HARMONIC MAPS ON ALMOST KENMOTSU MANIFOLDS

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In this paper, we first investigate harmonic maps and harmonic morphisms between almost Kenmotsu manifolds and almost Hermitian manifolds, extending some earlier results. Harmonic maps between almost Kenmotsu manifolds and contact metric manifolds are also investigated.

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1. INTRODUCTION

In 1964, J. Eells and J. H. Sampson [5] started the study of harmonic maps on Riemannian manifolds and proved that a holomorphic map between Kähler manifolds is harmonic. Motivated by their interesting result, since then many authors have investigated harmonic maps from both analytic and geometric points of view. Nowadays, the theory of harmonic maps becomes a very important field of research in differential geometry. In 1970, A. Lichnerowicz in [17] considered harmonic maps between Riemannian manifolds endowed with some special structures, namely, compact almost Kähler, semi-Kähler and quasi-Kähler structures, proving that a holomorphic map between two Kähler manifolds is not only a harmonic map but also attains the minimum of energy in its homotopy class. As a counterpart, S. Ianus and A. M. Pastore in [12] first studied harmonic maps on some odd dimensional smooth manifolds, namely, contact metric and Sasakian manifolds, proving that any $\pm\phi$-holomorphic map between two contact metric manifolds is harmonic. Recently, D. Chinea in [2] obtained some formula regarding the harmonicity of maps between two almost contact metric manifolds, extending the above result proved by Ianus and Pastore. Here we refer the readers to C. Gherghe [8, 9], C. Gherghe, S. Ianus and A. M. Pastore [10, 11] and N. A. Rehman [20, 21] for more recent results on harmonic maps between almost contact metric manifolds and almost Hermitian manifolds.

One object of this paper is to extend some earlier results regarding harmonic maps on Kenmotsu manifolds proved by Gherghe [8] and Rehman [20] to some types of almost Kenmotsu manifolds. In Section 3, it is proved that any (ϕ, J)-holomorphic map from a CR-integrable almost Kenmotsu manifold to a quasi-Kähler manifold is harmonic. Moreover, we also obtain that any (J, ϕ)-holomorphic map from a semi-Kähler manifold to a CR-integrable almost Kenmotsu manifold is harmonic if and only if it is a constant map. In Section 4, a characterization of the harmonicity of a $(\phi, \overline{\phi})$-holomorphic map between two almost Kenmotsu manifolds is obtained. Finally, we prove that a $(\phi, \overline{\phi})$-holomorphic map from a contact metric manifold to an almost Kenmotsu manifold is harmonic if and only it is a constant map. Some concrete examples of almost Kenmotsu manifolds are given.

2. PRELIMINARIES

The following notions regarding harmonic maps can be seen from Eells and Sampson [5] and Gherghe [8, 9]. Let us consider two smooth Riemannian manifolds (M, g) and (N, k) and a smooth map $f : M \to N$. If we denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of g and k, respectively, then the *second fundamental form* α_f of f is defined by

$$\alpha_f(X, Y) = \nabla^f_X f_* Y - f_*(\nabla_X Y) \tag{2.1}$$

for any vector fields $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ and ∇^f are the Lie algebra of all smooth vector fields on M and the pull-back connection of $\overline{\nabla}$ to the pull-back bundle $f^{-1}TN \to M$ (which is defined by $\nabla^f_X V = \overline{\nabla}_{f_* X} V$ for any vector field $X \in \mathfrak{X}(M)$ and any section V on the induced bundle $f^{-1}TN$), respectively. The *energy density* of f is a smooth function $e(f) : M \to [0, +\infty)$ defined by

$$e(f)_p = \frac{1}{2}\mathrm{trace}_g(f^* k)_p = \frac{1}{2}\sum_{i=1}^{\dim M} k(f_* e_i, f_* e_i),$$

where $\{e_1, \cdots, e_{\dim M}\}$ is a local orthonormal basis of the tangent space $T_p M$ at $p \in M$. If M is compact, then the integral of the energy density of f defined by

$$E(f) = \int_M e(f) v_g$$

is called the *energy* of f, where v_g is the volume measure associated to the metric g. A smooth map $f : M \to N$ is said to be *harmonic* if it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} E(f_t)|_{t=0} = 0$$

for all variations $\{f_t\}$ of f. The trace of the second fundamental form of f, denoted by

$$\tau(f)_p = \sum_{i=1}^{\dim M} \alpha_f(e_i, e_i)_p, \tag{2.2}$$

is called the *tension field* of f. Following Eells and Sampson [5], a map $f : M \to N$ is said to be harmonic if and only if $\tau(f) = 0$.

