.

$$f \in W^{-1,p'(x)}(\Omega,\rho^*). \tag{8}$$

Lemma 3.1. *Let $g \in L^{r(x)}(\Omega, \rho)$ and $g_n \in L^{r(x)}(\Omega, \rho)$ with $\|g_n\|_{r(x),\rho} \leq C$ for $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. on Ω, then $g_n \rightharpoonup g$ in $L^{r(x)}(\Omega, \rho)$.*

Proof. Let

$$E(N) = \{x \in \Omega : |g_n(x) - g(x)| \leq 1, \forall n \geq N\}.$$

Since $\text{meas}(E(N)) \to \text{meas}(\Omega)$ as $N \to \infty$, and setting

$$\mathcal{F} = \{\varphi_N \in L^{r'(x)}(\Omega, \rho^{1-r'(x)}) : \varphi_N \equiv 0 \text{ a.e. in } \Omega \backslash E(N)\},$$

where $L^{r'(x)}(\Omega, \rho^{1-r'(x)})$ is the conjugate space of $L^{r(x)}(\Omega, \rho)$ we shall show that $\mathcal{F}$ is dense in $L^{r'(x)}(\Omega, \rho^{1-r'(x)})$. Let $f \in L^{r'(x)}(\Omega, \rho^{1-r'(x)})$, we set

$$f_N(x) = \begin{cases} f(x) & \text{if } x \in E(N), \\ 0 & \text{if } x \in \Omega \backslash E(N). \end{cases}$$

Then

$$\begin{aligned} I_{r'(x),\rho}(f_N - f) &= \int_{E(N)} |f_N(x) - f(x)|^{r'(x)} \rho(x) \mathrm{d}x + \\ &\int_{\Omega \backslash E(N)} |f_N(x) - f(x)|^{r'(x)} \rho(x) \mathrm{d}x \\ &= \int_{\Omega \backslash E(N)} |f(x)|^{r'(x)} \rho(x) \mathrm{d}x \\ &= \int_{\Omega} |f(x)|^{r'(x)} \rho(x) \chi_{\Omega \backslash E(N)} \mathrm{d}x \end{aligned}$$

Taking $\psi_N(x) = |f(x)|^{r'(x)} \chi_{\Omega \backslash E(N)}$ for almost every x in Ω, we obtain

$$\psi_N \to 0 \text{ a.e. in } \Omega \quad \text{and} \quad |\psi_N| \leq |f|^{r'(x)}.$$

Using the dominated convergence theorem, we have $I_{r'(x),\rho}(f_N - f) \to 0$ as $N \to \infty$; therefore $f_N \to f$ in $L^{r'(x)}(\Omega, \rho^{1-r'(x)})$. Consequently $\mathcal{F}$ is dense in the space $L^{r'(x)}(\Omega, \rho^{1-r'(x)})$. Now we shall show that

$$\lim_{n \to \infty} \int_{\Omega} \varphi(x) \big(g_n(x) - g(x)\big) \rho(x) \mathrm{d}x = 0, \quad \forall\, \varphi \in \mathcal{F}.$$

Since $\varphi \equiv 0$ in $\Omega \setminus E(N)$, it suffices to prove that

$$\int_{E(N)} \varphi(x) (g_n(x) - g(x)) \rho(x) \mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

We set $\phi_n = \varphi\big(g_n - g\big)$. Since $|\varphi(x)||g_n(x) - g(x)| \leq |\varphi(x)|$ a.e. in $E(N)$ and $\phi_n \to 0$ *a.e.* in Ω, thanks to the dominated convergence theorem, we deduce $\phi_n \to 0$ in $L^1(\Omega)$. Which implies that

$$\lim_{n \to \infty} \int_{\Omega} \varphi(x) (g_n(x) - g(x)) \rho(x) \mathrm{d}x = 0, \quad \forall\, \varphi \in \mathcal{F}$$

