# SOME RESULTS ON THE REGULAR DIGRAPH OF IDEALS

FARZAD SHAVEISI

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The regular graph of ideals of the commutative ring $R$, denoted by $\Gamma_{reg}(R)$, is a graph whose vertex set is the set of all non-trivial ideals of $R$ and two distinct vertices $I$ and $J$ are adjacent if and only if either $I$ contains a $J$-regular element or $J$ contains an $I$-regular element. In this paper, a formula for the clique number of $\Gamma_{reg}(R)$ is given. Also, it is shown that both of the clique number and vertex chromatic number of $\Gamma_{reg}(R)$ are $n-1$, for every reduced ring $R$ with $n$ minimal prime ideals.

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## 1. INTRODUCTION

First we recall some definitions and notations on graphs and rings. Let $\Gamma$ be a digraph. An arc from a vertex $x$ to another vertex $y$ of $\Gamma$ is denoted by $x \longrightarrow y$. Also, we distinguish the *out-degree* $d_\Gamma^+(v)$, the number of arcs leaving a vertex $v$, and the *in-degree* $d_\Gamma^-(v)$, the number of arcs entering a vertex $v$. The degree $d_\Gamma(v)$ of a vertex $v$ is equal to the sum of its out- and in-degrees. We denote the vertex set of $\Gamma$, by $V(\Gamma)$. Let $G$ be a simple graph with the vertex set $V(G)$ and $A \subseteq V(G)$. We denote by $G[A]$ the subgraph of $G$ induced on $A$. If $|V(G)| = \mu$, for some cardinal number $\mu$, then the *complete graph* and its *complement* are denoted by $K_\mu$ and $\overline{K_\mu}$, respectively. The degree of a vertex $x$ of $G$ is denoted by $d(x)$ and the maximum degree of vertices of $G$ is denoted by $\Delta(G)$. A *bipartite graph* is a graph whose vertex set can be partitioned into two disjoint sets $U$ and $V$ such that every edge connects a vertex in $U$ to one in $V$. A *complete bipartite graph* is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. A complete bipartite graph with parts of sizes $\mu$ and $\nu$ is denoted by $K_{\mu,\nu}$. Moreover, if either $\mu = 1$ or $\nu = 1$, then the complete bipartite graph is said to be a *star graph*. The *center* of a star graph is a vertex that is adjacent to all other vertices. For every positive integer $n$, we denote the path and cycle with $n$ vertices by $P_n$ and $C_n$, respectively. A *clique* of a graph is a complete subgraph and the supremum of the sizes of cliques in $G$, denoted by $\omega(G)$, is called the *clique number* of $G$. If the graph has no vertex, then its clique number is defined to be 0. By $\chi(G)$, we denote the *vertex chromatic number* of $G$, i.e., the minimum number of colors which can be assigned to the vertices of $G$ in such a way that every two adjacent vertices have different colors. Let $G_1$ and $G_2$ be two arbitrary graphs. By $G_1 + G_2$ and $G_1 \vee G_2$, we mean the *disjoint union* of $G_1$ and $G_2$ and *join* of two graphs $G_1$ and $G_2$, respectively. For more details about the terminology of graphs used here, see [15].

Throughout this paper, $R$ is assumed to be a non-domain commutative ring with identity. An element $r \in R$ is called *$R$-regular* if $r \notin Z(R)$, where $Z(R)$ denotes the set of all zero-divisors of $R$. An $R$-sequence is a $d$-tuple $r_1, \ldots, r_d$ in $R$ such that for every $i \leq d$, $r_i$ is $\frac{R}{(r_1, r_2, \ldots, r_{i-1})}$-regular. The common length of maximal $R$-sequences in ideal $I$ of $R$ is denoted by $grade(I)$. In the case that $(R, \mathfrak{m})$ is a Noetherian local ring, the grade of $\mathfrak{m}$, denoted by $depth(R)$, is called the *depth* of $R$. By $Hom(M, N)$, we mean the set of all $R$-homomorphisms from $M$ to $N$. By $\mathbb{I}(R)$ ($\mathbb{I}(R)^*$), $Max(R)$ and $Min(R)$ we denote the set of all proper (non-trivial) ideals of $R$, the set of all maximal ideals of $R$ and the set of all minimal prime ideals of $R$, respectively. The ring $R$ is said to be *reduced*, if it has no non-zero nilpotent element. For every ideal $I$ of $R$, the *annihilator of $I$* is denoted by $Ann(I)$. Also, by $Var(I)$, we mean the set of all prime ideals $\mathfrak{p}$ of $R$ such that $\mathfrak{p} \supseteq I$. A subset $S$ of a commutative ring $R$ is called a *multiplicative closed subset* (m.c.s) of $R$ if $1 \in S$ and $x, y \in S$ implies that $xy \in S$. If $S$ is an m.c.s of $R$ and $M$ is an $R$-module, then we denote by $R_S$ and $M_S$, the ring of fractions of $R$ and the module of fractions of $M$ with respect to $S$, respectively. If $\mathfrak{p}$ is a prime ideal of $R$ and $S = R \setminus \mathfrak{p}$, we use the notation $M_{\mathfrak{p}}$, for the localization of $M$ at $\mathfrak{p}$. By $T(R)$, we mean the *total ring* of $R$ that is the ring of fractions, where $S = R \setminus Z(R)$.

