RINGS IN WHICH EVERY ZERO DIVISOR IS THE SUM OR DIFFERENCE OF A NILPOTENT ELEMENT AND AN IDEMPOTENT

MARJAN SHEBANI ABDOLYOUSEFI* and HUANYIN CHEN

Communicated by Vasile Brînzănescu

An element in a ring R is called uniquely weakly nil-clean if it can be uniquely written as the sum or difference of a nilpotent element and an idempotent. The structure of rings in which every zero-divisor is uniquely weakly nil-clean is completely determined. We prove that every zero-divisor in a ring R is uniquely weakly nil-clean if and only if R is a D-ring, or R is abelian, periodic, and $R/J(R)$ is isomorphic to a field F, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_3 \oplus B$ where B is Boolean, or a Boolean ring. As a specific case, rings in which every zero-divisor a or $-a$ is a nilpotent or an idempotent are characterized. Furthermore, we prove that every zero-divisor in a ring R can be uniquely written as the sum of a nilpotent element and an idempotent if and only if R is a D-ring, or R is abelian, periodic and $R/J(R)$ is Boolean.

AMS 2010 Subject Classification: 16L99, 16U60, 16U99, 16E50.

Key words: zero-divisor, uniquely weakly nil-clean ring, uniquely nil-clean ring.

1. INTRODUCTION

A ring R is clean provided that every element in R is the sum of a unit and an idempotent. Over the last ten to fifteen years there has been an explosion of interest in this class of rings as well as in their many generalizations and variations (cf. [7]). In [12], Diesl introduced an interesting subclass of clean rings: nil-clean rings. A ring R is nil-clean provided that every element in R is the sum of a nilpotent and an idempotent. If the decomposition is unique, R is called a uniquely nil-clean ring. The structure of such rings is very attractive (cf. [3, 8, 12] and [14–15]). In [2], Ahn and Anderson introduced weakly clean rings. A ring R is weakly clean provided that every element in R is the sum or difference of a nilpotent element and an idempotent. Very recently, uniquely weakly clean rings were studied by Tat (*cf.* [16]). On the other hand, Danchev and McGovern investigated weakly nil-clean rings, *i.e.*, rings in

* Corresponding author.

which every element is either the sum or difference of a nilpotent element and an idempotent (*cf.* [6] and [10]).

The motivation of this paper is to explore weak nil-cleanness of zero-divisors over noncommutative rings and its uniqueness. We say that $e \in R$ is a very idempotent if e or $-e$ is an idempotent. An element $a \in R$ is weakly nil-clean provided that it can be written as the sum of a nilpotent and a very idempotent. Thus, a ring R is weakly nil-clean provided that every element in R is weakly nil-clean. A weakly nil-clean $a \in R$ is called uniquely weakly nil-clean provided that $a = w_1 + e_1 = w_2 + e_2$ where w_1, w_2 are nilpotent and e_1, e_2 are very idempotents $\Longrightarrow e_1^2 = e_2^2$. In this sense, we say that a can be uniquely written as the sum or difference of a nilpotent element and an idempotent. A ring R is called uniquely weakly nil-clean if every element in R is uniquely weakly nil-clean. Clearly, $\mathbb{Z}_4$ is weakly uniquely nil-clean. Notice that in $\mathbb{Z}_4, -1 = 2 + 1$ is the sum of a nilpotent element and an idempotent and $-1 = 0 - 1$ is the difference of a nilpotent element and an idempotent, but $1 \neq -1$. Thus, the unique presentation of such decomposition is in the sense for very idempotents.

An element a of a ring R is a zero-divisor if there exist nonzero $b, c \in R$ such that $ab = 0 = ca$. Zero-divisors occur in many classes of rings. In this article, we are concerned on rings in which every zero-divisor is uniquely weakly nil-clean. The structure of such rings is completely determined. Furthermore, rings in which every zero-divisor can be uniquely written as the sum of a nilpotent element and an idempotent are also studied.

A ring R is called a periodic ring if for any $a \in R$ there exist distinct $m, n \in \mathbb{N}$ such that $a^m = a^n$. A ring R is called a D-ring if every zero-divisor in R is nilpotent ((*cf.* [1]). We call a ring R is uniquely weakly D-nil-clean provided that every zero-divisor in R is uniquely weakly nil-clean. We shall prove that a ring R is uniquely weakly D-nil-clean if and only if R is a D-ring, or R is abelian, periodic and $R/J(R)$ is isomorphic to a field F, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_3 \oplus B$ where B is Boolean, or a Boolean ring. As a specific case, we shall explore rings in which every zero-divisor is a nilpotent or a very idempotent. A ring R is called uniquely D-nil-clean provided that every zero-divisor in R is uniquely nil-clean. Moreover, we prove that a ring R is uniquely D-nil-clean if and only if R is a D-ring, or R is abelian, periodic; and $R/J(R)$ is Boolean.

Throughout, all rings are associative with an identity. We use $Id(R), N(R)$ and $J(R)$ to denote the sets of all idempotents, all nilpotent elements and the Jacobson radical of a ring R. $Z(R)$ and $NZ(R)$ stand for the sets of all zero-divisors and non zero-divisors of a ring R.

