REGULARITY OF LOCAL TIMES OF GAUSSIAN SELF-SIMILAR QUASI-HELICES

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We analyze the regularity in Sobolev-Watanabe spaces of the local times of Gaussian self-similar processes with a certain trajectorial regularity. The main purpose is to understand which of these parameters (the self-similarity index or the sample path regularity order) gives the regularity of the local time. We study several examples, such as fractional Brownian motion, bifractional Brownian motion or the solutions to the heat or wave equation with additive Gaussian noise.

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1. INTRODUCTION

The local time of a stochastic process measures the time spent by the process in a given Borel set. It is an important characteristic of a stochastic process. Especially for Gaussian processes, the local time has been widely studied. We refer, among others, to the recent monograph [18] for a complete exposition.

In the case of the Gaussian processes, one of the methods which has been widely employed to study their local time is the so-called chaos decomposition into multiple Wiener-Itô stochastic integrals. This approach has been first applied to the Brownian motion in [20, 21] and then to several other Gaussian processes, such as fractional Brownian motion [7, 11], bifractional Brownian motion [23] or the solution to the heat equation [25].

One of the problems analyzed via chaos expansion is the regularity of the local time in the so-called Sobolev-Watanabe spaces. This is a kind of regularity with respect to the variable of randomness ω. In particular, it extends the L^p- regularity of the local time since the Sobolev-Watanabe space $\mathbb{D}^{0,2}$ coincides with $L^2(\Omega)$. We refer to Section 2.2 for the definition of these spaces which have been originally introduced by Watanabe in [28]. For instance, the local time of the Wiener process belongs to the Sobolev-Watanabe space $\mathbb{D}^{\gamma,2}$ for every $\gamma < \frac{1}{2}$, see [20]. The result has been extended to fractional Brownian motion $(B_t^H)_{t\geq 0}$ (fBm in the sequel) in [11]. It has been showed that the local time of the fBm belongs to $\mathbb{D}^{\gamma,2}$ for every $\gamma < \frac{1}{2H} - \frac{1}{2}$. Note that the index H represents the self-similarity index of the fBm but also the order of the regularity of its sample paths since for every $s,t \geq 0$ one has $\mathbf{E}\left|B_t^H - B_s^H\right|^2 = |t-s|^{2H}$. Consider now the bifractional Brownian motion (see Section 4). This Gaussian process, denoted $(B_t^{H,K})_{t\geq 0}$, with $H \in (0,1)$ and $K \in (0,1]$ is HK-self-similar and it satisfies the quasi-helix property: for every $s,t \geq 0$,

$$C_1|t-s|^{2HK} \leq \mathbf{E}\left|B_t^{H,K} - B_s^{H,K}\right|^2 \leq C_2|t-s|^{2HK}$$

where C_1, C_2 are two strictly positive constants. In this case the local time of the bi-fBm belongs to the space $\mathbb{D}^{\gamma,2}$ for every $\gamma < \frac{1}{2HK} - \frac{1}{2}$ (see [23]) and the index HK can be interpreted as the self-similarity index and simultaneously, as the order of the sample paths regularity. The same phenomenon happens in the case of the solution to the heat equation (see [25]).

The purpose of this work is to understand which of these two parameters (the self-similarity index or the Hölder continuity index) gives the regularity of the local time. For certain Gaussian stochastic processes, these two parameters do no coincide. Such an example is the solution to the stochastic wave equation with additive noise which will be discussed later. We will show that the regularity of the local time in the Sobolev-Watanabe sense is rather related to the sample path regularity than to the self-similarity index.

We organized our paper as follows. In Section 2, we present our main assumptions and we introduce the multiple stochastic integrals and the local time. In Section 3, we prove our main result and finally, in Section 4 we discuss several examples.

2. PRELIMINARIES

2.1. ASSUMPTIONS

We consider throughout the paper a centered Gaussian process $(X_t)_{t\geq 0}$ with $X_0 = 0$ and we denote by $R(s,t) = \mathbf{E}X_tX_s$, $s,t \geq 0$ its covariance function. Fix an interval $I = [\varepsilon, 1]$ with $\varepsilon > 0$ small enough.

Let us make the following assumptions:

$C1$) The process $(X_t)_{t\geq 0}$ is self-similar of order $H \in (0,\infty)$, meaning that for every $c > 0$ the stochastic processes $(X_{ct})_{t\geq 0}$ and $(c^H X_t)_{t\geq 0}$ have the same finite dimensional-distributions.

$C2)$ The process $(X_t)_{t\geq 0}$ satisfies, for every $s,t\in I$,

$$C_1|t-s|^{2\alpha} \leq \mathbf{E}\,|X_t - X_s|^2 \leq C_2|t-s|^{2\alpha} \tag{1}$$

with some constants $0 < C_1 < C_2$ and with $\alpha > 0$. This property means that the process X is a quasi-helix in the sense of [15, 16].

$C3)$ For every $t\in I$,

$$\mathrm{Var} X_t \geq C > 0. \tag{2}$$

$C4)$ The covariance function is continuous in the sense that the mapping $(t,s)\to R(s,t) = \mathbf{E} X_t X_s$ is continuous on $[0,\infty)\times[0,\infty)$.