Next we collect some basics concerning almost Kenmotsu manifolds. If on a smooth Riemannian manifold (M^{2m+1}, g) of dimension $2m+1$, there exist a $(1,1)$-type tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.4}$$

for any vector fields $X, Y \in \mathfrak{X}(M)$, then we call M^{2m+1} an *almost contact metric manifold* denoted by $(M^{2m+1}, \phi, \xi, \eta, g)$. It follows from (2.3) and (2.4) that

$$\eta = g(\cdot, \xi), \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0. \tag{2.5}$$

The *fundamental 2-form* Φ on M^{2m+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields $X, Y \in \mathfrak{X}(M)$. By an *almost Kenmotsu manifold*, we mean an almost contact metric manifold M^{2m+1} such that η is closed and $d\Phi = 2\eta \wedge \Phi$ (see Janssens and Vanhecke [14]). Moreover, from Blair [1], an almost contact metric manifold satisfying $d\eta = \Phi$ is said to be a *contact metric manifold.*

We consider on an almost Kenmotsu manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ a $(1,1)$-type tensor field h defined by $h = \frac{1}{2}\mathcal{L}_\xi \phi$. From Kim and Pak [16] and Dileo and Pastore [3, 4] we have

$$h\xi = 0, \quad \mathrm{trace} h = 0, \quad h\phi + \phi h = 0, \tag{2.6}$$

$$\nabla_X \xi = X - \eta(X)\xi + h\phi X, \tag{2.7}$$

for any $X \in \mathfrak{X}(M)$, and h is symmetric with respect to the metric g. On the product of an almost contact metric manifold M^{2m+1} and $\mathbb{R}$, *i.e.*, $M^{2m+1} \times \mathbb{R}$, it is easy to check that $J(X, f\frac{d}{d_t}) = (\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t})$ defines an almost complex structure, where X denotes a vector field tangent to M^{2m+1}, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^{2m+1} \times \mathbb{R}$. Moreover, M^{2m+1} is said to be *normal* provided that J is integrable, *i.e.*, the Nijenhuis tensor of J vanishes. A normal almost Kenmotsu manifold is called a *Kenmotsu manifold* (see Kenmotsu [15]) and a normal contact metric manifold is said to be a *Sasakian manifold* (see Blair [1]).

On an almost contact metric manifold $(M^{2m+1}, \phi, \xi, \eta, g)$, we denote by $\mathcal{D}$ and $J_{\mathcal{D}}$ the distribution $\mathcal{D} = \ker \eta$ and the restriction of ϕ on $\mathcal{D}$, respectively. Therefore, M^{2m+1} admits an almost CR-structure $(\mathcal{D}, J_{\mathcal{D}})$. If the integral manifolds of $\mathcal{D}$ of an almost Kenmotsu manifold M^{2m+1} are Kählerian, then M^{2m+1} is said to be a *CR-integrable almost Kenmotsu manifold* (see Dileo and Pastore [4] and Wang et al. [22, 23]). Following Falcitelli and Pastore [6, Proposition 2.2] we see that M^{2m+1} is CR-integrable if and only if

$$(\nabla_X \phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX) \tag{2.8}$$

for any $X, Y \in \mathfrak{X}(M)$. By Dileo and Pastore [3, Proposition 2], we know that an almost Kenmotsu manifold is Kenmotsu if and only if it is CR-integrable and $h = 0$, that is, $(\nabla_X \phi)Y = g(Y, \phi X)\xi - \eta(Y)(\phi X)$ (see also Kenmotsu [15]).