Now, by the density of $\mathcal{F}$ in $L^{r'(x)}(\Omega, \rho^{1-r'(x)})$, we conclude that

$$\lim_{n\to\infty} \int_\Omega \varphi g_n \rho(x)\mathrm{d}x = \int_\Omega \varphi g \rho(x)\mathrm{d}x, \quad \forall \varphi \in L^{r'(x)}(\Omega, \rho^{1-r'(x)}).$$

Finally $g_n \rightharpoonup g$ in $L^{r(x)}(\Omega, \rho^{1-r'(x)})$. $\square$

LEMMA 3.2. *Assume that* (4)–(6), *and let* $(u_n)_n$ *be a sequence in* $W_0^{1,p(x)}(\Omega, \rho)$ *such that* $u_n \rightharpoonup u$ *weakly in* $W_0^{1,p(x)}(\Omega, \rho)$ *and a.e in* Ω *and*

$$\int_\Omega [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)]\nabla(u_n - u)\mathrm{d}x \to 0. \tag{9}$$

Then $u_n \to u$ *strongly in* $W_0^{1,p(x)}(\Omega, \rho)$.

Proof. Let $D_n = [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)]\nabla(u_n - u)$. D_n is a positive function, and by (9) $D_n \to 0$ in $L^1(\Omega)$ and for a subsequence $D_n \to 0$ *a.e.* in Ω.

Since $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega, \rho)$ and a.e in Ω, there exists a subset B of Ω, of zero measure, such that for $x \in \Omega \setminus B$, $|u(x)| < \infty$, $|\nabla u(x)| < \infty$, $k(x) < \infty$, $u_n(x) \to u(x)$, $D_n(x) \to 0$. Defining $\xi_n = \nabla u_n(x)$, $\xi = \nabla u(x)$, we have

$$\begin{aligned} D_n(x) &= [a(x, u_n, \xi_n) - a(x, u_n, \xi)](\xi_n - \xi) \\ &= a(x, u_n, \xi_n)\xi_n + a(x, u_n, \xi)\xi - a(x, u_n, \xi_n)\xi - a(x, u_n, \xi)\xi_n \\ &\geq \alpha\rho(x)|\xi_n|^{p(x)} + \alpha\rho(x)|\xi|^{p(x)} - \beta\rho(x)^{\frac{1}{p(x)}}(k(x) + |\xi_n|^{p(x)-1})|\xi| \\ &\quad - \beta\rho(x)^{\frac{1}{p(x)}}(k(x)|\xi|^{p(x)-1})|\xi_n| \\ &\geq \alpha|\xi_n|^{p(x)} - C_x\big[1 + |\xi_n|^{p(x)-1} + |\xi_n|\big], \end{aligned}$$

where C_x is a constant which depends on x, but does not depend on n. Since $u_n(x) \to u(x)$ we have $|u_n(x)| \leq M_x$, where M_x is some positive constant. Then by a standard argument $|\xi_n|$ is bounded uniformly with respect to n, we deduce that

$$D_n(x) \geq |\xi_n|^{p(x)}\big(\alpha - \frac{C_x}{|\xi_n|^{p(x)}} - \frac{C_x}{|\xi_n|} - \frac{C_x}{|\xi_n|^{p(x)-1}}\big).$$

If $|\xi_n| \to \infty$ (for a subsequence), then $D_n(x) \to \infty$ which gives a contradiction. Let now ξ^* be a cluster point of ξ_n. We have $|\xi^*| < \infty$ and by the continuity of a we obtain

$$[a(x, u, \xi^*) - a(x, u, \xi)](\xi^* - \xi) = 0.$$

In view of (5), we have $\xi^* = \xi$, which implies that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{a.e.in } \Omega.$$

Since the sequence $(a(x,u_n,\nabla u_n))_n$ is bounded in the space $(L^{p'(x)}(\Omega,\rho^*))^N$ and $a(x,u_n,\nabla u_n) \to a(x,u,\nabla u)$ a.e. in Ω, then by Lemma 3.1 we get

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \quad \text{in } (L^{p'(x)}(\Omega,\rho^*))^N \text{ a.e. in } \Omega.$$