2.2. MULTIPLE STOCHASTIC INTEGRALS AND WATANABE SPACES

Here we describe the elements from stochastic analysis that we will need in the paper. Consider $\mathscr{H}$ a real separable Hilbert space and $(B(\varphi), \varphi\in\mathscr{H})$ an isonormal Gaussian process on a probability space $(\Omega,\mathscr{A},P)$, that is a centered Gaussian family of random variables such that $\mathbf{E}\left(B(\varphi)B(\psi)\right) = \langle\varphi,\psi\rangle_{\mathscr{H}}$. Denote by I_n the multiple stochastic integral with respect to B (see [19]). This I_n is actually an isometry between the Hilbert space $\mathscr{H}^{\odot n}$ (symmetric tensor product *i.e.* the Hilbert space that contains the symmetrizations of the tensor products $h_1\otimes ...\otimes h_n$, with $h_i\in\mathscr{H}$, $i=1,..,n$)) equipped with the scaled norm $\frac{1}{\sqrt{n!}}\|\cdot\|_{\mathscr{H}^{\otimes n}}$ and the Wiener chaos of order n which is defined as the closed linear span of the random variables $H_n(B(\varphi))$ where $\varphi\in\mathscr{H}, \|\varphi\|_{\mathscr{H}} = 1$ and H_n is the Hermite polynomial of degree $n\geq 1$

$$H_n(x) = \frac{(-1)^n}{n!}\exp\left(\frac{x^2}{2}\right)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\exp\left(-\frac{x^2}{2}\right)\right), \qquad x\in\mathbb{R}.$$

The isometry of multiple integrals can be written as: for m,n positive integers,

$$\begin{aligned}\mathbf{E}\left(I_n(f)I_m(g)\right) &= n!\langle f,g\rangle_{\mathscr{H}^{\otimes n}} \quad \text{if } m=n,\\ \mathbf{E}\left(I_n(f)I_m(g)\right) &= 0 \quad \text{if } m\neq n.\end{aligned} \tag{3}$$

It also holds that

$$I_n(f) = I_n\big(\tilde{f}\big)$$

where $\tilde{f}$ denotes the symmetrization of f defined by $\tilde{f}(x_1,\dots,x_n) = \frac{1}{n!}\sum_{\sigma\in\mathscr{S}_n} f(x_{\sigma(1)},\dots,x_{\sigma(n)})$.

We recall that any square integrable random variable which is measurable with respect to the σ-algebra generated by B can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n\geq 0} I_n(f_n) \tag{4}$$

where $f_n \in \mathscr{H}^{\odot n}$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}[F]$.

Let L be the Ornstein-Uhlenbeck operator

$$LF = -\sum_{n\geq 0} nI_n(f_n)$$

if F is given by (4).

For $p > 1$ and $\alpha \in \mathbb{R}$ we introduce the Sobolev-Watanabe space $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|(I-L)^{\frac{\alpha}{2}}F\|_{L^p(\Omega)}$$

where I represents the identity. In this way, a random variable F as in (4) belongs $\mathbb{D}^{\alpha,2}$ if and only if

$$\sum_{n\geq 0}(1+n)^{\alpha}\|I_n(f_n)\|^2_{L^2(\Omega)} = \sum_{n\geq 0}(1+n)^{\alpha} n!\|f_n\|^2_{\mathscr{H}^{\otimes n}} < \infty. \tag{5}$$

Throughout this paper we will denote by $p_s(x)$ the Gaussian kernel of variance $s > 0$ given by $p_s(x) = \frac{1}{\sqrt{2\pi s}}e^{-\frac{x^2}{2s}}$, $x \in \mathbb{R}$ and for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ by $p_s^d(x) = \prod_{i=1}^d p_s(x_i)$.

2.3. THE LOCAL TIME

We first introduce the local time of a stochastic process $(X_t)_{t\in T}$. See [3,4] for a more complete exposition on local times for Gaussian processes. For any Borel set $I \subset T$ the occupation measure of X on I is defined as

$$\mu_I(A) = \lambda\left(t \in I, X_t \in A\right), \qquad A \in \mathscr{B}(\mathbb{R})$$

where λ denotes the Lebesque measure. If μ_I is absolutely continuous with respect to the Lebesque measure, we say that the process X has local time on I. The local time is defined as the Radon-Nykodim derivative of μ_I with respect to the Lebesque measure

$$L(I,x) = \frac{\mathrm{d}\mu_I}{\mathrm{d}\lambda}(x), \qquad x \in \mathbb{R}.$$

The local time $L(I,x)$ measures the time spent in I by the process at $x \in \mathbb{R}$. We will use the notation $L(t,x) := L([0,t],x), \quad t \in \mathbb{R}_+, x \in \mathbb{R}$. The local time satisfies the occupation time formula

$$\int_I f(X_t)\mathrm{d}t = \int_{\mathbb{R}} f(x)L(I,x)\mathrm{d}x \tag{6}$$

for any Borel set I in T and for any measurable non-negative function $f : \mathbb{R} \to \mathbb{R}$.

