THE CORE OF THE GAMES WITH FRACTIONAL LINEAR UTILITY FUNCTIONS

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We consider fractional linear programming production games for single-objective and multiobjective cases. We use the method of Chakraborty and Gupta (2002) in order to transform the fractional linear programming problems into linear programming problems. A cooperative game is attached and we prove the nonemptiness of the core by using the duality theory from linear programming. In the multiobjective case, we characterize the stable outcome of the associated cooperative game, which is balanced. By using a similar method as above, we also study the form of the elements belonging to the stable outcome of the cooperative game associated to an exchange economy with a finite number of agents and fractional linear utilities.

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1. INTRODUCTION

The purpose of the paper is twofold. Firstly, it attempts to determine the elements of the core, respectively of the stable outcome of two new types of generalized Owen models: fractional linear programming production games for single-objective and multiobjective cases. Secondly, it studies the form of elements of the stable outcome of the cooperative game associated to an exchange economy with a finite number of agents and fractional linear utilities.

Owen considered the linear programming production problem with n producers, who have m resources and cooperate in order to produce p goods. The producers' aim is the maximization of their income, which is modeled as objective function of the discussed problem. A cooperative game is attached, and the fair allocation of income is put into question. Relating methods from the duality theory with methods from cooperative games, we prove that the core is nonempty and we can find its elements. The condition of the nonemptiness of the core for a cooperative game is the balancedness, as it was proved in Bondareva (1963) or Shapley (1967).

The seminal work of Owen has many extensions. Samet and Zemel (1984) studied the relation between the core of a given LP-game and the set of payoff vectors generated by optimal dual solutions to the corresponding linear program. Granot (1986) generalized the Owen's model, so that the resources held by any subset of producers S is not restricted to be the vector sum of the resources held by the members of S. He also proved the non-emptiness of the core of the associated game. Curiel, Derks and Tijs (1989) considered linear production games with committee control. After that, Gellekom, Potters, Reijnierse, Engel and Tijs (2001) also studied linear production processes, while Nishizaki and Sakawa (1999, 2001) treated the multiobjective case.

In this paper, we consider that the producers want to maximize the average income on unit time, which is modeled by a fractional linear objective function. We generalize the Owen's model by introducing the fractional linear programming production games for the single-objective and multiobjective cases. The transformation of the fractional linear programming problems into linear programming problems is made by using the method of Chakraborty and Gupta (2002). We attach a cooperative game and we prove the non-emptiness of its core. The multiobjective game is balanced, but not superadditive, and for this case, we give a characterization of the stable outcome derived from the associate cooperative game.

Finally, we consider the cooperative game associated to an exchange economy with a finite number of agents. The agents' preferences are modeled by utilities, which are considered to be fractional linear functions. We consider the problem of allocations of goods among consumers, and we provide an answer by associating a cooperative game to the exchange economy. We study the properties of the game and describe the elements of its stable outcome. Even if the two problems presented in this paper seem to be different, the methods of approaching them are very similar. The unity of the models and ideas in this field of research can be emphasized. Our results generalize the ones obtained so far by considering the case of the fractional linear functions, which are used more and more often with a precise economic meaning. Our methods of research are quite new, in particularly because they concern the transformation of a fractional linear multiobjective linear programming problem in a linear multiobjective linear programming one. We can quote here the results obtained by Chakraborty and Gupta (2002).

The paper is organized in the following way: fractional linear programming production game is presented in Section 2, the multiobjective model is treated in Section 3 and the study of the exchange economies' model with a finite number of agents and fractional linear utility functions is the content of Section 4. Further research is outlined in Section 5.

2. FRACTIONAL LINEAR PROGRAMMING PRODUCTION GAMES

2.1. THE MODEL

This section is dedicated to defining a new model for a production problem which requires to maximize the average income on unit time, expressed by a unique fractional linear objective function.

We consider here the following model of fractional linear production game. There are m types of resources used for the production of p goods. For each $i \in N = \{1, 2, ..., n\}$, the player i is endowed with a vector b^i of resources, where $b^i = (b_1^i, b_2^i, ..., b_m^i)$. Any coalition S will use a total of $b_k(S) = \sum_{i \in S} b_k^i$ units of the kth resource. We assume that a unit of the jth good ($j = 1, ..., p$) requires a_{kj} units of the kth resource ($k = 1, ..., m$). A coalition S uses all its resources in order to produce a vector $(x_1, x_2, ..., x_p)$ of goods which satisfies

$$(1) \qquad \begin{array}{l} a_{11}x_1 + a_{12}x_2 + ... + a_{1p}x_p \leq b_1(S) \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2p}x_p \leq b_2(S) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mp}x_p \leq b_m(S)^T \\ x_1, x_2, ..., x_p \geq 0. \end{array}$$

The players of the coalition S want to maximize the average income on unit time, which is given by the objective function $\frac{N(x)}{D(x)} = \frac{c_1x_1+c_2x_2+...+c_px_p+c_0}{d_1x_1+d_2x_2+...+d_px_p+d_0}$.

We will denote by $x = (x_1, x_2, ..., x_p)^T$ the vector of goods, $c = (c_1, c_2, ..., c_p)^T \in \mathbb{R}^p$ and $d = (d_1, d_2, ..., d_p)^T \in \mathbb{R}^p$ the vectors which define the objective function, $A = (a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,p}}} \in \mathbb{R}^{m \times p}$ the matrix with the coefficients of the constraints and $b(S) = (b_1(S), b_2(S), ..., b_m(S))^T \in \mathbb{R}^m$ the vector of the resources used by coalition S. Let also denote by F_S the set of all feasible solutions of the problem: $F_S = \{x \in \mathbb{R}^p : Ax \leq b(S),\ x \geq \mathbf{0}\} \subseteq \mathbb{R}_+^p$. The problem can now be stated in the following concise form:

$$(2) \qquad \text{for each coalition } S, \text{ maximize } \frac{N(x)}{D(x)} \quad \text{subject to } F_S.$$

Our approach to solve the problem (2) consists of using the substitution proposed by Charnes and Cooper (1962): $y = tx$, $t = \frac{1}{d^Tx+d_0}$. Thus, Problem (2) becomes equivalent with (3), where

$$(3) \qquad \begin{array}{l} \text{Max } tN(\frac{y}{t}) \\ \text{subject to } tD(\frac{y}{t}) \leq 1 \\ \qquad A(\frac{y}{t}) - b(S) \leq \mathbf{0} \\ \qquad t > 0, y \geq \mathbf{0}. \end{array}$$

In order to find solutions for the problem above, we can make the following assumption, which is not a restrictive one:

for each $S \subseteq N$, $x \in F_S := \{x : Ax \leq b(S),\ x \geq \mathbf{0}\}$ implies $D(x) > 0$.

Furthermore, the fractional linear programming problem (3) can be reduced in a simple way to the linear programming problem (4) in its inequality standard form:

$$\text{(4)} \qquad \text{P(S):} \quad \begin{aligned} &\text{Max } c^T y + c_0 t \\ &\text{subject to } d^T y + d_0 t \leq 1 \\ &\qquad Ay - tb(S) \leq \mathbf{0} \\ &\qquad t > 0, y \geq \mathbf{0}. \end{aligned}$$

The relationships between the optimal solutions of the fractional linear programming problem (2) and the ones of the linear programming problem (4) were investigated by Schaible (1976, 1978) and expressed in the following results. Both theorems state a certain type of equivalence between the maximum points of the considered problems. Their importance consists in the fact that we can reduce our study to the linear programming case.