There is by now an extensive literature studying algebraic structures through associated graphs. See for instance [1, 6, 9, 10, 12, 13]. In particular, even though the notion of regular graph (digraph) of ideals of a commutative ring is only 6 years old. The *regular digraph of ideals* of a ring $R$, denoted by $\overrightarrow{\Gamma_{reg}}(R)$, is a digraph whose vertex set is the set of all non-trivial ideals of $R$ and for every two distinct vertices $I$ and $J$, there is an arc from $I$ to $J$ if and only if $I$ contains a $J$-regular element. The underlying graph of $\overrightarrow{\Gamma_{reg}}(R)$ is denoted by $\Gamma_{reg}(R)$. The regular digraph (graph) of ideals, first was introduced by Nikmehr and Shaveisi in [9]. In that paper, the authors have proved some results on the connectivity and the coloring of this graph. Then some other authors have followed the study of this graph, for example the reader can see [2,3,14]. In this paper, some results about the connectivity and the vertex coloring of the regular graph of ideals are proved. In Section 2, it is proved that paths and stars can't appear as regular graphs of ideals of rings. In Section 3, it is shown that $\chi(\Gamma_{reg}(R)) = \omega(\Gamma_{reg}(R)) = 2|\mathrm{Max}(R)| - f(R) - 1$, where $R$ is an Artinian ring and $f(R)$ denotes the number of fields, appeared in the decomposition of $R$ to direct product of local rings. Section 3 is devoted to the case that $R$ is a reduced ring. For example, for every reduced ring $R$ with $|\mathrm{Min}(R)| = n \geq 3$, we obtain that $\chi(\Gamma_{reg}(R)) = \omega(\Gamma_{reg}(R)) = n-1$. Finally, it is seen that the finiteness of the clique number and vertex chromatic number of the regular graph of ideals of $R$ depends on those of localizations of $R$ at maximal ideals.

## 2. PATHS AND STARS ARE NOT REGULAR GRAPHS OF IDEALS

Let $R$ be a ring. From [9], we know that the regular graph of ideals of $R$ is a cycle if and only if $\Gamma_{reg}(R) \cong C_6$. In this section, it is shown that $\Gamma_{reg}(R)$ is neither a path nor a star graph. First we need the following Lemma which shows that a vertex $I$ enters every other vertex of $\overrightarrow{\Gamma_{reg}}(R)$ if and only if $\mathrm{grade}(I) \neq 0$.

*Remark* 1. Let $I$ and $J$ be two distinct non-trivial ideals of $R$ such that $I \longrightarrow J$ is an arc of $\overrightarrow{\Gamma_{reg}}(R)$. Then from [5, Proposition 1.2.3], we deduce that $Hom_R(\frac{R}{I}, J) = 0$. Moreover, if $R$ is a Noetherian ring, then $Hom_R(\frac{R}{I}, J) = 0$ implies that there is an arc from $I$ to $J$ in $\overrightarrow{\Gamma_{reg}}(R)$.