Let now X be isonormal Gaussian process with variance $R(s,t) = \mathbf{E}X_tX_s$ as introduced in Section 2.2. The local time of X can be formally written as (see *e.g.* [20] or [21])

$$L(t,x) = \int_0^t \delta(x - X_s)\mathrm{d}s \tag{7}$$

where δ denotes the Dirac function and the quantity $\delta(x - X_s)$ can be understood as a distribution in the Watanabe spaces, see Section 2.2.

We will use the following decomposition of the delta Dirac function (see Nualart and Vives [20], Imkeller *et al.* [14], Eddahbi *et al.* [11,12]; see also [17] for a general theory) into orthogonal multiple Wiener-Itô integrals

$$\delta(x - X_s) = \sum_{n\geq 0} R(s)^{-\frac{n}{2}} p_{R(s)}(x) H_n\left(\frac{x}{R(s)^{\frac{1}{2}}}\right) I_n\left(1_{[0,s]}^{\otimes n}\right) \tag{8}$$

where $R(s) := R(s,s)$, $p_{R(s)}$ is the Gaussian kernel of variance $R(s)$, H_n is the Hermite polynomial of degree n and I_n represents the multiple Wiener-Itô integral of degree n with respect to the Gaussian process X as defined in Section 2.2.

3. REGULARITY OF THE LOCAL TIME

Consider X a Gaussian process that satisfies the assumptions stated in Section 2.1. We analyze in this paragraph the regularity of the local time $L(t,x)$ of X, with t,x fixed, viewed as a functional in the Sobolev-Watanabe spaces. Before proving our main result, let us state and prove two auxiliary lemmas.

LEMMA 1. *Let X be a Gaussian process that satisfies the quasi-helix property (1) and (2). Then there exists a constant $c > 0$ such that*

$$R(t,s)^2 \leq R(s)R(t) - c\mathbf{E}\left|X_t - X_s\right|^2$$

for every $s,t \in I$.

Proof. This follows from relation (3.15) in [6] by using assumptions (2) and (1). □

The following lemma is the key point for the obtention of the regularity of the local time.

LEMMA 2. *Assume that X is a Gaussian process satisfying conditions C1) –C4). Then, for n large enough, there exist $C > 0$ such that*

$$\int_0^1 \left| \frac{R(1,z)}{\sqrt{R(1)R(z)}} \right|^n R(z)^{-\frac{1}{2}} \mathrm{d}z \le Cn^{-\frac{1}{2\alpha}}.$$

Proof. Let us denote by $F(z) := \frac{|R(1,z|)}{\sqrt{R(1)R(z)}}$ for every $z \in [0,1]$.

We will divide the integral above upon the intervals $(0, 1-\delta)$ and $(1-\delta, 1)$ with δ an arbitrary point between 0 and 1. So

$$\begin{aligned}\int_0^1 \left(\frac{|R(1,z)|}{\sqrt{R(1)R(z)}} \right)^n R(z)^{-\frac{1}{2}} \mathrm{d}z &= \int_0^{1-\delta} |F(z)|^n R(z)^{-\frac{1}{2}} \mathrm{d}z \\ &\quad + \int_{1-\delta}^1 |F(z)|^n R(z)^{-\frac{1}{2}} \mathrm{d}z \\ &:= J_1 + J_2.\end{aligned}$$

Note that F takes values in the interval $[0,1]$, it is continuous, $F(0) = 0, F(1) = 1$. Moreover $F(z) = 1$ if and only if $z = 1$ from a well-known property of the covariance function and the trivial fact that X_1 and X_z are not proportional. This implies that there exists a constant $c_0 \in (0,1)$ such that $F(z) \le c_0$ for every $z \in (0, 1-\delta)$. Thus

$$J_1 \le C(H, \alpha, \delta) c_0^n.$$

Let us regard the term J_2. By using Lemma 1, we see that the term J_2 can be bounded as follows (c is a generic constant small enough):

$$\begin{aligned} J_2 &\le \int_{1-\delta}^1 \mathrm{d}z \left(1 - c\frac{(1-z)^{2\alpha}}{\sqrt{R(1)R(z)}} \right)^{\frac{n}{2}} R(z)^{-\frac{1}{2}} \\ &\le c(H,d,\alpha,\delta) \int_{1-\delta}^1 \mathrm{d}z \left(1 - c(1-z)^{2\alpha}\right)^{\frac{n}{2}} \\ &= c(H,d,\alpha,\delta) \int_{1-\delta}^1 \mathrm{d}z e^{\frac{n}{2}\log\left(1-c(1-z)^{2\alpha}\right)} \end{aligned}$$

where the constant c in the exponential function also depends on H, δ, d and we noticed the trivial fact that $R(z)$ is bounded below by a strictly positive constant for z outside the origin (condition C3)). Using the inequality $-\log z \geq 1-z$ for every $z \in (0,1]$ and hence for every $z \in (1-\delta, 1)$, we get

$$\begin{aligned} J_2 &\leq c(H,d,\alpha,\delta) \int_{1-\delta}^{1} \mathrm{d}z e^{-\frac{n}{2}c(1-z)^{2H-\frac{d-\alpha}{2}}} \\ &= c(H,d,\alpha,\delta) \int_{0}^{\delta} \mathrm{d}z e^{-\frac{n}{2}cz^{2H-\frac{d-\alpha}{2}}} \end{aligned}$$

and by $\frac{n}{2}z^{2\alpha} = y$ we find that

$$J_2 \leq c(H,\alpha,d,\delta) n^{-\frac{1}{2\alpha}}. \quad \square$$

Let us now state our main result. It gives the chaos expansion of the local time of the process X and its regularity in the Watanabe spaces.