2. UNIQUELY WEAKLY NIL-CLEAN RINGS

The aim of this section is to characterize uniquely weakly nil-clean rings which will be repeatedly used in the sequel. The necessary and sufficient conditions under which a group ring is uniquely weakly nil-clean are thereby obtained. We begin with

LEMMA 2.1 ([2, Theorem 2.28]). *Let R be a ring. Then every element in R is a very idempotent if and only if R is isomorphic to one of the following:*
(a) $\mathbb{Z}_3$,
(b) *a Boolean ring, or*
(c) $\mathbb{Z}_3 \oplus B$ *where B is a Boolean.*

THEOREM 2.2. *Let R be a ring. Then R is uniquely weakly nil-clean if and only if*
(1) *R is abelian;*
(2) *R is periodic;*
(3) *$R/J(R)$ is isomorphic to one of the following:*
  (a) $\mathbb{Z}_3$,
  (b) *a Boolean ring, or*
  (c) $\mathbb{Z}_3 \oplus B$ *where B is a Boolean.*

Proof. Suppose that R is uniquely weakly nil-clean. For any idempotent $e \in R$ and any $a \in R$, $e + ea(1-e) \in R$ is an idempotent. Since $\big(e + ea(1-e)\big) + 0 = e + ea(1-e)$, by the uniqueness, $\big(e + ea(1-e)\big)^2 = e^2$; hence, $ea(1-e) = 0$. This yields that $ea = eae$. Likewise, $ae = eae$, and so $ea = ae$. Thus, R is abelian. Let $a \in R$. Then there exists a central very idempotent $e \in R$ such that $w := a - e \in N(R)$. If $e^2 = e$, then $a - a^2 = w - 2ew - w^2 \in N(R)$. If $e^2 = -e$, then $a + a^2 = w + 2ew + w^2 \in N(R)$. In any case, we can find some $n \in \mathbb{N}$ such that $a^n = a^{n+1}f(a)$ where $f(t) \in R[t]$. In view of Herstein's Theorem, R is periodic, and then $N(R)$ forms an ideal of R. Therefore, $J(R) = N(R)$, and so every element in $R/J(R)$ is a very idempotent. In light of Lemma 2.1, (3) is satisfied.

Conversely, assume that (1) $-$ (3) hold. Let $a \in R$. Then $\overline{a}$ is a very idempotent, in terms of Lemma 2.1. As $J(R)$ is nil, every idempotent lifts modulo $J(R)$, and so every very idempotent lifts modulo $J(R)$. Thus, we can find a very idempotent $e \in R$ such that $\overline{a} = \overline{e}$. Hence, $v := a - e \in J(R) \subseteq N(R)$. If there exists a very idempotent $f \in R$ such that $w := a - f \in N(R)$, then $e^2 - f^2 = (a-v)^2 - (a-w)^2 = (-av - va + v^2) + (aw + wa - w^2)$. As $v \in J(R)$, we see that $-av - va + v^2 \in J(R)$. Furthermore, $aw + wa - w^2 \in N(R)$ since $aw = wa$. This implies that $1 - (e^2 - f^2) = -(-av - va + v^2) + \big(1 - (aw +$

$wa-w^2)\big) \in U(R)$. As $e^2, f^2 \in R$ are idempotents, we have $(e^2-f^2)^3 = e^2-f^2$, and so $(e^2-f^2)\big(1-(e^2-f^2)\big) = 0$. Accordingly, $e^2 = f^2$, as asserted. □

Corollary 2.3. *Let R be a ring. Then R is uniquely weakly nil-clean if and only if*

(1) *R is abelian;*
(2) *$J(R)$ is nil;*
(3) *$R/J(R)$ is isomorphic to one of the following:*
  (a) $\mathbb{Z}_3$,
  (b) *a Boolean ring, or*
  (c) *$\mathbb{Z}_3 \oplus B$ where B is a Boolean.*

Proof. $\Longrightarrow$ In view of Theorem 2.2, R is periodic. Thus, $J(R)$ is nil, as required.

$\Longleftarrow$ By (3), every element in $R/J(R)$ is a very idempotent. By (2), every idempotent lifts modulo $J(R)$. Let $a \in R$. Then $a-a^2 \in N(R)$. As in the proof of Theorem 2.2, R is periodic. This completes the proof, by Theorem 2.2. □

For a local ring R, we further derive that R is uniquely weakly nil-clean if and only if $J(R)$ is nil; $R/J(R)$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Corollary 2.4. *Let R be a ring. Then R is uniquely weakly nil-clean if and only if*

(1) *R is periodic;*
(2) *R is uniquely weakly D-nil-clean;*
(3) *$U(R) = \{x \pm 1 \mid x \in N(R)\}$.*

Proof. Suppose that R is uniquely weakly nil-clean. In view of Theorem 2.2, R is periodic. (2) is obvious. Let $x \in U(R)$. Then we have a very idempotent $e \in R$ such that $w := x - e \in N(R)$. As R is abelian, we see that $e = x - w$ and $ew = we$, and so $e = \pm 1$. Therefore $x = w \pm 1$, as desired.