LEMMA 2. *Let $R$ be a Noetherian ring, $\overrightarrow{\Gamma_{reg}}(R)$ contains at least one arc and $I$ be a vertex of $\overrightarrow{\Gamma_{reg}}(R)$. Then for every vertex $J$ of $\overrightarrow{\Gamma_{reg}}(R)$, $I \longrightarrow J$ is an arc of $\overrightarrow{\Gamma_{reg}}(R)$ if and only if grade$(I) \neq 0$.*

*Proof.* First suppose that $R$ is a local ring with the unique maximal ideal $\mathfrak{m}$. Let $I \neq \mathfrak{m}$. Since $I \longrightarrow \mathfrak{m}$ is an arc of $\overrightarrow{\Gamma_{reg}}(R)$, $I$ contains an $\mathfrak{m}$-regular element. Thus $\mathrm{grade}(I) \neq 0$. Now, suppose that $I = \mathfrak{m}$. If $\mathrm{Ann}(\mathfrak{m}) = 0$, then $\mathrm{grade}(I) = \mathrm{grade}(\mathfrak{m}) \neq 0$ and so we are done. If $\mathrm{Ann}(\mathfrak{m}) = \mathfrak{m}$, then $\mathfrak{m}^2 = 0$. This implies that $\overrightarrow{\Gamma_{reg}}(R)$ has no arc, a contradiction. Thus we can assume that $\mathrm{Ann}(\mathfrak{m})$ is a vertex of $\overrightarrow{\Gamma_{reg}}(R)$ distinct from $\mathfrak{m}$ and so $\mathfrak{m}$ contains an $\mathrm{Ann}(\mathfrak{m})$-regular element, a contradiction. Now, suppose that $R$ is not a local ring. Let $\mathfrak{p} \in \mathrm{Var}(I)$ and $JR_{\mathfrak{p}}$ be a vertex of $\overrightarrow{\Gamma_{reg}}(R_{\mathfrak{p}})$. Since there is an arc from $I$ to $J$, Remark 15 implies that $\mathrm{Hom}_R(\frac{R}{I}, J) = 0$. Thus by [11, Lemma 4.87], $\mathrm{Hom}_{R_{\mathfrak{p}}}(\frac{R_{\mathfrak{p}}}{IR_{\mathfrak{p}}}, JR_{\mathfrak{p}}) = 0$ and so by the local case, $\mathrm{grade}_{R_{\mathfrak{p}}}(IR_{\mathfrak{p}}, R_{\mathfrak{p}}) \neq 0$. Hence $\mathrm{depth}(R_{\mathfrak{p}}) \neq 0$, for every $\mathfrak{p} \in Var(I)$. So by using [5, Proposition 1.2.10 (i)], we have $\mathrm{grade}(I) = \inf\{\mathrm{depth}(R_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Var}(I)\} \neq 0$, as desired. The converse is trivial. $\square$

THEOREM 3. *For every Noetherian ring $R$, $\Gamma_{reg}(R)$ is not a star graph.*

*Proof.* Suppose to the contrary, $R$ is a ring such that $\Gamma_{reg}(R)$ is a star graph with the center $I$. If $R$ has exactly two non-trivial ideals, say $I$ and $J$, then it is not hard to check that $IJ = 0$ and so $\Gamma_{reg}(R) \cong \overline{K_2}$, a contradiction. Therefore, we can assume that $R$ contains at least three non-trivial ideals. Now, we consider the following three cases:

Case 1. $d^+(I) \neq 0$ and $d^-(I) \neq 0$. In this case, there exist two distinct non-trivial ideals $J$ an $K$ such that $I \longrightarrow J$ and $K \longrightarrow I$ are arcs of $\overrightarrow{\Gamma_{reg}}(R)$. Thus $K$ contains an $I$-regular element, say $x$ and $I$ contains a $J$-regular element, say $y$. Since $\Gamma_{reg}(R)$ is a star graph, there is no arc from $K$ to $J$. So there exists a non-zero element $z \in J$ such that $xz = 0$ and $yz \neq 0$. Thus $x(yz) = 0$, a contradiction.

Case 2. $d^+(I) = 0$ or $d^-(I) = 0$. In this case, since $\Gamma_{reg}(R)$ is a star graph, we deduce that either $I$ enters every other vertex of $\Gamma_{reg}(R)$ or there is an arc from every other vertex of $\Gamma_{reg}(R)$ to $I$. So, Lemma 2 and [9, Lemma 3.1] impliy that $I$ contains a regular element, say $x$. Thus $(x), (x^2), (x^3)$ form a triangle in $\Gamma_{reg}(R)$, a contradiction. $\square$