If on a smooth manifold N^{2n} of dimension $2n$ there exits a $(1,1)$-type tensor filed J satisfying $J^2 = -\mathrm{id}$, then N^{2n} is called an *almost complex manifold* and J is called an *almost complex structure*. On an almost complex manifold N^{2n} if there exists a Riemannian metric k satisfying $k(JX, JY) = k(X, Y)$ for any $X, Y \in \mathfrak{X}(N)$, k is called an *almost Hermitian metric* and in this case N^{2n} is called an *almost Hermitian manifold*, denoted by (N^{2n}, J, k). An almost Hermitian manifold (N^{2n}, J, k) is said to be a *quasi-Kähler manifold* if the following condition $(\nabla_X J)Y + (\nabla_{JX} J)JY = 0$ holds for any $X, Y \in \mathfrak{X}(N)$ and this is a generalization of a *Kähler manifold* (*i.e.*, $\nabla J = 0$). If on an almost Hermitian manifold (N^{2n}, J, k) there hold $\mathrm{div} J = 0$ or $(\nabla_X J)X = 0$ for any $X \in \mathfrak{X}(N)$, then N^{2n} is said to be a *semi-Kähler* or a *nearly-Kähler manifold*, respectively.

3. HARMONIC MAPS ON *CR*-INTEGRABLE ALMOST KENMOTSU MANIFOLDS

Suppose that $f : M \to N$ is a smooth map from an almost contact metric manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ into an almost Hermitian manifold (N^{2n}, J, k), then f is said to be a *(ϕ, J)-holomorphic map* if it satisfies

$$f_* \circ \phi = J \circ f_*. \tag{3.1}$$

Similarly, a map $f : N \to M$ is said to be a *(J, ϕ)-holomorphic* if it satisfies

$$f_* \circ J = \phi \circ f_*, \tag{3.2}$$

for more details see Ianus and Pastore [12] and Gherghe [8, 9].

Theorem 3.1. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a CR-integrable almost Kenmotsu manifold and (N^{2n}, J, k) a quasi-Kähler manifold, then any (ϕ, J)-holomorphic map $f : M \to N$ is harmonic.*

Proof. For any (ϕ, J)-holomorphic map $f : M \to N$, it follows from Gherghe [9] that

$$J(\tau(f)) = f_*(\mathrm{div}\phi) - \mathrm{trace}_g\beta, \tag{3.3}$$

where $\beta(X,Y) = (\nabla^f_X J)(f_*Y)$ for any vector fields $X, Y \in \mathfrak{X}(M)$. Since f is a (ϕ, J)-holomorphic map, then we have $f_*(\phi\xi) = J(f_*\xi) = 0$ and hence we get $f_*\xi = 0$. Considering a local orthonormal ϕ-basis $\{\xi, e_1, \ldots, e_m, \phi e_1, \ldots, \phi e_m\}$ on T_pM for any $p \in M$, using $f_*(\xi) = 0$ and omitting the summation symbol for the repeated indices $(1 \le i \le m)$, we obtain

$$\begin{aligned}\mathrm{trace}_g\beta =&(\overline{\nabla}_{f_*\xi}J)(f_*\xi) + (\overline{\nabla}_{f_*e_i}J)(f_*e_i) + (\overline{\nabla}_{f_*\phi e_i}J)(f_*\phi e_i)\\ =&(\overline{\nabla}_{f_*e_i}J)(f_*e_i) + (\overline{\nabla}_{Jf_*e_i}J)(Jf_*e_i)\\ =&0,\end{aligned} \tag{3.4}$$

where the last equality holds because N is a quasi-Kähler manifold. On the other hand, by Wang and Liu [23, Lemma 3.4] we see that on a CR-integrable almost Kenmotsu manifold there holds $\mathrm{div}\phi = 0$. Making use of this relation and (3.4) in equation (3.3) we obtain $J(\tau(f)) = 0$ and hence we get $\tau(f) = 0$. This completes the proof. □

Let $f : M \to N$ be a smooth map between two Riemannian manifolds (M, g) and (N, k). If for each open subset U of N with $f^{-1}(U) \neq \emptyset$ and each harmonic function $l : U \to \mathbb{R}$ the composition $l \circ f : f^{-1}(U) \to \mathbb{R}$ is harmonic, then f is said to be a *harmonic morphism*. Following Fuglede [7] and Ishihara [13], we know that f is a harmonic morphism if and only if it is a horizontally conformal harmonic map.