Conversely, assume that $(1)-(3)$ hold. Let $a \in R$. Then we have distinct $m, n \in \mathbb{N}(m > n)$ such that $a^m = a^n$. If a is a zero-divisor, then a is uniquely weakly nil-clean. If a is a non zero-divisor, $a^{m-n} = 1$. By (3), we see that a is uniquely weakly nil-clean. This completes the proof. □

Let $P(R)$ be the intersection of all prime ideals of R, *i.e.*, $P(R)$ is the prime radical of R. As is well known, $P(R)$ is the intersection of all minimal prime ideals of R.

Proposition 2.5. *Let R be a ring. Then R is uniquely weakly nil-clean if and only if*

(1) *R is abelian;*

(2) $R/P(R)$ *is uniquely weakly nil-clean.*

Proof. Suppose that R is uniquely weakly nil-clean. Then R is abelian. In view of Theorem 2.2, R is clean, and so it is an exchange ring. Thus, $R/P(R)$ is abelian. Obviously, $J\big(R/P(R)\big) = J(R)/P(R)$ is nil. Further, $\big(R/P(R)\big)/J\big(R/P(R)\big) \cong R/J(R)$. By Theorem 2.2 again, $R/P(R)$ is uniquely weakly nil-clean.

Conversely, assume that (1) and (2) hold. For any $x \in J(R)$, we see that $\overline{x} \in J\big(R/P(R)\big)$ is nilpotent. Since $P(R)$ is nil, we see that $x \in R$ is a nilpotent; hence that $J(R)$ is nil. As $R/J(R) \cong \big(R/P(R)\big)/J\big(R/P(R)\big)$, it follows from Theorem 2.2 that R is uniquely weakly nil-clean, as asserted. $\square$

Let R be a ring, and let G be a group. The augmentation ideal $I(R,G)$ of the group ring RG is the kernel of the homomorphism from RG to R induced by collapsing G to 1. That is, $I(R,G) = ker(\omega)$, where $\omega = \{\sum\limits_{g\in G} r_g g \mid \sum\limits_{g\in G} r_g = 0\}$.

LEMMA 2.6. *Let R be a ring, and let G be a group. If RG is uniquely weakly nil-clean, then so is R.*

Proof. Let $a \in R$. Then we have a very idempotent $e \in RG$ such that $a - e \in N(RG)$ and that such representation is unique. Hence, $a - \omega(e) = \omega(a - e) \in N(R)$. Obviously, $\omega(e) \in R$ is a very idempotent. If $a - f \in N(R)$ for a very idempotent $f \in R$, then $e = f$, as desired. $\square$

THEOREM 2.7. *Let R be a ring, and let G be a group. If $I(R,G)$ is nil, then RG is uniquely weakly nil-clean if and only if so is R.*

Proof. One direction is obvious by Lemma 2.6. Conversely, assume that R is uniquely weakly nil-clean. Let $x \in RG$. Then $x = \omega(x) + \big(x - \omega(x)\big)$. By hypothesis, there exists a very idempotent $e \in R$ such that $w := \omega(x) - e \in N(R)$. Hence, $x = e + \big(w + (x - \omega(x))\big)$. Since $ker(\omega)$ is nil, we see that $v := w + (x - \omega(x)) \in N(R)$. Assume that $x = f + w$ where $f \in RG$ is an very idempotent and $w \in N(RG)$. Then $f - \omega(f) \in ker(\omega)$ is nil. As R is uniquely weakly nil-clean, R is abelian. Hence, $\big(f - \omega(f)\big)\big(1 - (f - \omega(f))^2\big) = 0$, and so $f = \omega(f) \in R$. It is easy to verify that $vw = (x-e)(x-f) = (x-f)(x-e) = wv$, and then $e - f = w - v \in N(R)$. It follows from $(e-f)\big(1-(e-f)^2\big) = 0$ that $e = f$, as needed. $\square$

COROLLARY 2.8. *Let R be a ring with a prime $p \in J(R)$, and let G be a locally finite p-group. Then RG is uniquely weakly nil-clean if and only if R is uniquely weakly nil-clean.*

Proof. One direction is obvious. Conversely, assume that R is uniquely weakly nil-clean. Then $J(R)$ is nil by Corollary 2.3. We first suppose G is finite and prove the claim by induction on $|G|$. As the center of a nontrivial finite p-group contains more than one element, we may take $x \in G$ be an element in the center with the order p. Let (x) be the subgroup of G generated by x. Then $\overline{G} = G/(x)$ has smaller order. By induction hypothesis, $ker\big(\overline{\omega}\big)$ is nil, where $\overline{\omega} : R\overline{G} \to R, \sum\limits_{\overline{g} \in \overline{G}} r_{\overline{g}}\overline{g}$. Let $\varphi : RG \to R\overline{G}, \sum\limits_{g} r_g g \to \sum\limits_{g} r_g \overline{g}$. Then $ker(\varphi) = (1-x)RG$. Since $x^p = 1$, we see that $(1-x)^p \in pRG$; hence, $1 - x \in RG$ is nilpotent. But $\varphi\big(ker(\omega)\big) = ker\big(\overline{\omega}\big)$ is nil. For any $z \in ker(\omega)$, we have some $m \in \mathbb{N}$ such that $z^m \in ker(\varphi)$ is nilpotent. Thus, $z \in RG$ is nilpotent. We conclude that $ker(\omega)$ is nil, and therefore RG is uniquely weakly nil-clean, in terms of Theorem 2.7. $\square$

3. FACTORIZATION OF ZERO-DIVISORS

In this section, we work out the structure of uniquely weakly D-nil-clean rings. To do this, we need the connections between uniquely weakly nil-clean rings and uniquely weakly D-nil-clean rings.