Theorem 1. *For every $t \geq 0$ and $x \in \mathbb{R}$ the local time $L(t,x)$ of the process X admits the following chaos expansion into multiple Wiener-Itô integrals* (9)

$$L(t,x) = \int_0^t \delta(x - X_s)\mathrm{d}s = \int_0^t \sum_{n\geq 0} R(s)^{-\frac{n}{2}} p_{R(s)}(x) H_n\left(\frac{x}{R(s)^{\frac{1}{2}}}\right) I_n(1_{[0,s]^{\otimes n}})\mathrm{d}s.$$

Moreover $L(t,x)$ belongs to the Sobolev-Watanabe space $\mathbb{D}^{\gamma,2}$ for every

$$\gamma < \frac{1}{2\alpha} - \frac{1}{2}. \tag{10}$$

Proof. The decomposition (9) follows from the chaos expansion of the deta Dirac function (8) and (7). Let us compute the $\mathbb{D}^{\gamma,2}$ norm of the random variable $L(t,x)$ with $t \geq 0$ and $x \in \mathbb{R}$. The approach is classical and it has been already used in several works (*e.g.* [11,20,23]). By using the isometry of multiple Wiener-Itô integrals (3) and the expression of the Sobolev-Watanabe norm (5), we can write

$$\begin{aligned} &\|L(t,x)\|_{\gamma,2}^2 \\ &= \sum_{n\geq 0}(1+n)^\gamma \int_0^t\int_0^t \mathrm{d}u\mathrm{d}v R(u)^{-\frac{n}{2}} R(v)^{-\frac{n}{2}} p_{R(u)}(x) H_n\left(\frac{x}{R(u)^{\frac{1}{2}}}\right) \\ &\quad p_{R(v)}(x) H_n\left(\frac{x}{R(v)^{\frac{1}{2}}}\right) \mathbf{E} I_n(1_{[0,u]}^{\otimes n}) I_n(1_{[0,v]}^{\otimes n}) \\ &= \sum_{n\geq 0}(1+n)^\gamma n! \int_0^t\int_0^t \mathrm{d}u\mathrm{d}v R(u)^{-\frac{n}{2}} R(v)^{-\frac{n}{2}} p_{R(u)}(x) H_n\left(\frac{x}{R(u)^{\frac{1}{2}}}\right) \\ &\quad p_{R(v)}(x) H_n\left(\frac{x}{R(v)^{\frac{1}{2}}}\right) \times \langle 1_{[0,u]}^{\otimes n}, 1_{[0,v]}^{\otimes n}\rangle_{\mathscr{H}^{\otimes n}} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\geq 0}(1+n)^{\gamma}n!\int_0^t\int_0^t \mathrm{d}u\mathrm{d}v R(u)^{-\frac{n}{2}}R(v)^{-\frac{n}{2}}p_{R(u)}(x)H_n\left(\frac{x}{R(u)^{\frac{1}{2}}}\right)\\
&\quad p_{R(v)}(x)H_n\left(\frac{x}{R(v)^{\frac{1}{2}}}\right)R(u,v)^n\\
&= 2\sum_{n\geq 0}(1+n)^{\gamma}n!\int_0^t \mathrm{d}v\int_0^u \mathrm{d}v R(u)^{-\frac{n}{2}}R(v)^{-\frac{n}{2}}p_{R(u)}(x)H_n\left(\frac{x}{R(u)^{\frac{1}{2}}}\right)\\
&\quad p_{R(v)}(x)H_n\left(\frac{x}{R(v)^{\frac{1}{2}}}\right)R(u,v)^n.
\end{aligned}
$$

Above $\langle\cdot,\cdot\rangle$ denotes the scalar product in the Hilbert space associated with the Gaussian process X. We use the identity (see *e.g.* [7])

$$
H_n(y)e^{-\frac{y^2}{2}} = (-1)^{[n/2]}2^{\frac{n}{2}}\frac{2}{n!\pi}\int_0^\infty u^n e^{-u^2}g(uy\sqrt{2})\mathrm{d}u \tag{11}
$$

where $g(r)=\cos(r)$ if n is even and $g(r)=\sin(r)$ if n is odd. Since $|g(r)|\leq 1$, we have the bound

$$
\left|H_n(x)e^{-\frac{y^2}{2}}\right| \leq 2^{\frac{n}{2}}\frac{2}{n!\pi}\Gamma(\frac{n+1}{2}) := c_n \tag{12}
$$

Thus (with C a generic strictly positive constant)

(13)

$$
\begin{aligned}
\|L(t,x)\|^2_{\gamma,2} &\leq C\sum_{n\geq 0}(1+n)^{\gamma}n!c_n^2\int_0^t\int_0^u \mathrm{d}u\mathrm{d}v R(u)^{-\frac{n+1}{2}}R(v)^{-\frac{n+1}{2}}|R(u,v)|^n\\
&= C\sum_{n\geq 0}(1+n)^{\gamma}n!c_n^2\int_0^t \mathrm{d}u R(u)^{-\frac{n+1}{2}}u\int_0^1 \mathrm{d}z R(uz)^{-\frac{n+1}{2}}|R(uz,u)|^n
\end{aligned}
$$

where we made the change of variables $\frac{v}{u}=z$ in the integral $\mathrm{d}v$.