COROLLARY 3.1. *Let (M, ϕ, ξ, η, g) be a CR-integrable almost Kenmotsu manifold and (N, J, k) be an almost Hermitian manifold, then any (ϕ, J)-holomorphic morphism $f : M \to N$ is harmonic if and only if N is a semi-Kähler manifold.*

Proof. From Gherghe, Ianus and Pastore [10] we see that for a (ϕ, J)-holomorphic map f from an almost contact metric manifold M into an almost Hermitian manifold N, any two of the following statements imply the third: (i) $\mathrm{div}(J) = 0$, (ii) $f_*(\mathrm{div}\phi) = 0$, (iii) f is harmonic and so is harmonic morphism. Therefore, the following proof follows from Wang and Liu [23, Lemma 3.4], *i.e.*, in this case there holds $\mathrm{div}\phi = 0$. □

LEMMA 3.1. *Let $f : N \to M$ be a (J, ϕ)-holomorphic map from an almost Hermitian manifold (N, J, k) to an almost Kenmotsu manifold (M, ϕ, ξ, η, g), then we have*

$$\eta(\tau(f)) = -\mathrm{trace}_k f^*g. \tag{3.5}$$

Proof. Given a vector field $X \in \mathfrak{X}(N)$, as f is a (J, ϕ)-holomorphic map then we have

$$g(f_*X, \xi) = -g(f_*J^2X, \xi) = -g(\phi(f_*(JX)), \xi) = 0. \tag{3.6}$$

Considering a local orthonormal J-basis $\{u_1, \dots, u_n, Ju_1, \dots, Ju_n\}$ on T_pN for any $p \in N$, and omitting the summation symbol for repeated indices $(1 \le j \le n)$ we obtain

$$\begin{aligned} g(\tau(f), \xi) =& g((\nabla^f_{u_j} f_*)u_j + (\nabla^f_{Ju_j} f_*)Ju_j, \xi) \\ =& g(\nabla_{f_*u_j} f_*u_j + \nabla_{f_*Ju_j} f_*Ju_j, \xi) \\ =& -g(f_*u_j, \nabla_{f_*u_j}\xi) - g(f_*Ju_j, \nabla_{f_*Ju_j}\xi) \\ =& -g(f_*u_j, f_*u_j) - g(f_*Ju_j, f_*Ju_j), \end{aligned} \tag{3.7}$$

where we have used relations (3.2), (3.6), (2.3), (2.4) and (2.7). This completes the proof. □

THEOREM 3.2. *Let (N, J, k) be a semi-Kähler manifold and (M, ϕ, ξ, η, g) be a CR-integrable almost Kenmotsu manifold, then any (J, ϕ)-holomorphic map $f : N \to M$ is harmonic if and only if it is a constant map.*

Proof. For any (J, ϕ)-holomorphic map $f : N \to M$, it follows from Gherghe [9] that

$$\phi(\tau(f)) = f_*(\mathrm{div} J) - \mathrm{trace}_k \gamma, \tag{3.8}$$

where γ is defined by $\gamma(X, Y) = (\nabla^f_X \phi)(f_*Y)$ for any vector fields $X, Y \in \mathfrak{X}(N)$. By using relation (2.8) and a straightforward calculation we obtain

$$\begin{aligned} \mathrm{trace}_k \gamma =& (\nabla^f_{u_j}\phi) f_*u_j + (\nabla^f_{Ju_j}\phi) f_*Ju_j \\ =& (\nabla_{f_*u_j}\phi) f_*u_j + (\nabla_{f_*Ju_j}\phi) f_*Ju_j \\ =& g(hf_*u_j, f_*u_j)\xi + g(hf_*Ju_j, f_*Ju_j)\xi \\ =& 0, \end{aligned} \tag{3.9}$$

where in the last equality we have used relations (2.4), (2.6) and (3.1). Since N is a semi-Kähler manifold then we have $\mathrm{div} J = 0$. Therefore, making use of (3.9) and $\mathrm{div} J = 0$ in (3.8) we obtain

$$\tau(f) = g(\tau(f), \xi)\xi. \tag{3.10}$$

Thus, the proof follows from relations (3.5) and (3.10). □

Remark 1. Theorems 3.1 and 3.2 can be regarded as some generalizations of the corresponding results shown in Gherghe [8, 9] and Rehman [20].