LEMMA 3.1. *Every uniquely weakly D-nil-clean ring is abelian.*

Proof. Let $e \in R$ be an idempotent, and let $x \in R$. Then $e + ex(1-e) \in R$ is an idempotent. If $e = 1$, then $ex = exe$. If $1 - e = ex(1-e)$, then $ex = exe$. If $e \neq 1$ and $1 - e \neq ex(1-e)$, then $e + ex(1-e) \in R$ is a zero-divisor, as

$$(1-e)\big(e + ex(1-e)\big) = 0 = \big(e + ex(1-e)\big)\big(1 - e - ex(1-e)\big).$$

Since $e + ex(1-e) = e + ex(1-e) + 0$, by hypothesis, $e^2 = \big(e + ex(1-e)\big)^2$, and then $ex(1-e) = 0$. That is, $ex = exe$. Likewise, $xe = exe$. Thus, $ex = xe$. This completes the proof. $\square$

THEOREM 3.2. *Every uniquely weakly D-nil-clean ring is a D-ring or the product of two uniquely weakly nil-clean rings.*

Proof. Let R be a uniquely weakly D-nil-clean ring. In view of Lemma 3.1, R is abelian.

Case I. R is indecomposable. Then every zero-divisor is nilpotent or invertible. The later is impossible, and so R is a D-ring.

Case II. R is decomposable. Write $R = A \oplus B$. Let $a \in A$. Then $(a, 0) \in R$ is a zero-divisor. By hypothesis, there exists a very idempotent $(e, e') \in R$ such that $(a, 0) - (e, e') \in N(R)$, and that $(a, 0) - (f, f') \in N(R)$ with a very idempotent $(f, f') \in R$ implies that $(e, e')^2 = (f, f')^2$. Thus, $a - e \in N(R)$. If there exists a very idempotent $g \in A$ such that $a - g \in N(A)$. Then $(a, 0) - (g, 0) \in N(R)$. This implies that $(g, 0)^2 = (e, e')^2$, and so $g^2 = e^2$.

Therefore A is uniquely weakly nil-clean. Similarly, B is uniquely weakly nil-clean, as asserted. □

LEMMA 3.3. *Let R be a ring. Then every zero-divisor in R is a very idempotent if and only if R is isomorphic to one of the following:*

(1) *a domain,*

(2) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$*,*

(3) $\mathbb{Z}_3 \oplus B$ *where B is a Boolean, or*

(4) *a Boolean ring.*

Proof. Suppose that every zero-divisor in R is a very idempotent. By Lemma 3.1, R is abelian.

Case I. R is indecomposable. Then $Id(R) = \{0, 1\}$ and $-Id(R) = \{0, -1\}$. Thus, the only zero-divisor is zero. Hence, R is a domain.

Case II. R is decomposable. Then we have $S, T \neq 0$ such that $R = S \oplus T$. For any $t \in T$, $(0, t) \in R$ is a zero-divisor. By hypothesis, $(0, t)$ or $-(0, t)$ is an idempotent; hence that t or $-t$ is an idempotent in T. Therefore every element in T is a very idempotent. In light of Lemma 2.1, T is isomorphic to one of the following:

(a) $\mathbb{Z}_3$*,*

(b) *a Boolean ring, or*

(c) $\mathbb{Z}_3 \oplus B$ *where B is a Boolean.*

Likewise, S is isomorphic to one of the preceding. Thus, R is isomorphic to one of the following: R is isomorphic to one of the following:

(i) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$*,*

(ii) *a Boolean ring, or*

(iii) $\mathbb{Z}_3 \oplus B$ *where B is a Boolean.*

(iv) $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus B$ *where B is a Boolean.*

But in Case (iv), $(1, 2, 0) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus B$ is a zero-divisor, while it is not a very idempotent. Therefore Case (iv) will not appear, as desired.

Conversely, if R is a domain, then every zero-divisor is zero. If $R = \mathbb{Z}_3 \oplus \mathbb{Z}_3$, then $NZ(R) = \{(1,1), (1,2), (2,1)\}, Id(R) = \{(0,0), (0,1), (1,0), (1,1)\}$ and $-Id(R) = \{(0,0),\ (0,2),\ (2,0),\ (2,2)\}$. Therefore $R = NZ(R) \bigcup Id(R) \bigcup -Id(R)$. If $R = \mathbb{Z}_3 \bigoplus B$ where B is a Boolean, then $Id(R) = \{(0, b), (1, b) \mid b \in B\}$ and $-Id(R) = \{(0, b), (2, b) \mid b \in B\}$. Therefore $R = Id(R) \bigcup -Id(R)$. If R is a Boolean ring, then every element in R is an idempotent. In any case, every element in R is a very idempotent, the result follows. □

We will state now the main result of this section.