THEOREM 1 (Schaible 1976, 1978). *For some $\xi \in \Delta$, $N(\xi) \geq 0$, if (2) reaches a (global) maximum at $x = x^*$, then (3) reaches a (global) maximum at a point $(t, y) = (t^*, y^*)$, where $\frac{y^*}{t^*} = x^*$ and the objective functions at these points are equal.*

THEOREM 2 (Schaible 1976). *If (2) reaches a (global) maximum at a point x^*, then the corresponding transformed problem (3) attains the same maximum value at a point (t^*, y^*) where $x^* = \frac{y^*}{t^*}$. Moreover (3) has a concave objective function and a convex feasible set.*

2.2. THE ASSOCIATED COOPERATIVE GAME

We associate to the problem (4) the cooperative game (N, V), where $N = \{1, 2, ..., n\}$ is the set of players (or decision makers), $\mathscr{P}(N)$, the set of nonempty subsets of N, is the set of coalitions formed with players $1, 2, ..., n$, and $V : \mathscr{P}(N) \to \mathbb{R}$ describes the gain of a set of players by forming a coalition. The players have to choose which coalitions to form, by taking into account the appropriate payment methods of the coalition members.

We define the function V by

$V(S) = c_1 y_1 + c_2 y_2 + ... + c_p y_p + c_0 t$ if $S \subsetneq N$, where y is an optimal solution to problem $P(S)$,

$V(N) = \gamma(c_1 y_1 + c_2 y_2 + ... + c_p y_p + c_0 t)$, where y is an optimal solution to problem $P(N)$, and $\gamma > \max(\gamma^*, n)$ and $\gamma^* = \max\sum_{S\in\mathscr{B}}\gamma(S)$, $\mathscr{B}$ being any balanced coalition of N.

We recall that a collection $\mathscr{B}$ of coalitions is said to be balanced if there exists $\gamma(S) > 0$ for each $S \in \mathscr{B}$ such that, for each $i \in N$, $\sum_{\substack{S\in\mathscr{B}\\ i\in S}}\gamma(S) = 1$. The cooperative game (N, V) is called balanced if for each balanced collection $\mathscr{B}$, $\sum_{S\in\mathscr{B}}\gamma(S)V(S) \leq V(N)$.

The core of the game (N, V) is a set of payoff allocations $u \in \mathbb{R}^N$ with the property that no coalition can improve upon. Formally, the core is a set of payoff allocations $u \in \mathbb{R}^N$ satisfying

i) Efficiency: $\sum_{i\in N} u_i = V(N)$,

ii) Coalitional rationality: $\sum_{i\in S} u_i \geq V(S)$ for all coalitions $S \subseteq N$.

The Bondareva–Shapley theorem (Bondareva 1963, Shapley 1967) asserts that the core of a game is nonempty if and only if the game is balanced.

Firstly, we prove the nonemptiness of the core of the game (N, V), which is a consequence of the balancedness.

Theorem 3. *The game (N, V) is balanced.*

Proof. Let $\mathscr{B}$ be a balanced collection of N. Firstly, we know that $\sum_{S\in\mathscr{B}}\gamma(S)b_k(S) = b_k(N)$ for each $k \in \{1, 2, ..., m\}$, and then,

$$\begin{aligned}\sum_{S\in\mathscr{B}}\gamma(S)V(S) &= \sum_{S\in\mathscr{B}}\gamma(S)(c_1y_1(S)+c_2y_2(S)+...+c_py_p(S)+c_0t(S)) =\\ &= \sum_{j=1}^{p} c_j(\sum_{S\in\mathscr{B}}\gamma(S)y_j(S)) + c_0\sum_{S\in\mathscr{B}}\gamma(S)t(S) =\\ &= \gamma'(\sum_{j=1}^{p} c_j\widehat{y}_j + c_0\widehat{t}),\end{aligned}$$

where $\gamma' = \sum_{S\in\mathscr{B}} V(S)$, $\widehat{y}_j := \sum_{S\in\mathscr{B}}\frac{\gamma(S)}{\gamma'}y_j(S)$ and $\widehat{t} = \sum_{S\in\mathscr{B}}\frac{\gamma(S)}{\gamma'}t(S)$.

Let us assume that $\begin{pmatrix} y(S)\\ t(S)\end{pmatrix}$ is a feasible solution of Problem (4).

Since $Ay(S) \leq t(S)b(S)$ for each $S \in \mathscr{B}$ and $\frac{\gamma(S)}{\gamma'} \geq 0$, we obtain that

$A\frac{\gamma(S)}{\gamma'}y(s) \leq t(S)\frac{\gamma(S)}{\gamma'}b(S)$.

By adding, we obtain

$A(\sum_{S\in\mathscr{B}}\frac{\gamma(S)}{\gamma'}y(s)) \leq (\sum_{S\in\mathscr{B}}\frac{\gamma(S)}{\gamma'}b(S))t(S)$,

and then,

$A(\sum_{S\in\mathscr{B}}\frac{\gamma(S)}{\gamma'}y(s)) \leq b(N)(\sum_{S\in\mathscr{B}}\frac{\gamma(S)}{\gamma'}t(S))$.

Therefore, $A\widehat{y} \leq b(N)\widehat{t}$, that is $\begin{pmatrix}\widehat{y}\\ \widehat{t}\end{pmatrix}$ verifies $Ay - tb(N) \leq 0$.

Since $d^Ty(S) + d_0t(S) \leq 1$ for each $S \in \mathscr{B}$ and $\frac{\gamma(S)}{\gamma'} \geq 0$, it follows that

$d^T\sum_{S\in\mathscr{B}}\frac{\gamma(S)}{\gamma'}y(S) + d_0\sum_{S\in\mathscr{B}}\frac{\gamma(S)}{\gamma'}t(S) \leq 1$

and then,

$d^T\widehat{y}+d_0\widehat{t}\leq 1.$

We notice that $\widehat{y}\geq \mathbf{0},\widehat{t}\geq 0$ and we can also conclude that $\begin{pmatrix}\widehat{y}\\ \widehat{t}\end{pmatrix}$ is a feasible solution for the linear problem associated to the coalition N. Thus, the following inequality holds:

$V(N)\geq \gamma'(c_1\widehat{y}_1+c_2\widehat{y}_2+...+c_p\widehat{y}_p+c_0\widehat{t}).$

Consequently,

$\sum_{S\in\mathscr{B}}\gamma(S)V(S)\leq V(N),$

which means that the game (N,V) is balanced. $\square$

Shapley (1967) proved that a balanced game has a non-empty core. Based on this, we can state the following corollary:

Corollary 1. *The core of the game (N,V) is nonempty.*

Furthermore, we will find the elements of the core for the game (N,V).

For this purpose, we denote $e=(1,0,..,0,0)^T\in\mathbb{R}^{m+1}$, $c'=\begin{pmatrix}c\\ c_0\end{pmatrix}\in\mathbb{R}^{p+1}$, $y'=\begin{pmatrix}y\\ t\end{pmatrix}\in\mathbb{R}^{p+1}$, $b'(S)=\begin{pmatrix}d_0\\ -b(S)\end{pmatrix}\in\mathbb{R}^{m+1}$ and $A'(S)=\begin{pmatrix}d^T & d_0\\ A & -b(S)\end{pmatrix}\in\mathbb{R}^{(m+1)\times(p+1)}$.