We set $\bar{y}_n = a(x,\nabla u_n)\nabla u_n$ and $\bar{y} = a(x,\nabla u)\nabla u$. We can write

$$\bar{y}_n \to \bar{y} \quad \text{in} \quad L^1(\Omega).$$

We have

$$a(x,u_n,\nabla u_n)\nabla u_n \geq \alpha\rho(x)|\nabla u_n|^{p(x)}.$$

Let $z_n = \rho|\nabla u_n|^{p(.)}$, $z = \rho|\nabla u|^{p(.)}$, $y_n = \frac{\bar{y}_n}{\alpha}$, and $y = \frac{\bar{y}}{\alpha}$. By Fatou's lemma,

$$\int_\Omega 2y \,\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega y + y_n - |z_n - z| \,\mathrm{d}x;$$

i.e., $0 \leq -\limsup\limits_{n\to\infty} \int_\Omega |z_n - z|\mathrm{d}x$. Then

$$0 \leq \liminf_{n\to\infty} \int_\Omega |z_n - z|\mathrm{d}x \leq \limsup_{n\to\infty} \int_\Omega |z_n - z|\mathrm{d}x \leq 0,$$

this implies

$$\nabla u_n \to \nabla u \quad \text{in } (L^{p(x)}(\Omega,\rho))^N.$$

Hence $u_n \to u$ in $W_0^{1,p(x)}(\Omega,\rho)$, which completes the proof. $\square$

3.2. EXISTENCE OF WEAK SOLUTION

Definition 3.1. A bounded operator T from $W_0^{1,p(x)}(\Omega,\rho)$ to its dual is called pseudo-monotone if for all sequences $(u_k)_k$ in $W_0^{1,p(x)}(\Omega,\rho)$ satisfying:

$$\left\{\begin{array}{l} u_k \rightharpoonup u \quad \text{weakly in } \; W_0^{1,p(x)}(\Omega,\rho) \\ Tu_k \rightharpoonup \chi \text{weakly in } \; W^{-1,p'(x)}(\Omega,\rho^*) \\ \limsup\limits_{k\to\infty}\langle Tu_k, u_k\rangle \leq \langle \chi, u\rangle. \end{array}\right. \tag{10}$$

then

$$\chi = Tu \;\text{ and }\; \langle Tu_k, u_k\rangle \longrightarrow \langle \chi, u\rangle \;\text{ as }\; k \to +\infty.$$

Let us first define the weak solution of problem (1).

Definition 3.2. Let $f \in W^{-1,p'(x)}(\Omega,\rho^*)$ a weak solution of the problem (1) is a measurable function $u \in W_0^{1,p(x)}(\Omega,\rho)$ such that

$$\int_\Omega a(x,u,\nabla u)\nabla v \mathrm{d}x + \int_\Omega a_0(x,u,\nabla u) v \mathrm{d}x = < f, v > \tag{11}$$

for all $v \in W_0^{1,p(x)}(\Omega,\rho)$.

The main result of this paper is the following theorem.

THEOREM 3.1. *Under assumptions* (2), (3) *and* (4)–(8), *there exists at least a weak solution of* (1).

Let us define the operator $A_0 : W_0^{1,p(x)}(\Omega, \rho) \to W^{-1,p'(x)}(\Omega, \rho^*)$ by

$$\langle A_0 u, v\rangle = \int_\Omega a_0(x, u, \nabla u) v \mathrm{d}x,$$

and $A_1 u \equiv -\operatorname{div} a(x, u, \nabla u)$. Let us denote $A = A_1 + A_0$, we can write the equation (11) as $\langle Au, v\rangle = \langle f, v\rangle$.