THEOREM 4. *For every Noetherian ring $R$, $\Gamma_{reg}(R)$ is not a path.*

*Proof.* By contrary suppose that $R$ is a ring and $\Gamma_{reg}(R)$ is a path. Then by [9, Corollary 3.1], $R$ is not Artinian. So there exists an infinite descending chain of non-trivial ideals of $R$, say $I_1 \supsetneqq I_2 \supsetneqq \cdots$. By hypothesis, there exists two adjacent vertices $I_n$ $J$ in $\Gamma_{reg}(R)$, for every $n$, $n \geq 3$. If $I_n \longrightarrow J$ is an arc of $\overrightarrow{\Gamma_{reg}}(R)$, then $I_n$ contains a $J$-regular element and so $I_m$ contains a $J$-regular element, for every $m$, $1 \leq m \leq n$. Thus $d(J) \geq n$, a contradiction. If $J \longrightarrow I_n$ is an arc of $\overrightarrow{\Gamma_{reg}}(R)$, then $J$ contains an $I_m$-regular element, for every $m$, $m \geq n$, a contradiction. $\square$

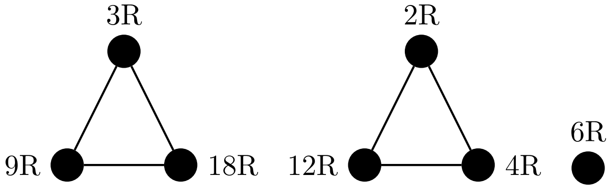
## 3. A FORMULA FOR THE CLIQUE NUMBER IN ARTINIAN RINGS

In this section, we give a formula for $\omega(\Gamma_{reg}(R))$, when $R$ is an Artinian ring. Also, it is shown that for every Artinian ring $R$, $|\mathrm{Max}(R)| - 1 \leq \omega(\Gamma_{reg}(R)) \leq 2|\mathrm{Max}(R)| - 1$ and the lower bound occurs if and only if $R$ is a reduced ring. If $R$ is an Artinian local ring which is not a field, then by [9, Theorem 2.1], $\omega(\Gamma_{reg}(R)) = 1$. Also, it is clear that for every field $F$, $\Gamma_{reg}(F)$ has no vertex. The following example shows that the upper bound occurs, too.

*Example* 5. Let $R = \mathbb{Z}_{36} \cong \mathbb{Z}_4 \times \mathbb{Z}_9$. Then it is clear that $R$ is an Artinian ring with two maximal ideals. On the other hand, we know that $C = \{\mathbb{Z}_4 \times (0), \mathbb{Z}_4 \times (3), (2) \times (0)\}$ is a clique in $\Gamma_{reg}(R)$ and so

$$\omega(\Gamma_{reg}(R)) \geq 3 = 2|\mathrm{Max}(R)| - 1.$$

In fact, the regular graph of $R = \mathbb{Z}_{36}$ is seen in the following figure:




$\Gamma_{reg}(\mathbb{Z}_{36})$

*Remark* 6. Let $R_1, \ldots, R_n$ be rings, $R \cong R_1 \times \cdots \times R_n$ and $I = I_1 \times \cdots \times I_n$ and $J = J_1 \times \cdots \times J_n$ be two distinct vertices of $\Gamma_{reg}(R)$. Then

(i) $I$ contains a $J$-regular element if and only if for every $i$, either $I_i$ contains a $J_i$-regular element or $J_i = (0)$.

(ii) Assume that every $R_i$ is an Artinian local ring. Then (i) and [9, Theorem 2.1] imply that if $I$ contains a $J$-regular element, then $J$ contains no $I$-regular element.

Lemma 7. *Let $S$ be an Artinian ring and $T$ be an Artinian local ring. If $R \cong S \times T$, then*

$$\omega(\Gamma_{reg}(R)) = \begin{cases} \omega(\Gamma_{reg}(S)) + 1; & T \text{ is a field} \\ \omega(\Gamma_{reg}(S)) + 2; & T \text{ is not a field.} \end{cases}$$

