MINIMAL DOUBLY RESOLVING SETS OF NECKLACE GRAPH

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Consider a simple connected undirected graph $G = (V, E)$, where V represents the vertex set and E represents the edge set, respectively. A subset D of V is called doubly resolving set if for every two vertices x, y of G, there are two vertices $u, v \in D$ such that $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. A doubly resolving set with minimum cardinality is called minimal doubly resolving set. This minimum cardinality is denoted by $\psi(G)$.

In this paper, we find the minimal doubly resolving set for necklace graph N_{e_n}, $n \geq 2$. Also, we prove that $\psi(N_{e_n}) = 3$ for $n \geq 2$.

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1. INTRODUCTION AND PRELIMINARY RESULTS

The concept of resolving set has been introduced by Slater [8] and also independently by Harary and Melter [4]. This concept has different applications in the areas of network discovery and verification [1], robot navigation [7], and chemistry.

Consider a simple connected undirected graph $G = (V, E)$, where V and E denote the sets of vertices and edges of G, respectively. Let $d(x, y)$ denote the distance between vertices x and y. A vertex v of graph G is said to resolve two vertices x and y of G if $d(v, x) \neq d(v, y)$. A vertex set $W = \{w_1, w_2, \ldots, w_k\}$ of G is a resolving set or locating set of G if every two distinct vertices of G are resolved by some vertex of W. The k-tuple $r(v, W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ is called the vector of metric coordinates of v with respect to W. A resolving set of minimum cardinality is called metric basis of G. The cardinality of metric basis, denoted by $\beta(G)$, is called metric dimension of G.

Cáceres *et al.* [2] introduced "doubly resolving sets" by showing its connection with metric dimension of the Cartesian product $G \square G$ of the graph G.

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These sets constitute a useful tool for obtaining upper bounds on the metric dimension of graphs. Vertices x, y of the graph G of order at least 2, are said to doubly resolve vertices u, v of G if $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. A vertex set D of G is a doubly resolving set of G if every two distinct vertices of G are doubly resolved by some two vertices of D, that is, if there are no two distinct vertices of G with the same difference between their corresponding metric coordinates with respect to D. The minimal doubly resolving set is a doubly resolving set with minimum cardinality. The cardinality of minimum doubly resolving set is denoted by $\psi(G)$. The problem of minimal doubly resolving set is NP-hard [5]. The minimal doubly resolving sets for Hamming and Prism graphs has been obtained in [6] and [3], respectively.

Note that $\beta(G) \leq \psi(G)$ always. Since if x, y doubly resolve u, v, then $d(u, x) - d(v, x) \neq 0$ or $d(u, y) - d(v, y) \neq 0$, and hence x or y resolve u, v, which follows that a doubly resolving set is also a resolving set. In [9], the metric dimension of necklace graph, $\beta(N_{e_n})$, has been determined. In this paper, we determine the minimal doubly resolving sets for necklace graph.

2. THE MINIMAL DOUBLY RESOLVING SETS FOR NECKLACE GRAPH N_{e_n}

In this section, we will find the minimal doubly resolving set for the necklace graph N_{e_n}. The necklace graph N_{e_n}, for $n \geq 2$, consists of the vertex set $V = \{v_0, v_1, \ldots, v_{n+1}, u_1, u_2, \ldots, u_n\}$ and the edge set $E = \{v_i v_{i+1}, u_i u_{i+1}, v_i u_i : 1 \leq i \leq n-1\} \cup \{v_0 v_1, v_0 u_1, v_0 v_{n+1}, v_n v_{n+1}, v_{n+1} u_n, v_n u_n\}$. The following Figure 1 displays the necklace graph N_{e_n}.