By using the above notations, the primal problem (4) can be written as

$$\begin{aligned} P(S):\quad & \text{Max } (c')^T y'\\ & \text{subject to } A'(S)y'\leq e\\ & y'\geq \mathbf{0}. \end{aligned}$$

The dual to $P(S)$ is the problem $D(S)$, where

$$\begin{aligned} D(S):\quad & \text{Min } \omega_1\\ & \text{subject to } (A'(S))^T\omega\geq c'\\ & \omega_i\geq 0 \text{ for each } i\in\{1,...,m+1\}. \end{aligned}$$

Explicitly, for each $S\subsetneq N$, $D(S)$ is

$$\begin{aligned} D(S):\quad & \text{Min } \omega_1\\ & \text{subject to } d\omega_1+A^T(\omega_2,\omega_3,...,\omega_{m+1})^T\geq c\\ & d_0\omega_1-b(S)(\omega_2,\omega_3,...,\omega_{m+1})^T\geq c_0\\ & \omega_i\geq 0 \text{ for each } i\in\{1,...,m+1\}. \end{aligned}$$

Suppose that the grand coalition N is formed. The dual to the problem $P(N)$ is $D(N)$, where

$$P(N): \quad \text{Max } \gamma(c')^T y' \\ \text{subject to } A'(N)y' \leq e \\ y' \geq \mathbf{0}.$$

$$D(N): \quad \text{Min } \omega_1 \\ \text{subject to } d\omega_1 + A^T(\omega_2, \omega_3, ..., \omega_{m+1})^T \geq \gamma c \\ d_0\omega_1 - b(N)(\omega_2, \omega_3, ..., \omega_{m+1})^T \geq \gamma c_0 \\ \omega_i \geq 0 \text{ for each } i \in \{1, ..., m+1\}.$$

We can formulate the results of our research concerning the form of the elements of the core of the game (N, V) in the next theorem.

THEOREM 4. *Let ω^* be an optimal solution for the dual problem of the associated linear programming problem $P(S)$ with $S = N$. Then the payoff $u = (u_1, u_2, ..., u_n) \in \mathbb{R}^n$, u_i defined by $u_i = \frac{1}{n}\omega_1^*$, $i = 1, 2, ..., n$ belongs to the core of the game (N, V).*

Let $\omega^* = (\omega_1^*, \omega_2^*, ..., \omega_{m+1}^*)^T$ be a solution of $D(N)$. Then, $V(N) = \omega_1^*$ and let $u = (u_1, u_2, ..., u_n)^T$ be the vector with the components $u_i = \frac{1}{n}\omega_1^*$ for each $i \in \{1, 2, ..., n\}$.

We will prove that the payoff vector $(u_1, u_2, ..., u_n)^T$ is an element of the core(N, V).

Firstly, we note that $\sum_{i \in N} u_i = v(N)$. We must show, in addition, that $\sum_{i \in S} u_i \geq v(S)$ for each $S \subset N$.

We also notice that $\sum_{i \in S} u_i = \frac{|S|}{n}\omega_1^*$.

The vector $(\frac{\omega_1^*}{\gamma}, \frac{\omega_2^*}{\gamma}, ..., \frac{\omega_{m+1}^*}{\gamma})^T$ verifies the restrictions of the problem $D(S)$:

$$d_0\frac{\omega_1^*}{\gamma} - \frac{b(S)}{\gamma}(\omega_2^*, ..., \omega_{m+1}^*)^T = \\ = d_0\frac{\omega_1^*}{\gamma} - \frac{b(N)}{\gamma}(\omega_2^*, ..., \omega_{m+1}^*)^T + \frac{b(N)-b(S)}{\gamma}(\omega_2^*, ..., \omega_{m+1}^*)^T \geq c_0$$

and

$$d\frac{\omega_1^*}{\gamma} - \frac{1}{\gamma}A^T(\omega_2^*, ..., \omega_{m+1}^*)^T \geq c.$$

Then, $V(S) \leq \frac{\omega_1^*}{\gamma}$.

Since $\gamma \geq n$, we have that $\frac{1}{\gamma} \leq \frac{1}{n} \leq \frac{|S|}{n}$ for each $S \subseteq N$ and then, $\sum_{i \in S} u_i = \frac{|S|}{n}\omega_1^* \geq \frac{1}{\gamma}\omega_1^* \geq V(S)$ for each $S \subseteq N$.

We conclude that $(u_1, u_2, ..., u_n)^T \in$core$(N, V)$. $\square$

3. MULTIOBJECTIVE FRACTIONAL LINEAR PROGRAMMING PRODUCTION GAMES

This section is designed to treating the case of the multiobjective fractional linear programming problem, where multiple decision makers, with different interests, are implied in the production process. A method due to Chakraborty and Gupta (2002) is used in order to obtain an equivalent multiobjective linear programming problem and it is shown that a cooperative game with values of coalitions arises from it. It is also proven that the associated cooperative game is balanced and a characterization of the Stable outcome of the associate cooperative game is given.

3.1. THE MODEL

In the following subsections, we will use the following notations. Let $\mathbb{R}^p_+ = \{x \in \mathbb{R}^p : x_j \geq 0,\ j = 1, 2, ..., p\}$ be the nonnegative orthant of the p-dimensional real space $\mathbb{R}$. If $A \subset \mathbb{R}^p$, we will denote by $\text{Max}A = \{a \in A : (A - a) \cap \mathbb{R}^p_+ = \{\mathbf{0}\}\}$ the the set of all Pareto maximal points.

We start by describing the multiobjective fractional linear production programming problem.

Let n be a fixed positive integer, let $N = \{1, 2, ..., n\}$ be the set of players (or decision makers) and $\mathscr{P}(N)$, the set of nonempty subsets of N being the set of coalitions formed with players $1, 2, ..., n$.

There are m types of resources used for the production of p goods. For each $i \in N$, the player i is endowed with a vector b^i of resources, where $b^i = (b^i_1, b^i_2, ..., b^i_m)$. Any coalition S will use a total of $b_k(S) = \sum_{i \in S} b^i_k$ units of the kth resource. We assume that a unit of the jth good $(j = 1, ..., p)$ requires a_{kj} units of the k th resource $(k = 1, ..., m)$. A coalition $S \subseteq N$ uses all its resources in order to produce a vector $(x_1, x_2, ..., x_p)$ of goods which satisfies the problem $P(S)$:

$$
\begin{array}{ll}
 & P(S) : \\
 & \max\ z_1(x) = \frac{N_1(x)}{D_1(x)} = \frac{(c_1)^T x + c_{10}}{(d_1)^T x + d_{10}} \\
 & \qquad z_2(x) = \frac{N_2(x)}{D_2(x)} = \frac{(c_2)^T x + c_{20}}{(d_2)^T x + d_{20}} \\
 & \qquad \ldots\ldots\ldots\ldots\ldots\ldots \\
 & \qquad z_r(x) = \frac{N_r(x)}{D_r(x)} = \frac{(c_r)^T x + c_{r0}}{(d_r)^T x + d_{r0}} \\
 & \text{subject to} \\
 & a_{11}x_1 + a_{12}x_2 + ... + a_{1p}x_p \leq b_1(S) \\
 & a_{21}x_1 + a_{22}x_2 + ... + a_{2p}x_p \leq b_2(S) \\
(5) & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}
$$

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mp}x_p \leq b_m(S)$$
$$x_1, x_2, ..., x_p \geq 0,$$

where $c_i, d_i \in \mathbb{R}^p$ for each $i \in \{1, 2, ..., r)$, $x \in \mathbb{R}^p$, $A = (a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,p}}} \in \mathbb{R}^{m \times p}$, $b(S) = (b_1(S), b_2(S), ..., b_m(S))^T \in \mathbb{R}^m$, $F_S = \{x \in \mathbb{R}^p : Ax \leq b(S),\ x \geq \mathbf{0}\} \subseteq \mathbb{R}^p_+$.