Notice that for every $u,z\geq 0$ it holds that

$$
R(uz,uz') = u^{2H}R(z,z'). \tag{14}
$$

This is an immediate consequence of the self-similarity property. From 14, we immediately get

$$
R(u,uz)=u^{2H}R(1,z),\quad R(uz)=u^{2H}R(z),\quad R(u)=u^{2H}R(1)
$$

and therefore, by (13)

$$
\|L(t,x)\|^2_{\gamma,2} \quad\leq\quad C\sum_{n\geq 0}(1+n)^{\gamma}n!c_n^2\int_0^t \mathrm{d}u u^{1-2H}
$$

$$\times \int_0^1 \left(\frac{|R(1,z)|}{\sqrt{R(1)R(z)}} \right)^n R(z)^{-\frac{1}{2}} \mathrm{d}z$$

and since $n!c_n^2$ behaves as $\sqrt{n}$ for n large enough, and using also Lemma 2, we obtain that, with some n_0 enough,

$$\sum_{n\geq 0} (1+n)^\gamma n! c_n^2 \int_0^t \mathrm{d}u u^{1-2H} \int_0^1 \left(\frac{R(1,z)}{\sqrt{R(1)R(z)}} \right)^n R(z)^{-\frac{1}{2}} \mathrm{d}z$$
$$\leq C \sum_{n\geq n_0} (1+n)^\gamma \sqrt{n} n^{-\frac{1}{2\alpha}}$$

which is convergent if $\frac{1}{2\alpha} - \gamma - \frac{1}{2} > 1$ which gives $\gamma < \frac{1}{2\alpha} - \frac{1}{2}$. $\square$

4. EXAMPLES

In this paragraph, we will discuss several examples of Gaussian processes that fulfill the conditions C1)-C4) in Section 2.1. In the first three examples treated below (the fractional Brownian motion, the bifractional Brownian motion and the solution to the heat equation with fractional-colored noise) the self-similarity index coincides with the order of the regularity of the sample paths. In the last two examples, these two parameters are different.

4.1. THE FRACTIONAL BROWNIAN MOTION

The fractional Brownian motion with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $(B_t^H)_{t\geq 0}$ with covariance

$$\mathbf{E} B_t^H B_s^H = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \qquad s,t \geq 0. \tag{15}$$

Notice that assumption C4) holds. The process B^H is self-similar of index H, so it satisfies condition C1). It is actually the only self-similar Gaussian process with stationary increments. Moreover, for every $s,t \geq 0$,

$$\mathbf{E} \left| B_t^H - B_s^H \right|^2 = |t-s|^{2H}$$

and consequently C2) holds. Clearly C3) is satisfied on every interval $I = [t_0, T]$ with $t_0 > 0$. Then the local time of the fBm belongs to the Watanabe space $\mathbb{D}^{\gamma,2}$ with $\gamma < \frac{1}{2H} - \frac{1}{2}$. This has been already proved in [11]. The order H corresponds to the order of Hölder regularity of B^H. It coincides in this case with the self-similarity index.

4.2. THE BIFRACTIONAL BROWNIAN MOTION

The bifractional Brownian motion $(B_t^{H,K})_{t\geq 0}$ is a centered Gaussian process, starting from zero, with covariance

$$R^{H,K}(t,s) = \frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K - |t-s|^{2HK}\right), \qquad s,t\geq 0 \tag{16}$$

with $H \in (0,1)$ and $K \in (0,1]$. We refer to [13] and [23] for the definition and the basic properties of this process. Note that, if $K=1$ then $B^{H,1}$ is the fractional Brownian motion with Hurst parameter $H \in (0,1)$. When $K=1$ and $H=\frac{1}{2}$ then it reduces to the standard Brownian motion. It follows from (16) that this stochastic process is self-similar of order HK. It has been proven in [13] that for every $s,t\geq 0$

$$2^{-K}|t-s|^{2HK} \leq \mathbf{E}\left|B_t^{H,K} - B_s^{H,K}\right|^2 \leq 2^{1-K}|t-s|^{2HK}.$$

Thus C2) and C3) are satisfied on every interval $I=[t_0,T]$ with $t_0>0$. From (16) we notice that C4) also holds. Consequently the local time of the fBm belongs to the Watanabe space $\mathbb{D}^{\gamma,2}$ with $\gamma < \frac{1}{2HK} - \frac{1}{2}$. We retrieve in this way the result in [23].

4.3. THE SUBFRACTIONAL BROWNIAN MOTION

This process has been introduced in [5]. The subfractional Brownian motion (sub-fBm) is defined as a centered Gaussian process $(S_t^H)_{t\geq 0}$ with covariance

$$R(t,s) = s^{2H} + t^{2H} - \frac{1}{2}\left((s+t)^{2H} + |t-s|^{2H}\right), \qquad s,t\geq 0$$

with $H \in (0,1)$.