4. HARMONIC MAPS BETWEEN ALMOST KENMOTSU MANIFOLDS AND CONTACT METRIC MANIFOLDS

A map $f : M \to \overline{M}$ between two almost contact metric manifolds (M, ϕ, ξ, η, g) and $(\overline{M}, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is said to be $(\phi, \overline{\phi})$-*holomorphic* if it satisfies

$$f_* \circ \phi = \overline{\phi} \circ f_*. \tag{4.1}$$

The proof of the following result is similar to that of Ianus and Pastore [12, Theorem 2.1].

Lemma 4.1. *Let (M, ϕ, ξ, η, g) and $(\overline{M}, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be two almost Kenmotsu manifolds and $f : M \to \overline{M}$ be a $(\phi, \overline{\phi})$-holomorphic map, then there exists a smooth function λ on M such that*

$$f_*\xi = \lambda\overline{\xi}, \quad f^*\overline{\eta} = \lambda\eta. \tag{4.2}$$

where λ is invariant along the distribution $\mathcal{D}$ of M.

Proof. Since f is a $(\phi, \overline{\phi})$-holomorphic map, we have $f_{*p}(\phi\xi_p) = \overline{\phi}_{f(p)}(f_*\xi_p) = 0$ and hence we obtain $f_*\xi_p = \lambda_p\overline{\xi}_{f(p)}$ for any point $p \in M$, where λ is given by $\lambda_p = \overline{g}_{f(p)}(f_*\xi_p, \overline{\xi}_{f(p)})$. Using this, we obtain directly that $f^*\overline{\eta} = \lambda\eta$, from which it follows that $f^*(d\overline{\eta}) = d\lambda \wedge \eta + \lambda d\eta$. As M and $\overline{M}$ are both almost Kenmotsu manifolds, then we get $d\lambda \wedge \eta = 0$. This completes the proof. □

Theorem 4.1. *Let (M, ϕ, ξ, η, g) and $(\overline{M}, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be two almost Kenmotsu manifolds and let $f : M \to \overline{M}$ be a $(\phi, \overline{\phi})$-holomorphic map, then f is harmonic if and only if*

$$\mathrm{trace}_g(f^*\overline{g}) = \lambda^2 + 2m\lambda + \xi(\lambda). \tag{4.3}$$

Proof. On an almost Kenmotsu manifold (M, ϕ, ξ, η, g) we have from Dileo and Pastore [3] that

$$(\nabla_X\phi)Y + (\nabla_{\phi X}\phi)\phi Y = -\eta(Y)\phi X - 2g(X, \phi Y)\xi - \eta(Y)hX \tag{4.4}$$

for any vector fields $X, Y \in \mathfrak{X}(M)$. It follows from (4.4) that

$$\nabla_X X + \nabla_{\phi X}\phi X = \phi[\phi X, X] - g(X, X)\xi - g(\phi X, \phi X)\xi \tag{4.5}$$

for any $X \in \mathcal{D}$. By (4.4) and a direct calculation we get

$$\delta\phi = (\nabla_{e_i}\phi)e_i + (\nabla_{\phi e_i}\phi)\phi e_i + (\nabla_\xi\phi)\xi = 0, \tag{4.6}$$

where $\{\xi, e_1, \ldots, e_m, \phi e_1, \ldots, \phi e_m\}$ is a local orthonormal basis on M and δ is the coderivative (the summation symbol for the repeated indices $(1 \le i \le m)$ is omitted). Since f is a $(\phi, \overline{\phi})$-holomorphic map, then we have