Firstly, we make the following notations:

Let $F(x) = (\frac{(c_1)^T x + c_{10}}{(d_1)^T x + d_{10}}, ..., \frac{(c_r)^T x + c_{r0}}{(d_r)^T x + d_{r0}})^T$. Let us define $I(S) = \{i : N_i(x) \geq 0$ for some $x \in F_S\ \}$ and $I^C(S) = \{i : N_i(x) \leq 0$ for every $x \in F_S\ \}$ for each $S \subseteq N$.

Suppose that for each $S \subseteq N$, F_S is nonempty and bounded.

We will assume further that $I^C(S) = \emptyset$ for each $S \subseteq N$.

In order to obtain an equivalent multiobjective linear programming problem, we shall use the method of Chakraborty and Gupta (2002).

Let $t = \cap_{i \in \{1,2,...,r\}} \frac{1}{d_i x + d_{i0}} \Leftrightarrow \frac{1}{d_i x + d_{i0}} \geq t$ for each $i \in \{1, 2, ..., r\}$ and $y = tx$.

The multiobjective linear fractional programming problem (5) is equivalent to the multiobjective linear programming problem (6):

$$(6) \qquad \begin{array}{l} \text{Max } \{tN_i(\frac{y}{t}) \text{ if } i \in \{1, 2, ..., r\} \\ tD_i(\frac{y}{t}) \leq 1 \text{ if } i \in \{1, 2, ..., r\} \\ A(\frac{y}{t}) - b(S) \leq \mathbf{0} \\ t \geq 0, y \geq \mathbf{0}. \end{array}$$

The problem (6) is equivalent to the problem (7)

$$(7) \qquad \begin{array}{l} \text{Max } (c_i)^T y + c_{i0}t,\ i \in \{1, 2, ..., r\} \\ (d_i)^T y + d_{i0}t \leq 1,\ i \in \{1, 2, ..., r\} \\ A(y) - tb(S) \leq \mathbf{0} \\ t \geq 0, y \geq \mathbf{0}. \end{array}$$

Chakraborty and Gupta (2002) proved that the constraint set of (6) is a non-empty convex set having feasible points.

We will use the following notations:

$T_S = \{\begin{pmatrix} y \\ t \end{pmatrix} : \begin{pmatrix} y \\ t \end{pmatrix}$ verifies the restriction of the problem (7)};

$\widehat{T}_S = \{(z \in \mathbb{R}^r : z = F\left(\begin{pmatrix} y \\ t \end{pmatrix}\right), \begin{pmatrix} y \\ t \end{pmatrix} \in T_S\}$

$V(S) = (\text{Max}\widehat{T}_S - \mathbb{R}^r_+) \cap \mathbb{R}^r_+$, where

$\text{Max}\widehat{T}_S$ is the set of all Pareto optimal values to the multiobjective linear production programming problem.

By following Nishizaki and Sakawa (2001), we note that if a feasible solution area T_S to the multiobjective linear production programming problem (6) is a nonempty bounded set, the set T_S is a bounded convex polyhedron and the characteristic set $V(S)$ is a comprehensive and compact subset of $\mathbb{R}^r$. We recall that the comprehensiveness of $V(S)$ means that if $u \leq v$ for $v \in V(S)$ and $u \in \mathbb{R}^r$, then, $u \in V(S)$.

3.2. THE ASSOCIATED COOPERATIVE GAME

In this subsection, we construct an associated multi-commodity game (N, V) and we prove that the game is balanced. We note that it is not superadditive. We recall that a game $G = (N, V)$ is called superadditive if $V(S \cup T) \geq V(S) + V(T)$ for any two disjoint coalitions S and T.

We also give a characterization of its stable outcome.

Let $N = \{1, 2, ..., n\}$ and $V : P(N) \to \mathbb{R}^r$ be defined by
$V(S) = (\text{Max}\widehat{T}_S - \mathbb{R}^r_+) \cap \mathbb{R}^r_+$ for each $S \subset N$ and
$V(N) = (\text{Max}\gamma\widehat{T}_N - \mathbb{R}^r_+) \cap \mathbb{R}^r_+$,

where $\gamma > \max(\gamma^*, n)$ and $\gamma^* = \max \sum_{S \in \mathscr{B}} \gamma(S)$, $\mathscr{B}$ being any balanced coalition of N.

The set of imputations of the game is defined by the set of payoff vectors satisfying the conditions of individual and collective rationality and it is expressed in the following way:

$$I(N, V) = \{u \in \mathbb{R}^{r \times n}_+ : u_N \in \text{Max}V(N),\ u_i \notin V(\{i\}) \backslash \text{Max}V(\{i\}),\ \forall i \in N\},$$

where u is the payoff vector $u = (u_1, u_2, ..., u_n) \in \mathbb{R}^{r \times n}_+$, $u_i = (u_i^1, u_i^2, ..., u_i^r)^T$ for each $i \in \{1, 2, ..., n\}$ and $u_N = \sum_{i \in N} u_i$.

The stable outcome is defined as

$$SO(N, V) = \{u \in \mathbb{R}^{r \times n}_+ : u_S \notin V(S) \backslash \text{Max}V(S),\ \forall S \subset N\},$$

where $u_S = \sum_{i \in S} u_i$.

Note that $SO(N, V)$ is the set of all feasible payoff vectors which no coalition S can improve on.

Furthermore, we prove the balancedness of the game (N, V).

THEOREM 5. *The game (N, V) is balanced.*

Proof. Let $\mathscr{B}$ be a balanced collection of N. Firstly, we know that
$\sum_{S \subset N} \gamma(S) b_k(S) = \sum_{S \subset N} \sum_{i \in S} \gamma(S) b_k^i(S) = \sum_{i \in N} \{\sum_{S \subset N, S \ni i} \gamma(S)\} b_k^i =$
$= \sum_{i \in N} b_k^i = b_k(N)$ for each $k \in \{1, 2, ..., m\}$.

Let $z(\begin{pmatrix} y(S) \\ t(S) \end{pmatrix}) = \begin{pmatrix} z_1(((y(S)^T, t(S))^T) \\ z_2(((y(S)^T, t(S))^T) \\ ... \\ z_r(((y(S)^T, t(S))^T) \end{pmatrix} \in V(S)$ and with this notation, we obtain

$$\sum_{S\in\mathscr{B}} \frac{\gamma(S)}{\gamma'} z(\begin{pmatrix} y(S) \\ t(S) \end{pmatrix}) = \sum_{S\in\mathscr{B}} \frac{\gamma(S)}{\gamma'} \begin{pmatrix} (c_1)^T y(S) + t(S)c_{10} \\ (c_2)^T y(S) + t(S)c_{20} \\ ... \\ (c_r)^T y(S) + t(S)c_{r0} \end{pmatrix} = \begin{pmatrix} \widehat{t}N_1(\frac{\widehat{y}}{\widehat{t}}) \\ \widehat{t}N_2(\frac{\widehat{y}}{\widehat{t}}) \\ ... \\ \widehat{t}N_r(\frac{\widehat{y}}{\widehat{t}}) \end{pmatrix}$$

$$= z(\begin{pmatrix} \widehat{y} \\ \widehat{t} \end{pmatrix}),$$

where $\gamma' = \sum_{S\in\mathscr{B}} \gamma(S)$, $\widehat{y} := \sum_{S\in\mathscr{B}} \frac{\gamma(S)}{\gamma'} y(S)$ and $\widehat{t} = \sum_{S\in\mathscr{B}} \frac{\gamma(S)}{\gamma'} t(S)$, $\widehat{y} \in \mathbb{R}^p_+$, $\widehat{t} \in \mathbb{R}_+$.

Assume that $\begin{pmatrix} y(S) \\ t(S) \end{pmatrix}$ is a feasible solution of Problem 7.