THEOREM 3.4. *Let R be a ring. Then R is uniquely weakly D-nil-clean if and only if R is a D-ring, or R satisfies the conditions:*

(1) *R is abelian;*
(2) *R is periodic;*
(3) *$R/J(R)$ is isomorphic to one of the following:*
 (a) *a field F,*
 (b) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$,
 (c) $\mathbb{Z}_3 \oplus B$ *where B is Boolean, or*
 (d) a Boolean ring.

Proof. $\Longrightarrow$ Suppose that R is not a D-ring. In view of Theorem 3.2, R is the product of two uniquely weakly nil-clean rings R_1 and R_2. By Theorem 2.2, R_1 and R_2 are abelian periodic rings, and then so is R. In view of [4, Theorem], $N(R)$ is an ideal of R. As R is periodic, $J(R)$ is nil; hence, $J(R) = N(R)$. As every idempotent lifts modulo $N(R)$, we see that $R/J(R)$ is abelian. Let $\overline{a} \in R/N(R)$ be a zero-divisor. If $a \in R$ is not a zero-divisor, then $a \in U(R)$, and so $\overline{a} \in U\big(R/N(R)\big)$, a contradiction. Thus, $a \in R$ is a zero-divisor. By hypothesis, a is the sum of a very idempotent and a nilpotent. Hence, $\overline{a}$ is a very idempotent. That is, every zero-divisor in $R/J(R)$ is a very idempotent. In light of Lemma 3.3, $R/J(R)$ is isomorphic to one of the following:
 (i) *a domain F,*
 (ii) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$,
 (iii) $\mathbb{Z}_3 \oplus B$ *where B is Boolean, or*
 (iv) *a Boolean ring.*

If $R = F$ is a domain, then for any $a \in R$, $a = 0$ or $a^m = 1$ for some $m \in \mathbb{N}$. This shows that R is a field, as required.

$\Longleftarrow$ In view of [4, Theorem], $N(R)$ forms an ideal of R. Let $a \in R$ be a zero divisor. Then $\overline{a} \in R/J(R)$ is a zero-divisor; otherwise, $\overline{a} \in R/J(R)$ is invertible, and so $a \in R$ is invertible, a contradiction. According to Lemma 3.5, $\overline{a}$ is a very idempotent in $R/J(R)$. As R is periodic, $J(R)$ is nilpotent, and so every idempotent modulo $J(R)$. This implies that $v := a - e \in N(R)$ for some very idempotent $e \in R$. Let $f \in R$ be a very idempotent such that $w := a-f \in N(R)$. Then $e^2-f^2 = (a-v)^2-(a-w)^2 = -av-va+v^2+aw+wa-w^2 \in N(R)$. As $e, f \in R$ are very clean, we se that $e^2, f^2 \in R$ are idempotents. It is easy to verify that $(e^2 - f^2)\big(1 - (e^2 - f^2)^2\big) = 0$, and so $e^2 = f^2$. Therefore we complete the proof. $\square$

We now consider a specific case and explore the structure of rings in which every zero-divisor is a very idempotent or a nilpotent element.

LEMMA 3.5. *Every ring in which every element is a very idempotent or a nilpotent element is abelian.*

Proof. Let $e \in R$ be an idempotent, and let $x \in R$. Then $1 - ex(1 - e) \in U(R)$. If $\big(1 - ex(1-e)\big)^2 = 1 - ex(1-e)$, then $ex(1-e) = 0$, and so $ex = exe$. If $\big(1 - ex(1-e)\big)^2 = -\big(1 - ex(1-e)\big)$, then $ex(1-e) = 2$. and so $ex(1-e) = 2e(1-e) = 0$. Hence, $ex = exe$. If $1 - ex(1-e) \in N(R)$, this will be a contradiction. Thus, $ex = exe$. Likewise, $xe = exe$. Therefore $ex = xe$, hence the result. □

Lemma 3.6. *Let R be a ring. Then the following are equivalent:*

(1) $R = N(R) \cup Id(R) \cup -Id(R)$

(2) $R = J(R) \cup Id(R) \cup -Id(R)$

(3) *R is isomorphic to one of the following:*

(a) $\mathbb{Z}_3$, $\mathbb{Z}_4$,

(b) *a Boolean ring, or*

(c) $\mathbb{Z}_3 \oplus B$ *where B is a Boolean.*

Proof. (1) $\Leftrightarrow$ This is proved by [10, Proposition 1.21]

(1) $\Leftrightarrow$ This is obvious by [10, Proposition 1.19] and Lemma 2.1. □

Theorem 3.7. *Let R be a ring. Then R is an abelian ring in which every zero-divisor in R is a very idempotent or a nilpotent element if and only if R is isomorphic to one of the following:*

(a) *a D-ring,*

(b) *a Boolean ring,*

(c) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$,

(d) $\mathbb{Z}_3 \oplus B$ *where B is a Boolean.*

Proof. Suppose that R is an abelian ring in which every zero-divisor in R is a very idempotent or a nilpotent element.

Case I. R is indecomposable. Then every very idempotent is $0, 1$ or -1. Hence, every zero-divisor in R is nilpotent. Hence, R is a D-ring.

