ON THE VALUE DISTRIBUTION OF DIFFERENCE POLYNOMIALS

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We investigate the value distribution of difference polynomials

\[ f^n - \sum_{\lambda \in I} a_{\lambda}(z) \prod_{\nu=1}^{m} f(z + c_{\nu})^{l_{\lambda,\nu}} - s(z) \]

and

\[ f^n - \sum_{\lambda \in I} a_{\lambda}(z) \prod_{\nu=1}^{m} (\Delta^{\nu} f)^{l_{\lambda,\nu}} - s(z), \]

where \( s(z) \) and the coefficients \( a_{\lambda}(z) (\lambda \in I) \) are small functions relative of \( f \), \( I = \{ \lambda : \lambda = (l_{\lambda,1}, \ldots, l_{\lambda,m}) | l_{\lambda,\nu} \in \mathbb{N} \} \) is a finite index set, \( c_{\nu}, \nu = 1, \ldots, m \) are distinct complex numbers, and \( \Delta \) is a difference operator.

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1. INTRODUCTION

We use standard notations from Nevanlinna theory. We denote by \( \sigma(f) \) the order of growth of the meromorphic function \( f \) on the complex plane \( \mathbb{C} \), and also use the notation \( \varsigma(f) \) to denote the hyper-order of \( f \),

\[ \sigma(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}, \quad \varsigma(f) = \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}, \]

where \( T(r, f) \) is the characteristic function of \( f \).

A well known result of Hayman [2] is stated that:

**Theorem A.** Let \( f \) be a transcendental meromorphic function and \( a \neq 0, b \) be finite complex constants. Then \( f^n + af' - b \) has infinitely many zeros for \( n \geq 5 \). If \( f \) is transcendental entire, this holds for \( n \geq 3 \), resp. \( n \geq 2 \), if \( b = 0 \).

Recently, Halburd-Korhonen [3, 4], and Chiang-Feng [1] have established the Nevanlinna theory for difference operators. As an application of this theory, Liu and Lain [6, 7] gave the following difference counterpart of Theorem A.

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Theorem B. Let $f$ be a transcendental meromorphic function of finite order $\rho(f) = \rho$, not of period $c$, $a$ be a non-zero complex constant. Then the difference polynomial $f^n(z) + a(f(z + c) - f(z)) - s(z)$ has infinitely many zeros in the complex plane, provided $n \geq 8$, where $s(z)$ is small function respective of $f$. If $f$ is transcendental entire function with finite order and $s(z)$ is entire function which is small function of $f$, then statement holds with $n \geq 3$.

A number of results on the value distribution of some difference polynomials were given by Qi-Ding-Yuan [10], Zheng-Chen [11], Liu-Yi [8]. However, these results have been still restricted to the case of meromorphic functions $f$ having finite order, and hence, hyper-order $\varsigma(f) = 0$. The purpose of this paper is to examine this problem in the case where meromorphic function $f$ having hyper-order $\varsigma(f) < 1$, and in the general case of difference polynomials.

For each meromorphic function $f$ on complex plane, by a difference product, we mean a difference monomial and its shifts, that is, an expression of type

$$\prod_{\nu=1}^{m} f(z + c_{\nu})^{l_{\nu}},$$

where $c_1, \ldots, c_m$ are distinct complex numbers and $l_1, \ldots, l_m$ are natural numbers. The difference polynomial of $f$ is given by

$$P(z, f) = \sum_{\lambda \in I} a_{\lambda}(z) \prod_{\nu=1}^{m} f(z + c_{\nu})^{l_{\lambda,\nu}},$$

where the coefficients $a_{\lambda}(z)(\lambda \in I)$ are small functions with respect to $f$ and $I = \{ \lambda : \lambda = (l_{\lambda,1}, \ldots, l_{\lambda,m})| l_{\lambda,\nu} \in \mathbb{N} \}$ is a finite index set and $c_{\nu}, \nu = 1, \ldots, m$ are distinct complex numbers. We denote $l_{\nu} = \max_{\lambda \in I} l_{\lambda,\nu}$ and $\ell = \sum_{\nu=1}^{m} l_{\nu}$.

Our results are stated as follows:

Theorem 1. Let $f$ be a transcendental meromorphic function with hyper-order $\varsigma := \varsigma(f) < 1$ and $s(z)$ be a small function with respect to $f$. Assume that $n \geq \ell + m + 3$ and $P(z, f) + s(z) \not\equiv 0$. Then the polynomial difference $f^n(z) - P(z, f) - s(z)$ has infinitely many zeros.