Since $Ay(S) \leq t(S)b(S)$ for each $S \in \mathscr{B}$ and $\frac{\gamma(S)}{\gamma'} \geq 0$, it follows that

$$A\frac{\gamma(S)}{\gamma'} y(s) \leq t(S)\frac{\gamma(S)}{\gamma'} b(S).$$

By adding, we obtain

$$A(\sum_{S\in\mathscr{B}} \frac{\gamma(S)}{\gamma'} y(s)) \leq (\sum_{S\in\mathscr{B}} \frac{\gamma(S)}{\gamma'} b(S))t(S),$$

and then,

$$A(\sum_{S\in\mathscr{B}} \frac{\gamma(S)}{\gamma'} y(s)) \leq b(N)(\sum_{S\in\mathscr{B}} \frac{\gamma(S)}{\gamma'} t(S)).$$

Therefore, $A\widehat{y} \leq b(N)\widehat{t}$, that is $\begin{pmatrix} \widehat{y} \\ \widehat{t} \end{pmatrix}$ verifies $Ay - tb(N) \leq 0$.

Since $(d_i)^T y(S) + d_{i0} t(S) \leq 1$ for each $S \in \mathscr{B}$, $i \in \{1, 2, ..., r\}$ and $\frac{\gamma(S)}{\gamma'} \geq 0$, it follows that

$$(d_i)^T \sum_{S\in\mathscr{B}} \frac{\gamma(S)}{\gamma'} y(S) + d_{i0} \sum_{S\in\mathscr{B}} \frac{\gamma(S)}{\gamma'} t(S) \leq 1 \text{ for each } i \in \{1, 2, ..., r\}$$

and then,

$$(d_i)^T \widehat{y} + d_{i0}\widehat{t} \leq 1 \text{ for each } i \in \{1, 2, ..., r\}.$$

We notice that $\widehat{y} \geq \mathbf{0}$ and $\widehat{t} \geq 0$ and conclude that $\begin{pmatrix} \widehat{y} \\ \widehat{t} \end{pmatrix} \in V(N)$ and therefore

$$\sum_{S\subset N} \gamma(S) \begin{pmatrix} y(S) \\ t(S) \end{pmatrix} \in V(N).$$

Since $\sum_{S\subset N} \gamma(S)V(S) \subset V(N)$, it follows that the game (N, V) is balanced. □

We consider the dual problem to the multiobjective linear problem in order to find a point belonging to the core. We present here some useful results concerning the duality of the multiobjective linear programming.

Let the primal and the dual problems be as follows:

$$\begin{aligned} &\max z(x) = Cx \\ &\text{subject to } z \in T_p = \{x : Ax = b,\, x \in \mathbb{R}^p_+\} \end{aligned} \tag{8}$$

and respectively

$$\begin{aligned} &\min g(w) = wb \\ &\text{subject to } w \in T_d = \{w : wAu \leq Cu \text{ for no } u \in \mathbb{R}^p_+\}, \quad) \end{aligned} \tag{9}$$

where $z(x) = (z_1(x), z_2(x), ..., z_r(x))^T$, $g(w) = (g_1(w), g_2(w), ..., g_r(w))^T$, $C \in \mathbb{R}^{r\times p}$, $A \in \mathbb{R}^{m\times p}$, $b \in \mathbb{R}^m, w \in \mathbb{R}^{r\times m}$.

In order to obtain Theorem 10, we will use the following theorems concerning the duality of multiobjective linear programming (see, for instance, Nishizaki and Sakawa (1999)).

THEOREM 6. *If x is a feasible solution for the primal problem (8) and w is a feasible solution for the dual problem (9), it is not the case that $g(w) \leq z(x)$.*

THEOREM 7. *Assume that x^* is a feasible solution of primal problem (8) and w is a feasible solution of dual problem (9). Also assume that $z(x^*) = g(w^*)$ is satisfied. Then, x^* is a Pareto optimal solution of primal problem (8), and w^* is a Pareto optimal solution of dual problem (9).*

THEOREM 8. *Considering main problem (8) and dual problem (9), the following two statements are equivalent.*

(1) *Each of the problems has a feasible solution.*

(2) *Each of the problems has a Pareto optimal solution, and there exists at least a pair of Pareto optimal solutions such that $z(x^*) = g(w^*)$.*

THEOREM 9. *The necessary and sufficient condition for x^* to be a Pareto optimal solution of primal problem (8) is that there exists a feasible solution w^* of dual problem (9) such that $z(x^*) = g(w^*)$. Then, w^* itself is a Pareto optimal solution of dual problem (9).*

The next theorem is the main result of this section. It gives an element of the stable outcome for the game (N, V).

THEOREM 10. *Let ω^* be a Pareto optimal solution for the dual problem of the associated multiobjective linear programming problem* (3) *with $S = N$. Then the payoff $u = (u_{1\cdot}, u_{2\cdot}, ..., u_{n\cdot}) \in \mathbb{R}^{r\times n}$, $u_{i\cdot} = (u_{i1}, u_{i2}, ..., u_{ir})$ defined by $u_{ik} = \frac{1}{n}\omega^*_{k1}$, $i = 1, 2, ..., n$ and $k = 1, 2, ..., r$ belongs to the stable outcome of the game (N, V).*

Proof. Firstly, we will reformulate the multiobjective linear production programming problems $P(S)$ for each $S \subseteq N$ equivalent to the multiobjective fractional linear production programming problems and the dual.

For this purpose, we make the following notations.

Let $c' \in \mathbb{R}^{r\times(p+1)}$, $A'(S) \in \mathbb{R}^{(r+m)\times(p+1)}$ for each $S \subseteq N$ and $b' \in \mathbb{R}^{r+m}$ be defined as $c' = \begin{pmatrix} c_1^T & c_{10} \\ c_2^T & c_{20} \\ \dots & \dots \\ c_r^T & c_{r0} \end{pmatrix}$, $A'(S) = \begin{pmatrix} d_1^T & d_{10} \\ d_2^T & d_{20} \\ \dots & \dots \\ d_r^T & d_{r0} \\ A & -I_m b(S) \end{pmatrix}$, $b' = \begin{pmatrix} \mathbf{1}_{\mathbb{R}^r} \\ \mathbf{0}_{\mathbb{R}^m} \end{pmatrix}$,

where

$\mathbf{1}_{\mathbb{R}^r} = (1, 1, ..., 1)^T \in \mathbb{R}^r$ and $\mathbf{0}_{\mathbb{R}^m} = (0, 0, ..., 0)^T \in \mathbb{R}^m$.

For each $S \subsetneq N$, the problem $P(S)$ is

$$P(S): \quad \text{Max } c' \cdot \begin{pmatrix} y \\ t \end{pmatrix}$$

$$\text{subject to } (y,t) \in T_p = \{\begin{pmatrix} y \\ t \end{pmatrix} : A'(S)\begin{pmatrix} y \\ t \end{pmatrix} \leq b', \begin{pmatrix} y \\ t \end{pmatrix} \in \mathbb{R}_+^{p+1}\}$$

and

$$P(N): \text{Max } \gamma c' \cdot \begin{pmatrix} y \\ t \end{pmatrix}$$

$$\text{subject to } \begin{pmatrix} y \\ t \end{pmatrix} \in T_p = \{\begin{pmatrix} y \\ t \end{pmatrix} : A'(N)\begin{pmatrix} y \\ t \end{pmatrix} \leq b', \begin{pmatrix} y \\ t \end{pmatrix} \in \mathbb{R}_+^{p+1}\}$$

Let $\widehat{T}_S$, $\widehat{T}_N$ be the feasible areas in the objective space of primal problems $P(S)$, respectively $P(N)$.