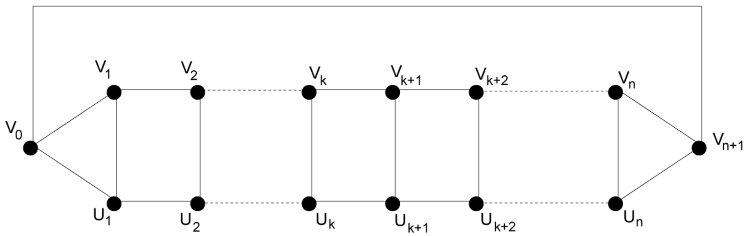

Fig. 1 – The necklace graph N_{e_n}.

It has been proved [9], that for $n \geq 2$, $\beta(N_{e_n}) = \begin{cases} 2, & \text{if } n \text{ is even;} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$

We will find here the minimal doubly resolving sets for necklace graphs. Also we will prove that $\psi(N_{e_n}) = 3$ for $n \geq 2$. Therefore the necklace graph is interesting to consider in the sense that, its metric dimension depends on parity of n, that is, whether n is even or odd, but the cardinality of minimal doubly resolving set is independent on the parity of n. Since $\psi(N_{e_n}) \geq \beta(N_{e_n})$, thus for $n \geq 2$, $\psi(N_{e_n}) \geq \begin{cases} 2, & \text{if } n \text{ is even;} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$

Define $S_i(v_0) = \{w \in V : d(v_0, w) = i\}$ be the set of vertices in V at distance i from v_0. For N_{e_n} with $n \geq 4$, we can easily find the sets $S_i(v_0)$ which are given in Table 1.

TABLE 1

$S_i(v_0)$ for N_{e_n}

n	i	$S_i(v_0)$
	1	$\{v_1, u_1, v_{n+1}\}$
	$2 \leq i \leq k$	$\{v_i, u_i, v_{n+2-i}, u_{n+2-i}\}$
$2k, (k \geq 2)$	$k+1$	$\{v_{k+1}, u_{k+1}\}$
$2k+1, (k \geq 2)$	$k+1$	$\{v_{k+1}, u_{k+1}, v_{k+2}, u_{k+2}\}$

It can be easily seen that $S_i(v_0) = \emptyset$ for $i \geq k+2$. Note that the sets $S_i(v_0)$, defined above can be used to determine the distance between two arbitrary vertices in V in the following way.

As symmetry of N_{e_n} displays the following fact that $d(v_i, v_j) = d(v_0, v_{j-i})$, $d(u_i, u_j) = d(v_0, v_{j-i})$ for $j > i$. If $n = 2k$ where $k \geq 2$, we have

$$d(v_i, u_j) = \begin{cases} d(v_0, v_{|i-j|}) + 1, & |i-j| \leq k, \ 1 \leq i, j \leq n; \\ d(v_0, v_{|i-j|}), & |i-j| > k, \ 1 \leq i, j \leq n; \\ d(v_0, v_{n+1-j}), & i = n+1; \\ d(v_0, v_j), & i = 0. \end{cases}$$

If $n = 2k+1$ where $k \geq 2$, we have

$$d(v_i, u_j) = \begin{cases} d(v_0, v_{|i-j|}) + 1, & |i-j| \leq k+1, \ 1 \leq i, j \leq n; \\ d(v_0, v_{|i-j|}), & |i-j| > k+1, \ 1 \leq i, j \leq n; \\ d(v_0, v_{n+1-j}), & i = n+1; \\ d(v_0, v_j), & i = 0. \end{cases}$$

Consequently, if we know the distance $d(v_0, w)$ for every $w \in V$ then we can reconstruct the distance between every two vertices from V.

LEMMA 2.1. *For $n = 2k$, $k \geq 2$, $\psi(N_{e_n}) > 2$.*

Proof. We know that $\psi(N_{e_n}) \geq 2$ and thus we should prove that every subset D of vertex set V with $|D| = 2$ is not a doubly resolving set for N_{e_n}.

In Table 2, one can find all possible types of such set D and the corresponding non-doubly resolving pair of vertices from V.