*Proof.* First note that for every clique $C$ of $\Gamma_{reg}(S)$, $C \times \{T\}$ is a clique of $\Gamma_{reg}(R)$. Also, for any clique $C' = \{I_i \times J_i\}_{i \in A}$ of $\Gamma_{reg}(R)$, from Remark 6 and [9, Theorem 2.1], we deduce that $\{J_i | \; I_i \times J_i \in C'\}_{i \in A}$ contains at most one nontrivial ideal. Therefore, $\omega(\Gamma_{reg}(S))$ is infinite if and only if $\omega(\Gamma_{reg}(R))$ is infinite. Now, assume that $\omega(\Gamma_{reg}(S))$ is finite and $C$ is a clique of $\Gamma_{reg}(S)$ with $|C| = \omega(\Gamma_{reg}(S))$. If $T$ is a field, then $C \times \{T\} \cup \{(0) \times T\}$ is a clique of $\Gamma_{reg}(S)$. Also, if $T$ is not a field, then for every nontrivial ideal $J$ of $T$, $C \times \{T\} \cup \{(0) \times J, (0) \times T\}$ is a clique of $\Gamma_{reg}(R)$. Therefore,

$$\omega(\Gamma_{reg}(R)) \geq \begin{cases} \omega(\Gamma_{reg}(S)) + 1; & T \text{ is a field} \\ \omega(\Gamma_{reg}(S)) + 2; & T \text{ is not a field.} \end{cases}$$

Next, we prove the inverse inequality. To see this, let $C' = \{I_i \times J_i | \; 1 \leq i \leq t\}$ be a maximal clique of $\Gamma_{reg}(R)$. Setting

$$C_1 = \{I_i \times J_i \in C' | \; J_i \text{ is a nontrivial ideal of } T\},$$

we deduce that there are sets $C_2, C_3$ such that

$$C' = C_1 \cup (C_2 \times \{(0)\}) \cup (C_3 \times \{T\}).$$

Hence

$$|C'| = |C_1| + |C_2| + |C_3|. \tag{1}$$

From [9, Theorem 2.1] and Remark 6, it follows that $|C_1| \leq 1$; moreover, if $T$ is a field, then $|C_1| = 0$. Now, we follow the proof in the following two cases:

Case 1. Either $C_2 = \varnothing$ or $C_3 = \varnothing$. If $C_2 = \varnothing$ (resp. $C_3 = \varnothing$), then by Remark 6, $C_3 \setminus \{(0)\}$ (resp. $C_2 \setminus \{T\}$) is a clique of $\Gamma_{reg}(S)$. This implies that $|C_2| + |C_3| \leq \omega(\Gamma_{reg}(S)) + 1$. Thus by (1), we have:

$$\omega(\Gamma_{reg}(R)) = |C'| \leq \begin{cases} \omega(\Gamma_{reg}(S)) + 1; & T \text{ is a field} \\ \omega(\Gamma_{reg}(S)) + 2; & T \text{ is not a field.} \end{cases}$$

Case 2. $C_2 \neq \varnothing$ and $C_3 \neq \varnothing$. In this case, one can easily check that $C_2$ and $C_3$ contain only nontrivial ideals. Also, it follows from Remark 6 that $C_2 \cup C_3$ is a clique of $\Gamma_{reg}(S)$, and this implies that $|C_2 \cup C_3| \leq \omega(\Gamma_{reg}(S))$. We claim that $|C_2 \cap C_3| \leq 1$. Suppose to the contrary, $I_1, J_1 \in C_2 \cap C_3$. Then it is clear that $I_1, J_1$ are nontrivial ideals of $S$, and $\{I_1 \times (0), I_1 \times T, J_1 \times (0), J_1 \times T\} \subseteq C'$. Thus $\overrightarrow{\Gamma_{reg}}(R)$ contains the arcs $I_2 \times T \longrightarrow I_1 \times (0)$ and $I_1 \times T \longrightarrow I_2 \times (0)$. Hence $I_1$ contains an $I_2$-regular element and $I_2$ contains an $I_1$-regular element, and this contradicts Remark 6(ii). So the claim is proved and hence,

$$|C_2| + |C_3| = |C_2 \cup C_3| + |C_2 \cap C_3| \leq \omega(\Gamma_{reg}(S)) + 1.$$

Thus again by (1), we have:

$$\omega(\Gamma_{reg}(R)) = |C'| \leq \begin{cases} \omega(\Gamma_{reg}(S)) + 1; & T \text{ is a field} \\ \omega(\Gamma_{reg}(S)) + 2; & T \text{ is not a field.} \end{cases}$$

Therefore, in any case, the assertion follows. $\square$

For any Artinian ring $R$, by $f(R)$, we denote the number of fields, appeared in the decomposition of $R$ to direct product of local rings.