$$\overline{\eta}(f_*X) = -\overline{g}(f_*\phi^2X, \overline{\xi}) = -\overline{g}(\overline{\phi}f_*\phi X, \overline{\xi}) = 0 \tag{4.7}$$

for any $X \in \mathcal{D}$. Making use of (4.1), (4.2), (4.4) and (4.7) we also have

$$\mathrm{trace}_g f^*(\overline{\nabla\phi}) = (\overline{\nabla}_{f_*e_i}\overline{\phi})f_*e_i + (\overline{\nabla}_{f_*\phi e_i}\overline{\phi})f_*\phi e_i + (\overline{\nabla}_{f_*\xi}\overline{\phi})f_*\xi = 0. \tag{4.8}$$

It follows from (2.6) and (2.7) that $\delta\eta = -2m$. Next we compute

$$\begin{aligned}\mathrm{trace}_g f^*(\overline{\nabla\eta}) =&(\overline{\nabla}_{f_*e_i}\overline{\eta})f_*e_i + (\overline{\nabla}_{f_*\phi e_i}\overline{\eta})f_*\phi e_i + (\overline{\nabla}_{f_*\xi}\overline{\eta})f_*\xi\\ =&-\overline{g}(\overline{\xi}, \overline{\nabla}_{f_*e_i}f_*e_i + \overline{\nabla}_{f_*\phi e_i}f_*\phi e_i)\\ =&\overline{g}(f_*e_i, f_*e_i) + \overline{g}(f_*\phi e_i, f_*\phi e_i)\\ =&\mathrm{trace}_g(f^*\overline{g}) - \lambda^2,\end{aligned} \tag{4.9}$$

where we have used relations (4.1) and (4.5) in the third equality. Notice that the tension field of a $(\phi, \overline{\phi})$-holomorphic map between two almost contact metric manifolds is given by (see Chinea [2])

$$\tau(f) = \overline{\phi}(\mathrm{trace}_g f^*(\overline{\nabla\phi})) - f_*(\phi\delta\phi + \delta\eta\xi) + (\xi(\lambda) - \mathrm{trace}_g f^*(\overline{\nabla\eta}))\overline{\xi}. \tag{4.10}$$

Thus, making use of $\delta\eta = -2m$, (4.6), (4.8) and (4.9) in relation (4.10), we get the theorem. $\square$

Remark 2. By Theorem 4.1, any $(\phi, \overline{\phi})$-holomorphic map f between two almost Kenmotsu manifolds is harmonic if and only if the energy density of f is given by $e(f) = \frac{1}{2}\lambda^2 + m\lambda + \frac{1}{2}\xi(\lambda)$.

Next we give some characterizations regarding the harmonicity of maps between almost contact metric manifolds and almost Kenmotsu manifolds.

THEOREM 4.2. *Let (M, ϕ, ξ, η, g) and $(\overline{M}, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be a contact metric manifold and an almost Kenmotsu manifold, respectively, and let $f : M \to \overline{M}$ be a $(\phi, \overline{\phi})$-holomorphic map, then f is harmonic if and only if it is a constant map.*

Proof. By Lemma 4.1, f being a $(\phi, \overline{\phi})$-holomorphic map implies that there exists a function λ such that $f^*(d\overline{\eta}) = d\lambda \wedge \eta + \lambda d\eta$, where λ is given in Lemma 4.1. Notice that M is a contact metric manifold and $\overline{M}$ is an almost Kenmotsu manifold, then we get from the above relation that $d\lambda \wedge \eta + \lambda\Phi = 0$. Using this, we get $\lambda g(X, \phi Y) = 0$ for any $X, Y \in \mathcal{D}$ and hence we have $\lambda = 0$. From Blair [1], on a contact metric manifold (M, ϕ, ξ, η, g) we have

$$(\nabla_X\phi)Y + (\nabla_{\phi X}\phi)\phi Y = 2g(X, Y)\xi - \eta(Y)(X + hX + \eta(X)\xi) \tag{4.11}$$

for any vector fields $X, Y \in \mathfrak{X}(M)$. Then the following relations are easy to check (see also Blair [1])

$$\delta\phi = 2m\xi, \quad \delta\eta = 0. \tag{4.12}$$

Making use of (4.12), (4.8) and (4.9) in relation (4.10) we get

$$\tau(f) = -(\overline{g}(f_*e_i, f_*e_i) + \overline{g}(f_*\phi e_i, f_*\phi e_i))\overline{\xi}. \tag{4.13}$$