Case II. R is decomposable. Write $R = S \oplus T$. For any $t \in T$, $(0, t)$ is a very idempotent or a nilpotent element. We infer that every element in T is a very idempotent or a nilpotent element. Similarly, every element in S is a very idempotent or a nilpotent element. By virtue of Lemma 3.6, S and T are both isomorphic to one of the following:

(a) $\mathbb{Z}_3$, $\mathbb{Z}_4$,

(b) *a Boolean ring, or*

(c) $\mathbb{Z}_3 \oplus B$ *where B is a Boolean.*

But one easily checks that $Z(R) \neq Id(R) \bigcup -Id(R) \bigcup N(R)$ for any of those types

(1) $\mathbb{Z}_3 \oplus \mathbb{Z}_4$,

(2) $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus B$ *where* B *is a Boolean ring,*
(3) $\mathbb{Z}_4 \oplus \mathbb{Z}_4$,
(4) $\mathbb{Z}_4 \oplus B$ *where* B *is a Boolean ring, and*
(5) $\mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus B$ *where* B *is a Boolean ring.*

Therefore R is isomorphic to one of $(a) - (d)$.

Conversely, R is abelian, as every D-ring is connected. One easily checks that any of these four types of rings satisfy the desired condition. □

In light of Theorem 3.7 and Theorem 3.4, every abelian ring in which every zero-divisor in R is a very idempotent or a nilpotent element is uniquely weakly D-nil-clean.

Corollary 3.8. *Let* R *be a ring. Then the following are equivalent:*
(1) R *is an abelian ring in which every zero-divisor in* R *is an idempotent or a nilpotent element.*
(2) R *is a D-ring or a Boolean ring.*

Proof. (1) $\Rightarrow$ (2) In view of Theorem 3.7, R is isomorphic to one of the following:
(a) *a D-ring,*
(b) *a Boolean ring,*
(c) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$,
(d) $\mathbb{Z}_3 \oplus B$ *where* B *is a Boolean.*

But in the case $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, $(0,2) \notin Id(R) \bigcup N(R)$. In the case $\mathbb{Z}_3 \oplus B$, $(2,0) \notin Id(R) \bigcup N(R)$. Therefore proving (2).

(2) $\Rightarrow$ (1) This is obvious. □

The abelian condition in Corollary 3.8 is necessary. For instance, every zero-divisor in $T_2(\mathbb{Z}_2)$ is an idempotent or a nilpotent element. But $T_2(\mathbb{Z}_2)$ is neither a Boolean ring nor a D-ring.

4. UNIQUELY D-NIL-CLEAN RINGS

The aim of this section is to describe the connection between uniquely D-nil-clean rings and uniquely nil-clean rings, and thereby characterize the structure of uniquely D-nil-clean rings.

Lemma 4.1. *Every uniquely D-nil-clean ring is abelian.*

Proof. This is similar to that in Lemma 3.1. □

Proposition 4.2. *A ring* R *is uniquely D-nil-clean if and only if for any zero-divisor* $a \in R$ *there exists a central idempotent* $e \in R$ *such that* $a - e \in N(R)$.

Proof. One direction is obvious from Lemma 4.1. Conversely, letting $e \in R$ be an idempotent, we have a central idempotent $f \in R$ such that $w := e - f \in N(R)$. Thus, $(e-f)^3 = e - f$, and so $(e-f)\big(1-(e-f)^2\big) = 0$. This implies that $e = f$, and then R is abelian. Let $a \in R$ be a zero-divisor. Then there exists a central $e \in R$ such that $a - e \in N(R)$. If there exists an idempotent $f \in R$ such that $a - f \in N(R)$, then $e - f = (a-f)-(a-e) \in N(R)$. It follows from $(e-f)^3 = e - f$ that $e = f$, which completes the proof. □

LEMMA 4.3. *Let R be a ring. Then R is a uniquely D-nil-clean ring if and only if R is a D-ring or R is uniquely nil-clean.*

Proof. $\Longrightarrow$ In view of Lemma 4.1, R is abelian.

Case I. R is indecomposable. Let $a \in R$ be a zero-divisor. Then $a \in R$ is nilpotent or $a \in U(R)$. This shows that every zero-divisor is nilpotent, *i.e.*, R is a D-ring.

Case II. R is decomposable. Write $R = A \oplus B$. For any $x \in A$, $(x,0) \in R$ is a zero-divisor. Hence, we can find a unique idempotent $(e,f) \in R$ such that $(x,0)-(e,f) \in N(R)$. Thus, $x - e \in N(R)$ for an idempotent $e \in R$. If there exists an idempotent $g \in R$ such that $x - g \in N(R)$. Then $(x,0)-(g,f) \in N(R)$. By the uniqueness, we get $g = e$, and therefore A is uniquely nil-clean. Similarly, B is uniquely nil-clean, and then R is uniquely nil-clean.

$\Longleftarrow$ If R is a D-ring, then R is a D-uniquely nil clean ring. So we assume that R is uniquely nil-clean, and therefore R is a uniquely D-nil-clean ring. □

THEOREM 4.4. *Let R be a ring. Then R is uniquely D-nil-clean if and only if R is a D-ring, or R satisfies the conditions:*

(1) *R is abelian;*
(2) *R is periodic;*
(3) *$R/J(R)$ is Boolean.*

Proof. $\Longrightarrow$ In view of Lemma 4.1, R is abelian. Suppose that R is not a D-ring. Then R is a uniquely nil-clean ring, in terms of Lemma 4.3. In view of [12, Theorem 5.9], $R/J(R)$ is Boolean and $J(R)$ is nil. Let $a \in R$. Then $a - a^2 \in J(R)$, and so $(a-a^2)^m = 0$ for some $m \in \mathbb{N}$. Similarly to Theorem 2.2, R is periodic, in terms of Herstein's Theorem.