The sub-fBm arises from occupation time fluctuations of branching particle systems (see [5]). It has properties analogous to those of fBm (self-similarity, long-range dependence, Hölder paths, variation and renormalized variation and it is neither a Markov processes nor a semimartingale). The increments of the process S^H behaves in the following way

$$(2-2^{2H-1})|t-s|^{2H} \leq \mathbf{E}\left(S_t^H - S_s^H\right)^2 \leq |t-s|^{2H}, \qquad \text{si } H>1/2$$

and

$$|t-s|^{2H} \leq \mathbf{E}\left(S_t^H - S_s^H\right)^2 \leq (2-2^{2H-1})|t-s|^{H}, \qquad \text{si } H<1/2.$$

See [5] or [24], see also [26, 27] for other results on this process. That means that the sub-fBm is, as the bi-fBm, a quasi-helix. Thus C2) and C3) are satisfied on every interval $I = [t_0, T]$ with $t_0 > 0$ and C4) also holds, so its local time is in the space $\mathbb{D}^{\gamma,2}$ with $\gamma < \frac{1}{2H} - \frac{1}{2}$.

4.4. THE SOLUTION TO THE HEAT EQUATION WITH FRACTIONAL-COLORED NOISE

First consider "the noise" W^H defined as a centered Gaussian process $W^H = \{W^H(t, A); t \geq 0, A \in \mathscr{B}_b(\mathbb{R}^d)\}$ with covariance:

$$\mathbf{E}(W^H(t, A)W^H(s, B)) = R_H(t, s) \int_A \int_B f(y - y') \mathrm{d}y \mathrm{d}y'. \tag{17}$$

where R_H denotes the covariance of the fractional Brownian motion (15) and f is the Riesz kernel of order α given by

$$f(x) = R_\alpha(x) := \gamma_{\alpha,d} |x|^{-d+\alpha}, \quad 0 < \alpha < d, \tag{18}$$

where $\gamma_{\alpha,d} = 2^{d-\alpha}\pi^{d/2}\Gamma((d-\alpha)/2)/\Gamma(\alpha/2)$. Note that f is the Fourier transform of the measure $\mu(d\xi) = |\xi|^{-\alpha} d\xi$.

Consider the stochastic process $(u(t, x), t \geq 0, x \in \mathbb{R}^d)$ given by

$$u(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t - u, x - y) W^H(\mathrm{d}s, \mathrm{d}y), \qquad t \geq 0, x \in \mathbb{R}^d \tag{19}$$

where the above integral is a standard Wiener integral with respect to the Gaussian noise W^H (see [1]) and G is the solution of $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$. The process u is actually the mild solution to the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u + \dot{W}^H, \quad t \geq 0, \ x \in \mathbb{R}^d \\ u(0, x) &= 0, \quad x \in \mathbb{R}^d, \end{aligned} \tag{20}$$

where Δ denotes the Laplacian on $\mathbb{R}^d$ and the noise W is defined by (17). Recall (see [1, 2]) that the process $(u(t, x))_{t \in [0,T], x \in \mathbb{R}^d}$ exists and satisfies

$$\sup_{t \in [0,T], x \in \mathbb{R}^d} \mathbf{E}\left(u(t, x)^2\right) < +\infty$$

if and only if

$$\int_{\mathbb{R}^d} \left(\frac{1}{1 + |\xi|^2}\right)^{2H} |\xi|^{-\alpha} \mathrm{d}\xi < \infty.$$

and this translates into

$$d < 4H + \alpha. \tag{21}$$

The covariance of the process $(u(t,x))_{t\geq 0}$ (here $x\in\mathbb{R}^d$ is fixed) is given by

(22)
$$R(t,s)=\mathbf{E}u(t,x)u(s,x)=d(\alpha,H)\int_0^t\int_0^s |u-v|^{2H-2}(t+s-u-v)^{-\frac{d-\alpha}{2}}dvdu.$$

with $d(\alpha,H)$ a strictly positive constant and as mentioned before $H\in(\frac{1}{2},1)$ and $0<\alpha<d<4H+\alpha$. From (22) we notice that the process $t\to u(t,x)$ is self-similar with parameter

$$H-\frac{d-\alpha}{4}.$$

The self-similarity index is strictly positive under (21). Therefore condition C1) is satisfied.

The increments of the solution to (20) satisfy the following (see [22]): there exists two strictly positive constants C_1,C_2 such that for any $t,s\geq 0$ and for any $x\in\mathbb{R}^d$

$$C_1|t-s|^{2H-\frac{d-\alpha}{2}}\leq \mathbf{E}\left|u(t,x)-u(s,x)\right|^2\leq C_2|t-s|^{2H-\frac{d-\alpha}{2}}. \tag{23}$$

Therefore again assumptions C2) and C3) are satisfied on intervals of the form $I=[t_0,T]$ with $t_0>0$. It is not difficult to see that C4) holds true.