By relation (4.13) we complete the proof. $\square$

THEOREM 4.3. *Any $(\phi, \overline{\phi})$-holomorphic map $f : M \to \overline{M}$ from an almost Kenmotsu manifold (M, ϕ, ξ, η, g) to a contact metric manifold $(\overline{M}, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is harmonic if and only if $\xi(\lambda) = 0$, where λ is invariant along $\mathcal{D}$ of M.*

Proof. Following the proof of Lemma 4.1 we see that in this context $f^*(d\overline{\eta}) = d\lambda \wedge \eta + \lambda d\eta$ is true. As M is an almost Kenmotsu manifold and $\overline{M}$ is a contact metric manifold, then it follows that $f^*\overline{\Phi} = d\lambda \wedge \eta$ and hence

$$f^*\overline{\Phi}(\phi X, \phi Y) = \overline{g}(f_*\phi X, \overline{\phi} f_*\phi Y) = 0 \tag{4.14}$$

for any vector fields $X, Y \in \mathfrak{X}(M)$. Replacing X by ϕX in (4.14) and using (4.1) and (4.7) we obtain

$$\overline{g}(f_*X, f_*Y) - \lambda^2\eta(X)\eta(Y) = 0 \tag{4.15}$$

for any $X, Y \in \mathfrak{X}(M)$. This implies that $f_*X = 0$ for any $X \in \mathcal{D}$. Therefore, using $f_*\phi X = 0$ for any $X \in \mathfrak{X}(M)$ and (4.1) in relation $f^*\overline{\Phi} = d\lambda \wedge \eta$ we obtain $d\lambda \wedge \eta = 0$. Thus, λ is invariant along the distribution $\mathcal{D}$ of M.

Moreover, using (4.11) we obtain $\nabla_X X + \nabla_{\phi X}\phi X + \phi[X, \phi X] = 0$ for any $X \in \mathcal{D}$. By using this relation and $f_*\phi X = 0$ in (2.2) we finally get

$$\tau(f) = \overline{\nabla}_{f_*\xi} f_*\xi - f_*(\nabla_{e_i} e_i + \nabla_{\phi e_i}\phi e_i) = \xi(\lambda)\overline{\xi}. \tag{4.16}$$

Therefore, the proof follows from (4.16). $\square$

Before closing this paper, we present some classes of almost Kenmotsu manifolds.

Example 4.1. Let N^{2n}, $n > 1$, be a strictly almost Kähler manifold. The warped product $\mathbb{R} \times_{ce^t} N^{2n}$ admits an almost Kenmotsu structure which is not Kenmotsu and not CR-integrable, where t is the coordinate of $\mathbb{R}$ and c is a constant. For more details we refer the reader to [3, pp. 345].

Example 4.2. Let M_1 be a strictly almost Kähler manifold and let M_2 be a non-Kenmotsu almost Kenmotsu manifold. The product manifold $M_1 \times M_2$ admits a strictly almost Kenmotsu structure. For more details we refer the reader to [19, Theorem 2.1.13].

On an almost Kenmotsu manifold M^{2n+1} we set $h' = h \circ \phi$. If the characteristic vector field ξ satisfies the generalized $(k, \mu)'$-nullity condition, *i.e.*,

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y)$$

for any vector fields X, Y and two functions k, μ, then M^{2n+1} is called a *generalized $(k, \mu)'$-almost Kenmotsu manifold.* Similarly, if ξ satisfies the generalized

(k,μ)-nullity condition, *i.e.*,

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

for any vector fields X, Y and two functions k, μ, then M^{2n+1} is called a *generalized* (k,μ)*-almost Kenmotsu manifold.*

Example 4.3. A generalized (k,μ) or $(k,\mu)'$-almost Kenmotsu manifold satisfying $h \neq 0$ is CR-integrable. For more details and concrete examples we refer the reader to Corollaries 4.1, 5.1 and Section 6 of [18].