For each $S \subsetneq N$, the duals $D(S)$ and $D(N)$ are

$$D(S): \quad \text{Min } \omega b'$$

$$\text{subject to } \omega \in T_d = \{\omega : \omega A'(S)u \leq c'u \text{ for no } u \in \mathbb{R}_+^{p+1}\}$$

and

$$D(N): \quad \text{Min } \omega b'$$

$$\text{subject to } \omega \in T_d = \{\omega : \omega A'(N)u \leq \gamma c'u \text{ for no } u \in \mathbb{R}_+^{p+1}\}.$$

Let ω^* and $\begin{pmatrix} y^* \\ t^* \end{pmatrix}$ be Pareto optimal solutions for the problems $D(N)$ and $P(N)$.

We conclude that $\omega^* b' = c' \cdot \begin{pmatrix} y^* \\ t^* \end{pmatrix}$ and then, $\omega^* b' \in \text{Max}\widehat{T}_N$. We have that

$$\sum_{i\in N} u_{i,\cdot} = \sum_{i\in N}(u_{i1}, u_{i2}, ..., u_{ir})^T = \sum_{i\in N}(\tfrac{1}{n}w_{11}^*, \tfrac{1}{n}w_{21}^*, ..., \tfrac{1}{n}w_{r1}^*)^T =$$
$$= (\omega_{11}^*, ..., \omega_{r1}^*)^T \in \text{Max}V(N).$$

For each $S \subsetneq N$, $\sum_{i\in S} u_{i,\cdot} = \frac{|S|}{n}\omega^*_{\cdot,1}$

$\omega^* A'(N)u \leq \gamma c'u$ for no $u \in \mathbb{R}^{p+1}_+$ implies $\frac{\omega^*}{\gamma}A'(S)u \leq c'u$ for no $u \in \mathbb{R}^{p+1}_+$. It follows that $\frac{\omega^* b'}{\gamma} \in V(S)$. Since $\gamma \geq n$, $\frac{1}{\gamma} \leq \frac{1}{n} \leq \frac{|S|}{n}$ for each $S \subseteq N$.

$\sum_{i\in S} u_{i,\cdot} = \frac{|S|}{n}\omega^*_{\cdot,1} \geq \frac{1}{\gamma}\omega^*_{\cdot,1}$.

Then, $\sum_{i\in S} u_{i,\cdot} \notin V(S)-\text{Max}V(S)$

We conclude that $u = (u_{1\cdot}, u_{2\cdot}, ..., u_{n\cdot})^T \in SO(N,V)$. □

4. EXCHANGE ECONOMIES

In this section, we associate a cooperative game (N,V) to a finite pure exchange economy, whose utilities are fractional linear functions, and we study its properties. The main results of this section state the balancedness of the cooperative game and give the form of the stable outcome's elements.

We start by defining the model of the economy which will be the object of our research.

We consider a pure exchange economy $\mathscr{E} = (X_i, e_i, U_i)_{i\in N}$ with a finite number of agents, $N = \{1, 2, ..., n\}$. The commodity space is the Euclidean space $\mathbb{R}^m$. Each agent $i \in N$ is characterized by her consumption set $X_i = \mathbb{R}^m$, her initial endowment $e_i \in \mathbb{R}^m_+$ and her utility function $U_i : \prod_{i\in N} X_i \to \mathbb{R}$. An allocation is an element $x_i \in \mathbb{R}^m_+$. An allocation x is defined as a feasible allocation if $\sum_{i\in N} x_i \leq \sum_{i\in N} e_i$.

Let $p = mn$. We will use the following notation: instead of $x = (x_1, x_2, ..., x_n) = (x^1_1, x^2_1, ..., x^m_1, x^1_2, x^2_2, ..., x^m_2, ..., x^1_n, x^2_n, ..., x^m_n) \in \mathbb{R}^p_+$, we will use $x = (x_1, x_2, ..., x_m, x_{1+m}, ..., x_{2m}, ..., x_{(n-1)m+1}, ..., x_{nm})$, where $(x^1_i, x^2_i, ..., x^m_i) = (x_{(i-1)m+1}, ..., x_{im})$.

The coefficients $a_{ij}(S)$ are defined as follows.

$$\text{For } S \subset N, a_{ij}(S) = \begin{cases} 1 \text{ if } j = (k-1)m + i,\ i \in \{1, 2, ..., m-1\} \text{ and } k \in S; \\ 0 \text{ if } j \in (k-1)m + i,\ i \in \{1, 2, ..., m-1\} \text{ and } k \notin S; \\ 1 \quad \text{if} \quad j = km \text{ and } k \in S; \\ 0 \quad \text{if} \quad j = km \text{ and } k \notin S \end{cases}$$

for each $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., p\}$.

For $S \subseteq N$, the coefficients $b_i(S) = \sum_{j\in S} e^i_j$ represent the initial endowment from the i^{th} good of the coalition S.

For each $S \subseteq N$, we define the following multiobjective fractional linear problem $P(S)$, associated to the exchange economy:

$$\max U_1(x) = \frac{N_1(x)}{D_1(x)} = \frac{c_{11}x_1 + c_{12}x_2 + ... + c_{1m}x_m + c_{10}}{d_{11}x_1 + d_{12}x_2 + ... + d_{1m}x_m + d_{10}}$$
$$U_2(x) = \frac{N_2(x)}{D_2(x)} = \frac{c_{21}x_{m+1} + c_{22}x_{m+2} + ... + c_{2m}x_{2m} + c_{20}}{d_{21}x_{m+1} + d_{22}x_{m+2} + ... + d_{2m}x_{m+2} + d_{20}}$$

...

$$U_n(x) = \frac{N_n(x)}{D_n(x)} = \frac{c_{n1}x_{(n-1)m+1}+c_{n2}x_{(n-1)m+2}+...+c_{nm}x_{mn}+c_{n0}}{d_{n1}x_{(n-1)m+1}+d_{n2}x_{(n-1)m+2}+...+d_{nm}x_{mn}+d_{n0}}$$

subject to

$$(10)\qquad \begin{array}{l} a_{11}(S)x_1 + a_{12}(S)x_2 + ... + a_{1p}(S)x_p \leq b_1(S) \\ a_{21}(S)x_1 + a_{22}(S)x_2 + ... + a_{2p}(S)x_p \leq b_2(S) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}(S)x_1 + a_{m2}(S)x_2 + ... + a_{mp}(S)x_p \leq b_m(S) \\ a_{11}(N)x_1 + a_{12}(N)x_2 + ... + a_{1p}(N)x_p \leq b_1(N) \\ a_{21}(N)x_1 + a_{22}(N)x_2 + ... + a_{2p}(N)x_p \leq b_2(N) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}(N)x_1 + a_{m2}(N)x_2 + ... + a_{mp}(N)x_p \leq b_m(N) \\ x_1, x_2, ..., x_p \geq 0, \end{array}$$

where $p = mn$.