The local time of the process $t\to u(t,x)$ ($x\in\mathbb{R}^d$ is fixed) belongs to the Sobolev-Watanabe space $\mathbb{D}^{\gamma,2}$ for every

$$\gamma<\frac{1}{2H-\frac{d-\alpha}{2}}-\frac{1}{2}. \tag{24}$$

Let us notice the following facts: the Hölder regularity index and the self-similarity index coincide once again as in the previous examples. On the other side, in this case the order γ may be always negative. Actually, the solution to the fractional-colored heat equation (20) admits a local time in $L^2(\Omega)=\mathbb{D}^{0,2}$ if

$$2H-\frac{d-\alpha}{2}\leq\frac{3}{2}.$$

4.5. THE SOLUTION TO THE WAVE EQUATION WITH WHITE NOISE IN TIME

Consider the linear stochastic wave equation driven by a white-colored noise W. That is

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x)+\dot{W}(t,x), \quad t\geq 0, x\in\mathbb{R}^d\\ u(0,x) &= 0, \quad x\in\mathbb{R}^d\end{aligned} \tag{25}$$

$$\frac{\partial u}{\partial t}(0,x) \quad = \quad 0, \quad x \in \mathbb{R}^d.$$

Here Δ is the Laplacian on $\mathbb{R}^d$ and $W = \{W_t(A); t \geq 0, A \in \mathscr{B}_b(\mathbb{R}^d)\}$ is a centered Gaussian field with covariance

$$\mathbf{E}(W_t(A)W_s(B)) = (t \wedge s) \int_A \int_B f(x-y)\mathrm{d}x\mathrm{d}y \tag{26}$$

where f is Riesz kernel (18). The solution of (25) is a square-integrable process $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ defined by the Wiener integral representation with respect to the noise (26)

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s, x-y)W(\mathrm{d}s, \mathrm{d}y) \tag{27}$$

with G_1 the fundamental solution of $\frac{\partial^2 u}{\partial t^2}(t,x) - \Delta u = 0$. The solution exists when the above integral is well-defined. As for the heat equation, it depends on the dimension d and on the spatial covariance. For example, when the noise is white both in time and in space the solution exists if and only if $d = 1$.

The necessary and sufficient condition for the existence of the solution follows from [8]. The stochastic wave equation (25) admits an unique mild solution $(u(t,x))_{t \geq 0, x \in \mathbb{R}^d}$ if and only if

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right) |\xi|^{-\alpha}\mathrm{d}\xi < \infty \tag{28}$$

which means $d < 2 + \alpha$.

Fix $x \in \mathbb{R}^d$. The covariance of $(u(t,x)_{t \geq 0}$ can be expressed as

$$\mathbf{E}u(t,x)u(s,x) \quad = \quad \int_0^{t \wedge s} \mathrm{d}u \int_{\mathbb{R}^d} \mathrm{d}\xi \frac{\sin((t-u)|\xi|)}{|\xi|} \frac{\sin((s-u)|\xi|)}{|\xi|} |\xi|^{-d}\mathrm{d}\xi.$$

It follows that the process $u(t,x), t \geq 0$ is self-similar of order $\frac{3-d+\alpha}{2}$, so condition C1) is satisfied. Also, let $t_0, M > 0$ and fix $x \in [-M,M]^d$. Then (see [9]) there exists two positive constants c_1, c_2 such that for every $s,t \in [t_0, T]$

$$c_1|t-s|^{2-d+\alpha} \leq \mathbf{E}\left|u(t,x) - u(s,x)\right|^2 \leq c_2|t-s|^{2-d+\alpha}$$

and this implies that C2)- C3) are satisfied on $I = [t_0, T]$ with $t_0 > 0$ for every $x \in [-M,M]^d$.

Let us point out an interesting fact: in the case of the solution to the wave equation, the Hölder regularity order and the self-similarity order are different. The local time belongs to the space $\mathbb{D}^{\gamma,2}$ with $\gamma < \frac{1}{2-d+\alpha} - \frac{1}{2}$. It is always a random variable in $L^2(\Omega)$ because $\frac{1}{2-d+\alpha} - \frac{1}{2} > 0$.

4.6. THE SOLUTION TO THE WAVE EQUATION WITH FRACTIONAL NOISE IN TIME

Now, the noise (26) is replaced by the fractional colored noise W^H whose covariance is defined by (17). The mild solution of (25) is a square-integrable process $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ defined by:

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s, x-y) W^H(\mathrm{d}s, \mathrm{d}y). \tag{29}$$

Again G_1 denotes the fundamental solution of the wave equation that already appeared in the previous paragraph. The above Wiener integral is well-defined if

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^{H+\frac{1}{2}} |\xi|^{-\alpha} < \infty. \tag{30}$$

This means that

$$d < \alpha + 2H + 1. \tag{31}$$

Note that the condition is different from the case of the heat equation (compare (31) and (21)).

The covariance of u can be expressed as

$$\mathbf{E}u(t,x)u(s,x) = a(H) \int_0^t \mathrm{d}u \int_0^s \mathrm{d}v |u-v|^{2H-2} \\ \int_{\mathbb{R}^d} \mathrm{d}\xi \frac{\sin((t-u)|\xi|)}{|\xi|} \frac{\sin((s-v)|\xi|)}{|\xi|} |\xi|^{-d+\beta} \mathrm{d}\xi.$$

It is easy to note that this process is self-similar of order

$$H + 1 - \frac{d-\alpha}{2}$$

which is positive, so assumption C1) holds true. From [10], there exists a positive constants c_1, c_2 such that for every $s, t \in [t_0, T]$

$$c_1 |t-s|^{2H+1-\beta} \leq \mathbf{E} \left| u(t,x) - u(s,x) \right|^2 \leq c_2 |t-s|^{2H+1-\beta}.$$

The above inequality gives C2) and C3) for every time interval outside the origin. Finally, we note that C4) is clearly satisfied.