Proposition 8. *For any Artinian ring $R$, $\omega(\Gamma_{reg}(R)) = 2|\mathrm{Max}(R)| - f(R) - 1$.*

*Proof.* If $R$ is a field, then there is nothing to prove. So, assume that $R$ is an Artinian ring which is not a field. Then [4, Theorem 8.7] implies that $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $n = |\mathrm{Max}(R)|$ and every $R_i$ is an Artinian local ring. We prove the assertion, by induction on $n$. If $n = 1$, then the assertion follows from [9, Theorem 2.1]. Thus we can assume that $n \geq 2$. Now, setting $S = R_1 \times R_2 \times \cdots \times R_{n-1}$, we follow the proof in the following two cases:

Case 1. $R_n$ is a field. In this case, the induction hypothesis implies that $\omega(\Gamma_{reg}(R')) = 2|\mathrm{Max}(R')| - f(R') - 1 = 2(n-1) - (f(R)-1) - 1 = 2n - f(R) - 2$; Thus by Lemma 7, we have:

$$\omega(\Gamma_{reg}(R)) = \omega(\Gamma_{reg}(R')) + 1 = 2n - f(R) - 1.$$

Case 2. $R_n$ is not a field. In this case, the induction hypothesis implies that

$\omega(\Gamma_{reg}(R')) = 2|\mathrm{Max}(R')| - f(R') - 1 = 2(n-1) - f(R) - 1 = 2n - f(R) - 3$; Thus again by Lemma 7, we have:

$$\omega(\Gamma_{reg}(R)) = \omega(\Gamma_{reg}(R')) + 2 = 2n - f(R) - 1.$$

Therefore, in any case, the assertion follows. □

From [9, Theorem 2.3] and Proposition 8, we have the following corollary.

COROLLARY 9. *Let $R$ be an Artinian ring. Then*

(i) $\omega(\Gamma_{reg}(R)) = \chi(\Gamma_{reg}(R))$.

(ii) *If $R$ is reduced, then $\omega(\Gamma_{reg}(R)) = |\mathrm{Max}(R)| - 1$.*

Now, we state the correct version of Theorem 2.2 from [9].

THEOREM 10. *If $R$ is an Artinian ring, then $|\mathrm{Max}(R)|-1 \leq \omega(\Gamma_{reg}(R)) \leq 2|\mathrm{Max}(R)|-1$. Moreover, $\omega(\Gamma_{reg}(R)) = |\mathrm{Max}(R)|-1$ if and only if $R$ is reduced.*

## 4. REGULAR GRAPH OF IDEALS AND LOCALIZATION

In this section, the clique number and the vertex chromatic number of $\Gamma_{reg}(R)$ are determined, when $R$ is a reduced ring. First, we recall the following interesting result, due to Eben Matlis.

PROPOSITION 11 ([8], Proposition 1.5). *Let $R$ be a ring and $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ be a finite set of distinct minimal prime ideals of $R$. Let $S = R \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i$. Then $R_S \cong R_{\mathfrak{p}_1} \times \cdots \times R_{\mathfrak{p}_n}$.*

THEOREM 12. *Let $R$ be a reduced ring, $|\mathrm{Min}(R)| = n \geq 3$ and $\omega(\Gamma_{reg}(R)) < \infty$. Then*

$$\omega(\Gamma_{reg}(R)) = \chi(\Gamma_{reg}(R)) = \chi(\Gamma_{reg}(T(R))) = \omega(\Gamma_{reg}(T(R))) = n - 1.$$

*Proof.* Assume that $\omega(\Gamma_{reg}(R)) < \infty$. First we show that every element of $R$ is an either zero-divisor or unit. By contrary, suppose that $x \in R$ is neither zero-divisor nor unit. Then it is not hard to check that $\{(x^n)\}_{n\geq 1}$ is an infinite clique of $\Gamma_{reg}(R)$, a contradiction. Suppose that $\mathrm{Min}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, for some positive integer $n$. If $S = R \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i$, then [7, Corollary 2.4] implies that