TABLE 2

Non-doubly resolved pairs of N_{e_n} for $n = 2k$, $k \geq 2$

D	Non-doubly resolved pair
$\{v_0, v_i;\ 1 \leq i \leq k\} \cup \{v_0, u_i;\ 1 \leq i \leq k\}$ $\cup\{v_{n+1}, v_i;\ k+1 \leq i \leq n\} \cup \{v_{n+1}, u_i;\ k+1 \leq i \leq n\}$	$\{u_k, u_{k+1}\}$
$\{v_0, v_i;\ k+1 \leq i \leq n-1\} \cup \{v_0, u_i;\ k+1 \leq i \leq n-1\}$	$\{u_k, u_{i+1}\}$
$\{v_0, v_n\} \cup \{v_0, u_n\} \cup \{v_0, v_{n+1}\}$	$\{u_1, v_1\}$
$\{v_{n+1}, v_1\} \cup \{v_{n+1}, u_1\}$	$\{u_n, v_n\}$
$\{v_{n+1}, v_i;\ 2 \leq i \leq k\} \cup \{v_{n+1}, u_i;\ 2 \leq i \leq k\}$	$\{u_{k+1}, u_{i-1}\}$
$\{v_i, v_j;\ 1 \leq i, j \leq n,\ i \neq j\} \cup \{u_i, u_j;\ 1 \leq i, j \leq n,\ i \neq j\}$	$\{v_{k+1}, u_{k+1}\}$
$\{v_i, u_j;\ 1 \leq i, j \leq k\} \cup \{v_i, u_j;\ k+1 \leq i, j \leq n\}$	$\{v_0, v_{n+1}\}$
$\{v_i, u_j;\ 1 \leq i \leq k,\ k+1 \leq j \leq n\}$	$\{v_{k+1}, u_k\}$
$\{v_i, u_j;\ k+1 \leq i \leq n,\ 1 \leq j \leq k\}$	$\{v_k, u_{k+1}\}$

Let us prove, for example, the vertices v_{k+1}, u_k are not doubly resolved by any two vertices to the set $\{v_i, u_j;\ 1 \leq i \leq k,\ k+1 \leq j \leq n\}$. We have

(i) $d(v_i, v_{k+1}) = d(v_0, v_{k+1-i}) = k + 1 - i$,

(ii) $d(v_i, u_k) = d(v_0, v_{k-i}) + 1 = k - i + 1$,

(iii) $d(u_j, v_{k+1}) = d(v_0, v_{j-k-1}) + 1 = j - k - 1 + 1 = j - k$,

(iv) $d(u_j, u_k) = d(v_0, v_{j-k}) = j - k$.

From (i), (ii), (iii) and (iv), we have $d(v_i, v_{k+1}) - d(v_i, u_k) = d(u_j, v_{k+1}) - d(u_j, u_k) = 0$, that is, $\{v_i, u_j;\ 1 \leq i \leq k,\ k+1 \leq j \leq n\}$ is not a resolving set of N_{e_n}. Similarly, we can consider all other types of D from Table 2 and verify their corresponding non-doubly resolved pairs of vertices. $\square$

Lemma 2.2. *For $n = 2k$, $k \geq 2$, $\psi(Ne_n) = 3$.*

Proof. From Lemma 2.1, it is clear that $\psi(N_{e_n}) \geq 3$. Consider the following table which shows the vectors of metric coordinates of vertices of N_{e_n} with respect to the set $D^* = \{v_0, v_k, v_{k+1}\}$ in a special way.

Starting from Table 3, note that $v_0 \in D^*$ therefore the first metric coordinate of a vector from $S_i(v_0)$ with respect to D^* is equal to i. It is easy to check that for each $i \in \{1, 2, \ldots, k+1\}$, there do not exist two vertices $x, y \in S_i(v_0)$ such that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to 0. Also, it can be easily seen that for each $i, j \in \{1, 2, \ldots, k+1\}$, $i \neq j$, there do not exist two vertices $x \in S_i(v_0)$ and $y \in S_j(v_0)$ such that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to $i - j$. In this way, the set $D^* = \{v_0, v_k, v_{k+1}\}$ becomes the minimal doubly resolving set for N_{e_n} with $n = 2k$, $k \geq 2$ and hence Lemma 2.2 holds. □