Step 1. Let's show that the operator A is bounded. Firstly, by using (H_2), (7) and Proposition 2.2 we can easily prove that $\|a_0(x, u, \nabla u)\rho(x)^{\frac{-1}{p(x)}}\|_{L^{p'(x)}(\Omega)}$ is bounded for all $u \in W_0^{1,p(x)}(\Omega, \rho)$. Therefore, thanks to Hölder's inequality, we have for all $u, v \in W_0^{1,p(x)}(\Omega, \rho)$,

$$\begin{aligned}
&\langle A_0 u, v\rangle = \Big| \int_\Omega a_0(x, u, \nabla u) v \mathrm{d}x \Big| = \\
&\Big| \int_\Omega a_0(x, u, \nabla u)\rho(x)^{\frac{-1}{p(x)}} v\rho(x)^{\frac{1}{p(x)}} \mathrm{d}x \Big| \\
&\leq C(\int_\Omega \|a_0(x, u, \nabla u)\| \|_{L^{p'(x)}(\Omega,\rho^*)} \|v\rho(x)^{\frac{1}{p(x)}}\|_{L^{p(x)}(\Omega)} \\
&\leq C\|v\|_{L^{p(x)}(\Omega,\rho)},
\end{aligned}$$

which implies that the operator A_0 is bounded. Similarly by using (4) and Hölder's inequality, we can see that A_1 is bounded and by consequent A is bounded.

Step 2. In this step, we show that A is coercive. For that, let $v \in W_0^{1,p(x)}(\Omega, \rho)$. From (6), we have by using the Proposition (2.2) and Remark (2.1)

$$\frac{< A_1 v, v >}{||v||_{1,p(x),\rho}} \geq \frac{C}{||v||_{1,p(x)}} \alpha I_\rho(\nabla v) \geq C' ||v||_{1,p(x)}^r$$

for some $r > 1$. On the other hand, $\dfrac{< A_0 v, v >}{||v||_{1,p(x)}}$ is bounded, by consequent

$$\frac{< Av, v >}{||v||_{1,p(x),\rho}} \longrightarrow \infty \quad \text{as } ||v||_{1,p(x),\rho} \to \infty.$$

Step 3. It remains to show that the operator A is pseudo-monotone from $W_0^{1,p(x)}(\Omega, \rho)$ into $W^{-1,p'(x)}(\Omega, \rho^*)$. Let $(u_k)_k$ be a sequence in $W_0^{1,p(x)}(\Omega, \rho)$

such that

$$
(12) \qquad \begin{cases} u_k \rightharpoonup u \quad \text{ weakly in } W_0^{1,p(x)}(\Omega,\rho) \\ Au_k \rightharpoonup \chi \text{ weakly in } W^{-1,p'(x)}(\Omega,\rho^*) \\ \limsup_{k\to\infty} \langle Au_k, u_k\rangle \leq \langle \chi, u\rangle. \end{cases}
$$

We will prove that

$$
\chi = Au \text{ and } \langle Au_k, u_k\rangle \longrightarrow \langle \chi, u\rangle \text{ as } k \to +\infty.
$$

Since $(u_k)_k$ is a bounded sequence in $W_0^{1,p(x)}(\Omega,\rho)$, then by using the Propositions 2.2 and 2.3, there is a subsequence still denoted by $(u_k)_k$ such that

$$
u_k \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega,\rho),
$$
$$
u_k \to u \text{ in } L^{p(x)}(\Omega) \text{ and a.e in } \Omega.
$$

By the growth condition, the sequence $(a(x,u_k,\nabla u_k))_k$ is bounded in $(L^{p'(x)}(\Omega,\rho^*))^N$, therefore there exists a function $\varphi \in (L^{p'(x)}(\Omega,\rho^*))^N$ such that

$$
(13) \qquad a(x,u_k,\nabla u_k) \rightharpoonup \varphi \text{ in } (L^{p'(x)}(\Omega,\rho^*))^N \text{ as } k\to\infty.
$$

Similarly, since $(a_0(x,u_k,\nabla u_k))_k$ is bounded in $L^{p'(x)}(\Omega,\rho^*)$, then there exists a function $\psi \in L^{p'(x)}(\Omega,\rho^*)$ such that

$$
(14) \qquad a_0(x,u_k,\nabla u_k) \rightharpoonup \psi \text{ in } L^{p'(x)}(\Omega,\rho^*) \text{ as } k\to\infty.
$$