$T(R) = R_S$. So by Proposition 11, we have $T(R) \cong R_{\mathfrak{p}_1} \times \cdots \times R_{\mathfrak{p}_n}$. Since $R$ is reduced, by [8, Proposition 1.1], Part (1), every $R_{\mathfrak{p}_i}$ is a field. We claim that if $I$ and $J$ are two distinct vertices of $\overrightarrow{\Gamma_{reg}}(R)$, then $I \longrightarrow J$ is an arc in $\overrightarrow{\Gamma_{reg}}(R)$ if and only if $I_S \longrightarrow J_S$ is an arc in $\overrightarrow{\Gamma_{reg}}(R_S)$. First suppose that $I$ and $J$ are two distinct non-trivial ideals of $R$ and there is an arc from $I$ to $J$ in $\overrightarrow{\Gamma_{reg}}(R)$. Since $S$ contains no zero-divisor, we deduce that $I_S$ and $J_S$ are two non-trivial ideals of $R_S$. We show that $I_S \neq J_S$. Suppose to the contrary, $I_S = J_S$. Then for every $x \in I$, there exists an element $t \in S$ such that $tx \in J$. Since every element in $S$ is a unit, we deduce that $x \in J$. So $I \subseteq J$. Similarly, one can show that $J \subseteq I$. Thus $I = J$, a contradiction. Therefore, $I_S \neq J_S$. Now, let $x \in I$ be a $J$-regular element. Then one can easily show that $\frac{x}{1} \in I_S$ is a $J_S$-regular element and so there is an arc from $I_S$ to $J_S$ in $\overrightarrow{\Gamma_{reg}}(R_S)$. Conversely, let $\frac{x}{s} \in I_S$ be a $J_S$-regular element. Then we show that $x \in I$ is a $J$-regular element. Suppose to the contrary, $xy = 0$, for some $0 \neq y \in J$. Then we deduce that $\frac{x}{s}.\frac{y}{1} = 0$, a contradiction. So the claim is proved. Therefore, the graphs $\Gamma_{reg}(R)$ and $\Gamma_{reg}(T(R))$ are isomorphic. Now, since $T(R)$ is the direct product of $n$ fields, the assertion follows from Proposition 8. $\square$

The following corollary is an immediate consequence of Theorem 12.

Corollary 13. *Let $R$ be a reduced ring with finitely many minimal prime ideals such that $\omega(\Gamma_{reg}(R)) < \infty$. Then*

$$|\mathrm{Min}(R)| = |\mathrm{Max}(T(R))| = \chi(\Gamma_{reg}(T(R))) + 1 = \omega(\Gamma_{reg}(R)) + 1.$$

Finally, in the remaining of this paper, we see that the finiteness of the clique number and vertex chromatic number of the regular graph of ideals of $R$ depends on those of localizations of $R$ at maximal ideals. Before this, we need to recall the following lemma from [1].

Lemma 14 (See [1], Lemma 9). *Let $R$ be a ring, $I$ and $J$ be two non-trivial ideals of $R$. If for every $\mathfrak{m} \in \mathrm{Max}(R)$, $I_{\mathfrak{m}} = J_{\mathfrak{m}}$, then $I = J$.*

*Remark* 15. Let $I$ and $J$ be two distinct non-trivial ideals of $R$. If $I \longrightarrow J$ is an arc of $\overrightarrow{\Gamma_{reg}}(R)$, then it is not hard to show that $\mathrm{Hom}_R(\frac{R}{I}, J) = 0$. Moreover, if $R$ is a Noetherian ring, then $\mathrm{Hom}(\frac{R}{I}, J) = 0$ implies that $I \longrightarrow J$ is an arc of $\overrightarrow{\Gamma_{reg}}(R)$.

Theorem 16. *Let $R$ be a Noetherian ring with finitely many maximal ideals. If for every $\mathfrak{m} \in \mathrm{Max}(R)$, $\omega(\Gamma_{reg}(R_{\mathfrak{m}}))$ is finite, then $\omega(\Gamma_{reg}(R))$ is finite.*

*Proof.* Let $\mathrm{Max}(R) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$. Suppose to the contrary, $C = \{J_i\}_{i=1}^{\infty}$ is an infinite clique of $\Gamma_{reg}(R)$. Then by Remark 15, for every $i$ and