TABLE 3

Vectors of metric coordinates for N_{e_n}, $n = 2k$, $k \geq 2$

i	$S_i(v_0)$	$D^* = \{v_0, v_k, v_{k+1}\}$
0	v_0	$(0, k, k+1)$
1	v_1	$(1, k-1, k)$
	u_1	$(1, k, k+1)$
	v_{n+1}	$(1, k+1, k)$
$2 \leq i \leq k$	v_i	$(i, k-i, k+1-i)$
	u_i	$(i, k+1-i, k+2-i)$
	v_{n+2-i}	$(i, k+2-i, k+1-i)$
	u_{n+2-i}	$(i, k+3-i, k+2-i)$
$k+1$	v_{k+1}	$(k+1, 1, 0)$
	u_{k+1}	$(k+1, 2, 1)$

LEMMA 2.3. *For $n = 2k+1$, $k \geq 2$, $\psi(N_{e_n}) = 3$.*

Proof. For n odd we have $3 = \beta(N_{e_n}) \leq \psi(N_{e_n})$. Consider the following Table 4 which shows the vectors of metric coordinates of vertices of N_{e_n} with respect to the set $D^* = \{v_0, v_1, v_{k+2}\}$ in a special way.

TABLE 4

Vectors of metric coordinates for N_{e_n}, $n = 2k+1$, $k \geq 2$

i	$S_i(v_0)$	$D^* = \{v_0, v_1, v_{k+2}\}$
0	v_0	$(0, 1, k+1)$
1	v_1	$(1, 0, k+1)$
	u_1	$(1, 1, k+2)$
	v_{n+1}	$(1, 2, k)$
$2 \leq i \leq k$	v_i	$(i, i-1, k+2-i)$
	u_i	$(i, i, k+3-i)$
	v_{n+2-i}	$(i, i+1, k+1-i)$
	u_{n+2-i}	$(i, i+1, k+2-i)$
$k+1$	v_{k+1}	$(k+1, k, 1)$
	u_{k+1}	$(k+1, k+1, 2)$
	v_{k+2}	$(k+1, k+1, 0)$
	u_{k+2}	$(k+1, k+2, 1)$

Starting from Table 4, note that $v_0 \in D^*$ therefore the first metric coordinate of a vector from $S_i(v_0)$ with respect to D^* is equal to i. It is easy to check that for each $i \in \{1, 2, \ldots, k+1\}$, there do not exist two vertices $x, y \in S_i(v_0)$ such that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to 0. Also, it can be easily seen that for each $i, j \in \{1, 2, \ldots, k+1\}$,

$i \neq j$, there do not exist two vertices $x \in S_i(v_0)$ and $y \in S_j(v_0)$ such that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to $i - j$. In this way, the set $D^* = \{v_0, v_1, v_{k+2}\}$ becomes the minimal doubly resolving set for N_{e_n} with $n = 2k + 1$, $k \geq 2$ and hence Lemma 2.3 holds. $\square$

A total enumeration technique shows that $\psi(N_{e_n}) = 3$ for $n = 2, 3$. The sets $\{v_0, v_1, v_2\}$ and $\{v_0, v_1, v_3\}$ are the minimal doubly resolving sets for N_{e_2} and N_{e_3}, respectively. Combining this fact with Lemma 2.2 and Lemma 2.3, we get the following main theorem of this section.

THEOREM 2.1. *Let N_{e_n} be the necklace graph, $n \geq 2$. Then $\psi(N_{e_n}) = 3$.*

3. CONCLUSION

In this paper, we determined the minimal doubly resolving set and its cardinality for necklace graph N_{e_n}. This family of graphs is interesting to consider, in the sense, that its metric dimension $\beta(N_{e_n})$ depends on the parity of n. On the other hand $\psi(N_{e_n}) = 3$ for every n.

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