It is clear that, for all $v \in W_0^{1,p(x)}(\Omega,\rho)$,

$$
(15) \qquad \begin{aligned} \langle \chi, v\rangle &= \lim_{k\to\infty} \langle Au_k, v\rangle, \\ &= \lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla v \,\mathrm{d}x + \lim_{k\to\infty}\int_\Omega a_0(x,u_k,\nabla u_k) v\,\mathrm{d}x. \\ &= \int_\Omega \varphi\nabla v\,\mathrm{d}x + \int_\Omega \psi v\,\mathrm{d}x. \end{aligned}
$$

On the one hand, by (14) we have

$$
(16) \qquad \int_\Omega a_0(x,u_k,\nabla u_k)u_k\,\mathrm{d}x \longrightarrow \int_\Omega \psi u\,\mathrm{d}x \text{ as } k\to\infty.
$$

by combining (12) and (15), we have

$$
\begin{aligned} \limsup_{k\to\infty}\langle A(u_k),u_k\rangle = \limsup_{k\to\infty} \int_\Omega a(x,u_k,\nabla u_k)\nabla u_k \mathrm{d}x + \int_\Omega a_0(x,u_k,\nabla u_k)u_k\mathrm{d}x, \\ \leq \int_\Omega \varphi\nabla u\mathrm{d}x + \int_\Omega \psi u\mathrm{d}x. \end{aligned}
$$

Therefore

$$\limsup_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,\mathrm{d}x \le \int_\Omega \varphi\nabla u\,\mathrm{d}x. \tag{17}$$

On the other hand, thanks to (6), we have

$$\int_\Omega (a(x,u_k,\nabla u_k)-a(x,u_k,\nabla u))(\nabla u_k-\nabla u)\,\mathrm{d}x > 0. \tag{18}$$

then

$$\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,\mathrm{d}x \ge -\int_\Omega a(x,u_k,\nabla u)\nabla u\,\mathrm{d}x + \int_\Omega a(x,u_k,\nabla u_k)\nabla u\,\mathrm{d}x + \int_\Omega a(x,u_k,\nabla u)\nabla u_k\,\mathrm{d}x,$$

by (13), we get

$$\liminf_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,\mathrm{d}x \ge \int_\Omega \varphi\nabla u\,\mathrm{d}x.$$

This implies, by using (17) that

$$\lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,\mathrm{d}x = \int_\Omega \varphi\nabla u\,\mathrm{d}x. \tag{19}$$

By combining (15), (16), and (19), we obtain

$$\langle Au_k,u_k\rangle \longrightarrow \langle \chi,u\rangle \quad \text{as} \quad k\to+\infty.$$

On the other hand, by (19), and the fact that $a(x,u_k,\nabla u)\longrightarrow a(x,u,\nabla u)$ in $(L^{p'(x)}(\Omega,\rho^*))^N$ we deduce that

$$\lim_{k\to+\infty}\int_\Omega (a(x,u_k,\nabla u_k)-a(x,u_k,\nabla u))(\nabla u_k-\nabla u)\,\mathrm{d}x = 0,$$

by Lemma 3.2, we obtain

$$u_k \longrightarrow u \qquad \text{in} \quad W_0^{1,p(x)}(\Omega,\rho),$$

then, for a subsequence denoted u_k we have $\nabla u_k \to \nabla u$ *a.e.* in Ω.

Since a and a_0 are Carathéodory functions we can write

$$a(x,u_k,\nabla u_k) \rightharpoonup a(x,u_k,\nabla u) \qquad \text{in} \qquad (L^{p'(x)}(\Omega))^N,$$

and

$$a_0(x,u_k,\nabla u_k) \longrightarrow a_0(x,u_k,\nabla u) \qquad \text{in} \qquad L^{p'(x)}(\Omega),$$

by consequent, $\chi = Au$ which allows to conclude that the operator A is pseudo-monotone. By using the theory of pseudo-monotone mappings (see [10]) there exists at least a weak solution of (1), which complete the proof of Theorem 3.1.

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