We will use the following notations:

$$c_i = (0, ...0, c_{i1}, c_{i2}, ..., c_{im}, 0, ...0)^T \in \mathbb{R}^p,$$

$$d_i = (0, ...0, d_{i1}, d_{i2}, ..., d_{im}, 0, ..., 0)^T \in \mathbb{R}^p \text{ for each } i \in \{1, 2, ..., n\},\ x \in \mathbb{R}^p,$$

$$A(S) = (a_{ij}(S))_{\substack{i=\overline{1,m} \\ j=\overline{1,p}}} \in \mathbb{R}^{m\times p},\ b(S) = (b_1(S), b_2(S), ..., b_m(S))^T \in \mathbb{R}^m,$$

$$F_S = \{x \in \mathbb{R}^p : A(S)x \leq b(S),\ A(N)x \leq b(N),\ x \geq \mathbf{0}\} \subseteq \mathbb{R}^p_+.$$

The multiobjective linear fractional programming problem (10) is equivalent to the multiobjective linear programming problem (11):

$$(11)\qquad \begin{array}{l} \text{Max } \{tN_i(\frac{y}{t}),\ i \in N\} \\ \text{subject to } tD_i(\frac{y}{t}) \leq 1 \text{ if } i \in N \\ \qquad A(S)(\frac{y}{t}) - b(S) \leq 0 \\ \qquad A(N)(\frac{y}{t}) - b(N) \leq 0 \\ \qquad t \in \mathbb{R}_+,\ y \in \mathbb{R}^p_+, \end{array}$$

or, explicitly,

$$(12)\qquad \begin{array}{l} \text{Max } (c_i)^T y + c_{i0}t,\ i \in \{1, ..., n\}; \\ \text{subject to } (d_i)^T y + d_{i0}t \leq 1,\ i \in \{1, ..., n\}; \\ \qquad A(S)y - tb(S) \leq 0 \\ \qquad A(N)y - tb(N) \leq 0 \\ \qquad t \in \mathbb{R}_+,\ y \in \mathbb{R}^m_+. \end{array}$$

We will attach the following cooperative game (N, V) to the economy $\mathscr{E}$: $N = \{1, 2, ..., n\}$ and $V : \mathscr{P}(N) \to \mathbb{R}^n$, $V(\emptyset) = \{\mathbf{0}\}$ is defined by

$$V(S)=\begin{cases}(\mathrm{Max}\widehat{T}_S-\mathbb{R}^n_+)\cap\mathbb{R}^n_+, \text{ if } S\subset N;\\(\mathrm{Max}\gamma\widehat{T}_N-\mathbb{R}^n_+)\cap\mathbb{R}^n_+, \text{ if } S=N,\end{cases}$$

where $\gamma>\max(\gamma^*,n)$ and $\gamma^*=\max\sum_{S\in\mathscr{B}}\gamma(S)$, $\mathscr{B}$ being any balanced coalition of N, $T_S=\{\begin{pmatrix}y\\t\end{pmatrix}:\begin{pmatrix}y\\t\end{pmatrix}$ verifies the restrictions of the problem (11)$\}$;
$\widehat{T}_S=\{z\in\mathbb{R}^n : z=(tN_i(\frac{y}{t}))_{i\in\{1,2,\ldots,n\}}, \begin{pmatrix}y\\t\end{pmatrix}\in T_S\}$.

Our aim is to obtain characterizations of elements of the stable outcome of the game (N,V) associated to a pure finite exchange economy with fractional linear utility functions. For this, we state a preliminary result concerning the game (N,V).

Theorem 11. *The game (N,V) is balanced.*

Proof. Let $\mathscr{B}$ be a balanced collection of N. First, we know that

$$\sum_{S\subset N}\gamma(S)b_k(S)=\sum_{S\subset N}\sum_{i\in S}\gamma(S)b^i_k(S)=\sum_{i\in N}\{\sum_{S\subset N,S\ni i}\gamma(S)\}b^i_k=$$
$$=\sum_{i\in N}b^i_k=b_k(N) \text{ for each } k\in\{1,2,...,m\}.$$

Let $z(\begin{pmatrix}y(S)\\t(S)\end{pmatrix})=\begin{pmatrix}z_1(((y(S)^T,t(S))^T)\\z_2(((y(S)^T,t(S))^T)\\\ldots\\z_n(((y(S)^T,t(S))^T)\end{pmatrix}\in V(S)$ and then,

$$\sum_{S\in\mathscr{B}}\frac{\gamma(S)}{\gamma'}z(\begin{pmatrix}y(S)\\t(S)\end{pmatrix})=\sum_{S\in\mathscr{B}}\frac{\gamma(S)}{\gamma'}\begin{pmatrix}(c_1)^Ty(S)+t(S)c_{10}\\(c_2)^Ty(S)+t(S)c_{20}\\\ldots\\(c_n)^Ty(S)+t(S)c_{n0}\end{pmatrix}=$$
$$=\begin{pmatrix}\widehat{t}N_1(\frac{\widehat{y}}{\widehat{t}})\\\widehat{t}N_2(\frac{\widehat{y}}{\widehat{t}})\\\ldots\\\widehat{t}N_n(\frac{\widehat{y}}{\widehat{t}})\end{pmatrix}=z(\begin{pmatrix}\widehat{y}\\\widehat{t}\end{pmatrix}),$$

where $\gamma'=\sum_{S\in\mathscr{B}}\gamma(S)$, $\widehat{y}:=\sum_{S\in\mathscr{B}}\frac{\gamma(S)}{\gamma'}y(S)$ and $\widehat{t}=\sum_{S\in\mathscr{B}}\frac{\gamma(S)}{\gamma'}t(S)$, $\widehat{y}\in\mathbb{R}^p_+$, $\widehat{t}\in\mathbb{R}_+$.

Assume that $\begin{pmatrix}y(S)\\t(S)\end{pmatrix}$ is a feasible solution of Problem (11).

We have that

$$A(S)y(S)\leq t(S)b(S)$$

and

$$A(N)y(S)\leq t(S)b(N) \text{ for each } S\in\mathscr{B},\ \frac{\gamma(S)}{\gamma'}\geq 0.$$

Consequently,

$$\frac{\gamma(S)}{\gamma'}A(N)y(s) \leq t(S)\frac{\gamma(S)}{\gamma'}b(N).$$

By adding, we obtain

$$\sum_{S\in\mathscr{B}} A(N)\tfrac{\gamma(S)}{\gamma'}y(s)) \leq \sum_{S\in\mathscr{B}} \tfrac{\gamma(S)}{\gamma'}A(N)y(s) \leq \sum_{S\in\mathscr{B}} \tfrac{\gamma(S)}{\gamma'}t(S)b(N) \leq$$
$$\leq b(N)\sum_{S\in\mathscr{B}} t(S)\tfrac{\gamma(S)}{\gamma'},$$

and then,

$$A(N)(\sum_{S\in\mathscr{B}} \frac{\gamma(S)}{\gamma'}y(s)) \leq b(N)(\sum_{S\in\mathscr{B}} \frac{\gamma(S)}{\gamma'}t(S)).$$

Therefore,

$$A(N)\widehat{y} \leq b(N)\widehat{t},$$

that is, $\begin{pmatrix} \widehat{y} \\ \widehat{t} \end{pmatrix}$ verifies $A(N)y - tb(N) \leq 0$.

Since $(d_i)^T y(S) + d_{i0}t(S) \leq 1$ for each $S \in \mathscr{B}$ and $i \in \{1, 2, ..., n\}$ and also $\frac{\gamma(S)}{\gamma'} \geq 0$, it follows that

$(d_i)^T \sum_{S\in\mathscr{B}} \frac{\gamma(S)}{\gamma'}y(S) + d_{i0}\sum_{S\in\mathscr{B}} \frac{\gamma(S)}{\gamma'}t(S) \leq 1$ for each $S \in \mathscr{B}$ and $i \in \{1, 2, ..., n\}$

and then, $(d_i)^T\widehat{y} + d_{i0}\widehat{t} \leq 1$ for each $i \in \{1, 2, ..., n\}$.