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REFERENCES

[1] R. Balan and C.A. Tudor, *The stochastic heat equation with fractional-colored noise: existence of the solution.* ALEA Lat. Am. J. Probab. Math. Stat. **4** (2008), 57–87.

[2] R. Balan and C.A. Tudor, *The stochastic wave equation with fractional noise: A random field approach.* Stochastic Process. Appl. **120** (2010), 2468–2494.

[3] S.M. Berman, *Local nondeterminism and local times of Gaussian processes.* Indiana Univ. Math. J. **23** (1973), 69–94.

[4] S.M. Berman, *Local times and sample function properties of stationary Gaussian processes.* Trans. Amer. Math. Soc. **137** (1969), 277–299.

[5] T. Bojdecki, L. Gorostiza and A. Talarczyk, *Sub-fractional Brownian motion and its relation to occupation times.* Statist. Probab. Lett. **69** (2004), 405–419.

[6] H. Biermé, C. Lacaux and Y. Xiao, *Hitting probabilities and the Hausdorff dimension of the inverse images of anisotropic Gaussian random fields.* Bull. Lond. Math. Soc. **41** (2009), 253–273.

[7] L. Coutin, D. Nualart and C.A. Tudor, *The Tanaka formula for the fractional Brownia motion.* Stochastic Process. Appl. **94** (2001), 301–315.

[8] R. Dalang, *Extending the martingale measure stochastic integral with applications to spatially homogeneous SPDE's.* Electron. J. Probab. **4** (1999), 1–29. Erratum in Electron. J. Probab. **6** (2001), 5.

[9] R. Dalang and M. Sanz-Solé, *Hölder-Sobolev regularity of the solution to the stochastic wave equation in dimension three.* Mem. Amer. Math. Soc. **199** (2009), 1–70.

[10] J. Clarke De la Cerda and C.A. Tudor, *Hitting probabilities for the solution of the wave equation with fractional-colored noise.* Rev. Mat. Iberoam. **30** (2014), 685–709.

[11] M. Eddahbi, R. Lacayo, J.L. Sole, C.A. Tudor and J. Vives, *Regularity of the local time for the d-dimensional fractional Brownian motion with N-parameters.* Stoch. Anal. Appl. **23** (2005), 383–400.

[12] M. Eddahbi, R. Lacayo, J.L. Sole, C.A. Tudor and J. Vives, *Renormalization of the local times for d dimensional fractional Brownian motion with N parameters.* Nagoya Math. J. **207** (2006).

[13] C. Houdré and J. Villa, *An example of infinite dimensional quasi-helix.* Contemp. Math. **336** (2003), 195–201.

[14] P. Imkeller and P. Weisz, *The asymptotic behavior of local times and occupation integrals of the N-parameter Wiener process in $\mathbb{R}^d$.* Probab. Theory Related Fields **98** (2004), 47–75.

[15] J.P. Kahane, *Hélices et quasi-hélices.* Adv. Math. **7B** (1981), 417–433.

[16] J.P. Kahane, *Some Random Series of Functions.* Cambridge Univ. Press, 1985.

[17] H.H. Kuo, *White Noise Distribution Theory.* CRC Press, Boca Raton, 1996.

[18] M. Marcus and J. Rosen, *Markov Processes, Gaussian Processes, and Local Times.* Cambridge Studies in Advanced Mathematics, 2006.

[19] D. Nualart, *Malliavin Calculus and Related Topics.* Springer, 1995.

[20] D. Nualart and J. Vives, *Smoothness of Brownian local times and related functionals.* Potential Anal. **1** (1992), 257-263.

[21] D. Nualart and J. Vives, *Chaos expansions and local times.* Publ. Mat. **36** (1992), 827–836.

[22] H. Ouahhabi and C.A. Tudor, *Additive functionals of the solution to fractional stochastic heat equation.* J. Fourier Anal. Appl. **19** (2012), 777–791.

[23] F. Russo and C.A. Tudor, *On the bifractional Brownian motion.* Stochastic Process. Appl. **5** (2006), 830–856.

[24] C. Tudor, *Some properties of the sub-fractional Brownian motion.* Stochastics **79** (2007), 431–448.

[25] C.A. Tudor, *Chaos expansion and regularity of the local time of the solution to the stochastic heat equation with additive fractional-colored noise.* Taiwanese J. Math. **17** (2013), 1765–1777.

[26] M. Tudor, *Double subfractional integrals and mollifier approximation.* Math. Rep. (Bucur.) **9(59)** (2007), 385–390.

[27] M. Tudor, *A strong approximation for double subfractional integrals.* Appl. Anal. **86** (2007), 1037–1048.

[28] S. Watanabe, *Lectures on Stochastic Differential Equations and Malliavin Calculus.* Springer-Verlag, 1994.

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