$j$ with $i \neq j$, either $\mathrm{Hom}_R(\frac{R}{J_i}, J_j) = 0$ or $\mathrm{Hom}_R(\frac{R}{J_j}, J_i) = 0$. Thus after localization we deduce that either $\mathrm{Hom}_{R_{\mathfrak{m}_1}}(\frac{R_{\mathfrak{m}_1}}{(J_i)_{\mathfrak{m}_1}}, (J_j)_{\mathfrak{m}_1}) = 0$ or $\mathrm{Hom}_{R_{\mathfrak{m}_1}}(\frac{R_{\mathfrak{m}_1}}{(J_j)_{\mathfrak{m}_1}}, (J_i)_{\mathfrak{m}_1}) = 0$, for every $i$ and $j$ with $i \neq j$. Since $\omega(\Gamma_{reg}(R_{\mathfrak{m}_1})) < \infty$, we deduce that there exists an infinite subset $A_1 \subseteq \mathbb{N}$ such that for every $i, j \in A_1$, $(J_i)_{\mathfrak{m}_1} = (J_j)_{\mathfrak{m}_1}$. Now, using $\omega(\Gamma_{reg}(R_{\mathfrak{m}_2})) < \infty$, we conclude that there exists an infinite subset $A_2 \subseteq A_1$ such that for every $i, j \in A_2$, $(J_i)_{\mathfrak{m}_2} = (J_j)_{\mathfrak{m}_2}$. By continuing this procedure one can see that there exists an infinite subset $A_n \subseteq A_{n-1}$ such that for every $i, j \in A_n$, $(J_i)_{\mathfrak{m}_l} = (J_j)_{\mathfrak{m}_l}$, for every $l$, $l = 1, \ldots, n$. Therefore, by Lemma 14, we get a contradiction. □

THEOREM 17. *Let $R$ be a ring with finitely many maximal ideals. If for every $\mathfrak{m} \in \mathrm{Max}(R)$, $\chi(\Gamma_{reg}(R_{\mathfrak{m}}))$ is finite, then $\chi(\Gamma_{reg}(R))$ is finite and moreover,*

$$\chi(\Gamma_{reg}(R) \leq \prod_{\mathfrak{m} \in \mathrm{Max}(R)} (\chi(\Gamma_{reg}(R_{\mathfrak{m}}) + 2) - 2.$$

*Proof.* Let $\mathrm{Max}(R) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ and $f_i : V(\Gamma_{reg}(R_{\mathfrak{m}_i})) \longrightarrow \{1, \ldots, \chi(\Gamma_{reg}(R_{\mathfrak{m}_i}))\}$ be a proper vertex coloring of $\Gamma_{reg}(R_{\mathfrak{m}_i})$, for every $i$, $1 \leq i \leq n$. We define a function $f$ on $\mathbb{I}(R) \setminus \{R\}$ by $f(I) = (g_1(I_{\mathfrak{m}_1}), \ldots, g_n(I_{\mathfrak{m}_n}))$, where

$$g_i(I_{\mathfrak{m}_i}) = \begin{cases} 0; & I_{\mathfrak{m}_i} = (0) \\ -1; & I_{\mathfrak{m}_i} = R_{\mathfrak{m}_i} \\ f_i(I_{\mathfrak{m}_i}); & \text{otherwise.} \end{cases}$$

Using Lemma 14, it is not hard to check that $f$ is a proper vertex coloring of $\Gamma_{reg}(R)$ and this completes the proof. □

Recall that the length of a longest chain of prime ideals contained in the ideal $I$ is called the height of $I$ and it is denoted by *height*$(I)$. Also, the ring $R$ is called Cohen-Macaulay if for every ideal $I$ of $R$, $\mathrm{grade}(I) = \mathrm{height}(I)$. We finish this paper with the following result which shows that Cohen-Macaulay rings, whose regular graphs of ideals have finite clique numbers, are Artinian.

THEOREM 18. *Let $R$ be a Cohen-Macaulay ring. Then $\omega(\Gamma_{reg}(R)) < \infty$ if and only if $R$ is Artinian.*

*Proof.* First suppose that $\omega(\Gamma_{reg}(R)) < \infty$. We claim that every element of $R$ is either zero-divisor or unit. Suppose to the contrary, $x$ is an element of $R$ which is neither zero-divisor nor unit. Then one can easily check that $C = \{(x^n)\}_{n \geq 1}$ is an infinite clique of $\Gamma_{reg}(R)$, a contradiction. So, the claim is proved. Since $R$ is Cohen-Macaulay, for every non-trivial ideal $I$ of $R$, we have $\mathrm{height}(I) = \mathrm{grade}(I) = 0$. Hence $R$ is Artinian. The converse follows from [9, Theorem 2.2]. □

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*Razi University,*
*Faculty of Sciences,*
*Department of Mathematics,*
*P.O.Box 67149-67346,*
*Kermanshah, Iran*
*f.shaveisi@razi.ac.ir*