By virtue of [4, Theorem], $N(R)$ forms an ideal of R. Hence, $J(R) = N(R)$. Let $\overline{a} \in R/J(R)$ is a zero-divisor. Then $a \in R$ is a divisor; otherwise, $a \in U(R)$ as R is periodic, a contradiction. Hence, a is the sum of an idempotent and a nilpotent element. This shows that $\overline{a}$ is an idempotent. Therefore, every zero-divisor in $R/J(R)$ is an idempotent.

Set $S = R/J(R)$. Suppose that S has a nonzero zero-divisor. Then we have some $x, y \in R$ such that $xy = 0, x, y \neq 0$. Hence, $(yx)^2 = 0$. If

$yx \neq 0$, then $yx \in R$ is a zero-divisor. So $yx \in R$ is an idempotent. Thus, $yx = (yx)^2 = 0$. This implies that $x \in R$ is a zero-divisor, and so $x = x^2$. It follows that $1 - x \in R$ is a zero-divisor; hence that $1 - x = (1 - x)^2$. Therefore $x^2 = x$.

Let $a \in R$. Then $\big(xa(1 - x)\big)^2 = 0$. Hence, $xa(1 - x) = 0$; otherwise, $xa(1 - x) \in R$ is an idempotent, and so $xa(1 - x) = 0$, a contradiction. Thus, $xa(1 - x) = 0$, hence, $xa = xax$. Likewise, $ax = xax$. Thus, $xa = ax$. If $xa = 0$, then $a \in R$ is a zero-divisor, and so it is an idempotent. If $xa \neq 0$, it follows from $xa(1 - x) = 0$ that $xa \in R$ is a zero-divisor, and so $xa = (xa)^2$. Hence, $xa(1 - a) = 0$. This implies that $1 - a \in R$ is a zero-divisor, and then $1 - a = (1 - a)^2$. Thus, $a = a^2$. Therefore $a \in R$ is an idempotent. Consequently, $R/J(R)$ is Boolean or $R/J(R)$ is a domain. If $R/J(R)$ is a domain, the periodic property implies that R is a field. Thus, R is local. But $J(R)$ is nil, and so every zero-divisor is nilpotent. We infer that R is a D-ring, an absurd. This shows that $R/J(R)$ is Boolean, as desired.

$\Longleftarrow$ If R is a D-ring, then R is uniquely D-nil-clean. We now assume that (1) – (3) hold. Let $a \in R$ be a zero-divisor. As $R/J(R)$ is Boolean, $a - a^2 \in J(R) \subseteq N(R)$. Thus, we can find an idempotent $e \in R$ such that $a - e \in N(R)$. Since R is abelian, we see that such idempotent e is unique. Therefore R is uniquely D-nil-clean. $\square$

In light of Theorem 4.4 and Theorem 3.4, every uniquely D-nil-clean ring is a uniquely weakly D-nil-clean ring.

LEMMA 4.5. *Let R be a ring. Then R is uniquely nil-clean if and only if*

(1) *R is abelian.*

(2) *$R/J(R)$ is Boolean and $J(R)$ is nil.*

Proof. $\Longrightarrow$ This is obvious by [12, Lemma 5.5 and Theorem 5.9].

$\Longleftarrow$ For any $a \in R$, $a - a^2 \in J(R)$, and so we have an idempotent $e \in R$ such that $a - e \in J(R)$, as $J(R)$ is nil. Write $a = e + v$. Then $v \in J(R) \subseteq N(R)$. If there exists an idempotent $f \in R$ and a $w \in N(R)$ such that $a = f + w$, then $e - f = (a - v) - (a - w) = w - v$. Clearly, $wv = (a - f)(a - e) = (a - e)(a - f) = vw$, and so $e - f \in N(R)$. Since $(e - f)^3 = e - f$, we see that $e - f = 0$, and then $e = f$. Therefore R is uniquely nil clean. $\square$

THEOREM 4.6. *Let R be a ring. Then R is uniquely nil-clean if and only if*

(1) *$2 \in R$ is nilpotent;*

(2) *R is uniquely weakly nil-clean.*

Proof. Suppose that R is uniquely nil clean. In view of Lemma 4.5, $\overline{2}^2 = \overline{2}$ in $R/J(R)$, and so $2 \in J(R)$ is nilpotent. By Lemma 4.5 and Theorem 2.2, we observe that every uniquely nil-clean ring is uniquely weakly nil-clean.

Conversely, assume that (1) and (2) hold. As $2 \in \mathbb{Z}_3$ is not nilpotent. In view of Theorem 2.2, R is abelian, $J(R)$ is nil, and that $R/J(R)$ is Boolean. The result follows by Lemma 4.5. $\square$

COROLLARY 4.7. *Let R be a ring. Then R is uniquely nil-clean if and only if*

(1) *R is abelian;*

(2) *$R/P(R)$ is uniquely nil-clean.*

Proof. One direction is obvious, by Theorem 4.6 and Proposition 2.5.