We notice that $\widehat{y} \geq \mathbf{0}, \widehat{t} \geq 0$ and conclude that $\begin{pmatrix} \widehat{y} \\ \widehat{t} \end{pmatrix} \in V(N)$ and therefore,

$$\sum_{S\subset N} \gamma(S) \begin{pmatrix} y(S) \\ t(S) \end{pmatrix} \in V(N).$$

We obtain that $\sum_{S\subset N} \gamma(S)V(S) \subset V(N)$, resulting that the game is balanced. □

We consider the dual problem to the multiobjective linear problem in order to find a point belonging to the stable outcome

$SO(N, V) = \{x \in \mathbb{R}_+^{n\times n} : x_S \notin V(S)\backslash \mathrm{Max}V(S), \forall S \subset N\}$.

The next theorem is the main result of this section and it gives the form of the elements of the stable outcome.

Theorem 12. *Let ω^* be a Pareto optimal solution for the dual problem of the associated multiobjective linear programming problem* (12) *with $S = N$. Then the payoff $u = (u_{1\cdot}, u_{2\cdot}, ..., u_{n\cdot}) \in \mathbb{R}^{n\times n}$, $u_{i\cdot} = (u_{i1}, u_{i2}, ..., u_{in})$ defined by $u_{ik} = \frac{1}{n}\omega^*_{k1}$, $i = 1, 2, ..., n$ and $k = 1, 2, ..., n$ belongs to the stable outcome of the game (N, V).*

Proof. Firstly, we formulate the multiobjective linear production programming problems $P(S)$ for each $S \subseteq N$ equivalent to the multiobjective fractional linear production programming problems and the dual.

Let $p = nm$, $c' \in \mathbb{R}^{n\times(p+1)}$, $A'(S) \in \mathbb{R}^{(n+2m)\times(p+1)}$ and $b' \in \mathbb{R}^{n+2m}$ be defined as $c' = \begin{pmatrix} c_1^T & c_{10} \\ c_2^T & c_{20} \\ \dots & \dots \\ c_n^T & c_{n0} \end{pmatrix}$, $A'(S) = \begin{pmatrix} d_1^T & d_{10} \\ d_2^T & d_{20} \\ \dots & \dots \\ d_n^T & d_{n0} \\ A(S) & -I_m b(S) \\ A(S) & -I_m b(N) \end{pmatrix}$, $b' = \begin{pmatrix} \mathbf{1}_{\mathbb{R}^n} \\ \mathbf{0}_{\mathbb{R}^m} \\ \mathbf{0}_{\mathbb{R}^m} \end{pmatrix}$,

where $\mathbf{1}_{\mathbb{R}^n} = (1, 1, ..., 1)^T \in \mathbb{R}^n$ and $\mathbf{0}_{\mathbb{R}^m} = (0, 0, ..., 0)^T \in \mathbb{R}^m$.

For each $S \subsetneq N$, the problem $P(S)$ is

$$P(S): \ \text{Max } c' \cdot \begin{pmatrix} y \\ t \end{pmatrix}$$

$$\text{subject to} \begin{pmatrix} y \\ t \end{pmatrix} \in T_p = \{\begin{pmatrix} y \\ t \end{pmatrix} : A'(S) \begin{pmatrix} y \\ t \end{pmatrix} \leq b', \begin{pmatrix} y \\ t \end{pmatrix} \in \mathbb{R}_+^{p+1}\}.$$

Suppose that the grand coalition N is formed. Then, the multiobjective linear production programming problem is represented as problem $P(S)$ with $S = N$.

$$P(N): \ \text{Max } \gamma c' \cdot \begin{pmatrix} y \\ t \end{pmatrix}$$

$$\text{subject to} \begin{pmatrix} y \\ t \end{pmatrix} \in T_p = \{\begin{pmatrix} y \\ t \end{pmatrix} : A'(N) \begin{pmatrix} y \\ t \end{pmatrix} \leq b', \begin{pmatrix} y \\ t \end{pmatrix} \in \mathbb{R}_+^{p+1}\}.$$

Let $\widehat{T}_S$, $\widehat{T}_N$ be the feasible areas in the objective space of primal problems $P(S)$, resp. $P(N)$.

For each $S \subsetneq N$, the corresponding dual $D(S)$ to the mathematical programming problem $P(S)$ is expressed as:

$$D(S): \quad \text{Min } \omega b'$$
$$\text{subject to } \omega \in T_d = \{\omega : \omega A'(S)u \leq c'u \text{ for no } u \in \mathbb{R}_+^{p+1}\}$$

and the dual to $P(N)$ is $D(N)$.

$$D(N): \quad \text{Min } \omega b'$$
$$\text{subject to } \omega \in T_d = \{\omega : \omega A'(N)u \leq \gamma c'u \text{ for no } u \in \mathbb{R}_+^{p+1}\}.$$

Thus, we can obtain a point in the stable outcome of the game (N, V) by solving the above linear programming problem.

Let ω^* and (y^*, t^*) be Pareto optimal solutions for the problems $D(N)$ and $P(N)$.

Theorem 9 guarantees that $\omega^* b' = \gamma c' \cdot \begin{pmatrix} y^* \\ t^* \end{pmatrix}$ and then, $\omega^* b' \in \mathrm{Max}\widehat{T}_N$. We have that

$$\begin{aligned}\textstyle\sum_{i\in N} u_{i,\cdot} &= \textstyle\sum_{i\in N}(u_{i1}, u_{i2}, ..., u_{in})^T = \sum_{i\in N}(\frac{1}{n}w^*_{11}, \frac{1}{n}w^*_{21}, ..., \frac{1}{n}w^*_{n1})^T = \\ &= (\omega^*_{11}, ..., \omega^*_{n1})^T \in \mathrm{Max}V(N).\end{aligned}$$

Note that for each $S \subsetneq N$,

$$\sum_{i\in S} u_{i,\cdot} = \frac{|S|}{n}\omega^*_{\cdot,1}$$

We also obtain that $\omega^* A'(N)u \leq \gamma c' u$ for no $u \in \mathbb{R}^{p+1}_+$ implies that

$$\tfrac{\omega^*}{\gamma} A'(S)u \leq c'u \text{ for no } u \in \mathbb{R}^{p+1}_+.$$

It follows that $\frac{\omega^* b'}{\gamma} \in V(S)$.
Since $\gamma \geq n$, then, $\frac{1}{\gamma} \leq \frac{1}{n} \leq \frac{|S|}{n}$ for each $S \subseteq N$.
Furthermore, we find a lower limit for the sum of $u_{i,\cdot}$, as it can be seen:

$$\begin{aligned}&\textstyle\sum_{i\in S} u_{i,\cdot} = \frac{|S|}{n}\omega^*_{\cdot,1} \geq \frac{1}{\gamma}\omega^*_{\cdot,1} \text{ and then,} \\ &\textstyle\sum_{i\in S} u_{i,\cdot} \notin V(S) - \mathrm{Max}V(S)\end{aligned}$$

We conclude that $u = (u_{1\cdot}, u_{2\cdot}, ..., u_{n\cdot}) \in SO(N, V)$. □

5. CONCLUDING REMARKS

Our paper focused on the examination of the properties of the cooperative games associated to some new types of production games. The last games were introduced in order to model the situations when the agents want to maximize their average income on unit time. For this purpose, it was needed to use fractional linear functions. Owen's model was generalized in two directions, by defining the fractional linear programming production games for the single-objective and multiobjective cases. The methods and the approach used for the description of the elements of the core of the associated cooperative game proved to be suitable for the study of the cooperative game associated to a finite exchange economy with fractional linear utilities. Open problems may refer to new approaches meant to reduce a multiobjective fractional linear programming problem to a multiobjective linear programming one. Or they may refer to the fuzzy extensions of the obtained results and to the use of the fuzzy methods for the mentioned problems. Another significant way to develop further this topic of research is enrichment of the notions depicting the image of the problems coming from the real world.

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