Finally, we remark that a 3-dimensional almost Kenmotsu manifold is always CR-integrable. For more details we refer the reader to [4, Section 5].

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REFERENCES

[1] D.E. Blair, *Riemannian geometry of contact and symplectic manifolds.* Progr. Math. **203**, Birkhäuser, New York, 2010.

[2] D. Chinea, *Harmonicity on maps between almost contact metric manifolds.* Acta Math. Hungar. **126** (2010), 352–362.

[3] G. Dileo and A.M. Pastore, *Almost Kenmotsu manifolds and local symmetry.* Bull. Belg. Math. Soc. Simon Stevin **14** (2007), 343–354.

[4] G. Dileo and A.M. Pastore, *Almost Kenmotsu manifolds and nullity distributions.* J. Geom. **93** (2009), 46–61.

[5] J. Eells and J.H. Sampson, *Harmonic mapping of Riemannian manifolds.* Amer. J. Math. **86** (1964), 109–160.

[6] M. Falcitelli and A.M. Pastore, *Almost Kenmotsu f-manifolds.* Balkan J. Geom. Appl. **12** (2007), 32–43.

[7] B. Fuglede, *Harmonic morphisms between Riemannian manifolds.* Ann. Inst. Fourier **28** (1978), 107–144.

[8] C. Gherghe, *Harmonic map on Kenmotsu manifolds.* Rev. Roumaine Math. Pures Appl. **45** (2000), 447–454.

[9] C. Gherghe, *Harmonicity on some almost contact metric manifolds.* Rend. Circ. Mat. Palermo (2) **49** (2000), 415–424.

[10] C. Gherghe, S. Ianus and A.M. Pastore, *Harmonic maps, harmonic morphism and stability.* Bull. Math. Soc. Sci. Math. Roumanie **43** (2000), 247–254.

[11] C. Gherghe, S. Ianus and A.M. Pastore, *CR-manifolds, harmonic maps and stability.* J. Geom. **71** (2001), 42–53.

[12] S. Ianus and A.M. Pastore, *Harmonic maps on contact metric manifolds.* Ann. Math. Blaise Pascal **2** (1995), 43–53.

[13] T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions.* J. Math. Kyoto Univ. **19** (1979), 215–225.

[14] D. Janssens and L. Vanhecke, *Almost contact structures and curvature tensors.* Kodai Math. J. **4** (1981), 1–27.

[15] K. Kenmotsu, *A class of almost contact Riemannian manifolds.* Tohoku Math. J. **24** (1972), 93–103.

[16] T.W. Kim and H.K. Pak, *Canonical foliations of certain classes of almost contact metric structures.* Acta Math. Sin. (Engl. Ser.) **21** (2005), 841–846.

[17] A. Lichnerowicz, *Applications harmoniques et varieés kahleriennes.* Symp. Math. **3** (1970), 341–402.

[18] A.M. Pastore and V. Saltarelli, *Generalized nullity distributions on almost Kenmotsu manifolds.* Int. Electron. J. Geom. **4** (2011), 168–183.

[19] G. Pitiş, *Geometry of Kenmotsu Manifolds.* Publ. House Transilv. Univ. Braşov, Braşov, 2007.

[20] N.A. Rehman, *Harmonic maps on Kenmotsu manifolds.* An. Ştiint. Univ. "Ovidius" Constanţa Ser. Mat. **21** (2013), 197–208.

[21] N.A. Rehman, *Stability on generalized Sasakian space form.* Math. Reports (Bucur.) **17** (2015), 57–64.

[22] Y. Wang, *Conformally flat CR-integrable almost Kenmotsu manifolds.* Bull. Math. Soc. Sci. Math. Roumanie **59** (2016), 375–387.

[23] Y. Wang and X. Liu, *Locally symmetric CR-integrable almost Kenmotsu manifolds.* Mediterr. J. Math. **12** (2015), 159–171.

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