Conversely, assume that (1) and (2) hold. By virtue of Theorem 4.6, $\overline{2} \in R/P(R)$ is nilpotent. We infer that $2 \in R$ is nilpotent. Furthermore, $R/P(R)$ is uniquely weakly nil-clean. According to Proposition 2.5, R is uniquely weakly nil-clean. By using Theorem 4.6 again, R is uniquely nil-clean. $\square$

COROLLARY 4.8. *Let R be a ring, and G be a group. Then RG is uniquely nil-clean if and only if R is uniquely nil-clean and $I(R,G)$ is nil.*

Proof. Suppose RG is uniquely nil-clean. Then RG is uniquely weakly nil-clean and $2 \in N(RG)$, by Theorem 4.6. Hence, R is uniquely weakly nil-clean and $2 \in N(R)$. By using Theorem 4.6 again, R is uniquely nil-clean. Thanks to Lemma 4.5, $RG/J(RG)$ is Boolean. For any $g \in G$, we see that $(1-g)-(1-g)^2 \in J(RG)$; hence, $1-g \in J(RG)$. This implies that $ker(\omega) \subseteq J(RG)$ is nil, as desired.

Conversely, assume that R is uniquely nil-clean and $ker(\omega)$ is nil. By virtue of Theorem 4.6 and Theorem 2.7, $2 \in N(R)$ and RG is uniquely weakly nil-clean. Therefore RG is uniquely nil-clean, in terms of Theorem 4.6. $\square$

Let G be a 3-group. Then $\mathbb{Z}_3G$ is not uniquely nil-clean by Corollary 4.8, while it is uniquely weakly nil-clean.

COROLLARY 4.9. *Let R be a ring in which 2 is nonnilpotent. Then R is uniquely D-nil-clean if and only if R is uniquely weakly D-nil-clean.*

Proof. $\Longrightarrow$ This is obvious.

$\Longleftarrow$ In light of Theorem 3.2, R is a D-ring, or the product of two uniquely weakly nil-clean rings. As 2 is nonnilpotent in R, every R is uniquely weakly nil-clean is uniquely nil-clean, by Theorem 4.6. Therefore R is uniquely D-nil-clean, in terms of Lemma 4.3. $\square$

Acknowledgments. The authors are grateful to the referee for his/her careful reading of the manuscript and the useful comments made. Huanyin Chen was supported by the Natural Science Foundation of Zhejiang Province, China (No. LY17A010018).

REFERENCES

[1] H. Abu-Khuzam and A. Yaqub, *Structure of rings with certain conditions on zero-divisors.* Int. J. Math. Math. Sci. **15** (2006), 1–6.

[2] M.S. Ahn and D.D. Anderson, *Weakly clean rings and almost clean rings.* Rocky Mountain J. Math. **36** (2006), 783–798.

[3] D. Andrica and G. Călugăreanu, *A nil-clean* 2×2 *matrix over the integers which is not clean.* J. Algebra Appl. **13**(2014), 1450009 [9 pages] DOI: 10.1142/S0219498814500091.

[4] A. Badawi, *On abelian* π*–regular rings.* Comm. Algebra **25** (1997), 1009–1021.

[5] S. Breaz, G. Galugareanu, P. Danchev and T. Micu, *Nil-clean matrix rings.* Linear Algebra Appl. **439** (2013), 3115–3119.

[6] S. Breaz, P. Danchev and Y. Zhou, *Weakly nil-clean rings.* Preprint, 2015.

[7] H. Chen, *Rings Related Stable Range Conditions.* Series in Algebra **11**, World Scientific, Hackensack, NJ, 2011.

[8] H. Chen, *Strongly nil-clean matrices over* $R[x]/\left(x^2-1\right)$. Bull. Korean Math. Soc. **49** (2012), 589–599.

[9] H. Chen, *On strongly nil-clean matrices.* Comm. Algebra **41** (2013), 1074–1086.

[10] P.V. Danchev and W.W. McGovern, *Commutative weakly nil-clean unital rings.* J. Algebra **425** (2015), 410–422.

[11] A.J. Diesl, *Classes of Strongly Clean Rings.* Ph.D. Thesis, University of California, Berkeley, 2006.

[12] A.J. Diesl, *Nil clean rings.* J. Algebra **383** (2013), 197–211.

[13] V.A. Hiremath and S. Hegde, *Using ideals to provide a unified approach to uniquely clean rings.* J. Aust. Math. Soc. **96** (2014), 258–274.

[14] M.T. Kosan, T.K. Lee and Y. Zhou, *When is every matrix over a division ring a sum of an idempotent and a nilpotent?* Linear Algebra Appl. **450** (2014), 7–12.

[15] A. Stancu, *On some constuctions of nil-clean, clean, and exchange rings.* arXiv: 1404. 2662v1 [math.RA], 10 Apr 2014.

[16] Q.K. Tat, *Weakly Clean Rings and Related Rings.* Ph.D. Thesis, University of Malaya, Kualalum pur, 2015.

Received 6 March 2016

Women's University of Semnan,
(Farzanegan),
Semnan, Iran
sheibani@@fgusem.ac.ir

Hangzhou Normal University,
Department of Mathematics,
Hangzhou 310036, China
huanyinchen@@aliyun.com