If in Theorem 1 we take $\ell = m$ and $P(z, f) = \sum_{j=1}^{m} a_{j}(z)f(z + c_{j})$, then we obtain the following corollary.

Corollary 1. Let $f$ be a transcendental meromorphic function with hyper-order $\varsigma(f) = \varsigma < 1$, $s(z)$, $a_{j}(z)$, $j = 1, \ldots, m$ be small functions with respect to $f$, and $c_{j}$, $j = 1, \ldots, m$ be complex distinct constants. Assume that $n \geq 2m + 3$ and $\sum_{j=1}^{m} a_{j}(z)f(z + c_{j}) + s(z) \not\equiv 0$. Then the polynomial difference $f^n(z) - \sum_{j=1}^{m} a_{j}(z)f(z + c_{j}) - s(z)$ has infinitely many zeros.
If in Corollary 1 we take $m = 1$, $a_1(z) = \frac{1}{a}, a \neq 0$, $s(z) = -\frac{b}{a}, a, b \in \mathbb{C}$, we get the following corollary.

**COROLLARY 2.** Let $f$ be a transcendental meromorphic function with hyper-order $\varsigma(f) = \varsigma < 1$, and $a, c$ be non-zero complex constants. Then for any integer $n \geq 5$,

$$\Psi_1(z) = f(z + c) - af^n(z)$$

assume all finite $b \in \mathbb{C}$ infinitely often.

For each meromorphic function $f$, then $\Delta(f) = f(z+c) - f(z)$ ($c \in \mathbb{C}\{0\}$) is called a difference of $f(z)$. Set $\Delta^m f = \Delta(\Delta^{m-1} f)$, for each integer $m \geq 2$. By an easy computation, we have

$$\Delta^m f(z) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} f(z + (m - i)c).$$

Now we consider the difference polynomial

$$Q(z, f) = \sum_{\lambda \in I} a_\lambda(z) \prod_{\nu=1}^{m} (\Delta^\nu f)^{l_{\lambda,\nu}},$$

where the coefficients $a_\lambda(z)(\lambda \in I)$ are small functions with respect to $f$ and $I = \{\lambda : \lambda = (l_{\lambda,1}, \ldots, l_{\lambda,m})|l_{\lambda,\nu} \in \mathbb{N}\}$ is a finite index set. We also denote $\ell_\nu = \max_{\lambda \in I} l_{\lambda,\nu}$ and $\ell = \sum_{\nu=1}^{m} \ell_\nu$.

**THEOREM 2.** Let $f$ be a transcendental meromorphic function with hyper-order $\varsigma(f) = \varsigma < 1$ and $s(z)$ be a small function with respect to $f$. Assume that $n \geq \sum_{\nu=1}^{m} \ell_\nu (\nu + 1) + m + 4$ and $Q(z, f) + s(z) \neq 0$. Then the polynomial difference $f^n(z) - Q(z, f) - s(z)$ has infinitely many zeros.

2. SOME LEMMAS

In order to prove the two theorems, we need the following lemmas, due to Halburd-Korhonen-Tohge [4].

**LEMMA 2.1.** Let $T : [0, +\infty) \to [0, +\infty)$ be a non-decreasing continuous function, and let $s \in (0, +\infty)$. If the hyper-order of $T$ is strictly less than one, i.e.,

$$\limsup_{r \to \infty} \frac{\log \log T(r)}{\log r} = \varsigma < 1,$$

then

$$T(r + s) = T(r) + o(T(r) e^{-\varepsilon}),$$

where $\varepsilon > 0$ and $r \to \infty$ outside of a set of finite logarithmic measure.
Lemma 2.2. Let $f$ be a non-constant meromorphic function, $\varepsilon > 0$ and $c \in \mathbb{C}$.

If $f$ is of finite order, then there exists a set $E = E(f, \varepsilon)$ satisfying
\[
\limsup_{r \to \infty} \frac{\int_{E \cap [1, r)} \frac{dt}{t}}{\log r} \leq \varepsilon
\]
(i.e. of logarithmic density at most $\varepsilon$), such that
\[
m\left(r, \frac{f(z + c)}{f(z)}\right) = O\left(\frac{\log r}{r} T(r, f)\right)
\]
for all $r$ outside the set $E$.

If $\varsigma(f) = \varsigma < 1$ and $\varepsilon > 0$, then
\[
m\left(r, \frac{f(z + c)}{f(z)}\right) = o\left(\frac{T(r, f)}{r^{1-\varsigma-\varepsilon}}\right)
\]
for all $r$ outside of a set of finite logarithmic measure.

Lemma 2.3. Let $f$ be a non-constant meromorphic function with $\varsigma(f) = \varsigma < 1$, and $c \in \mathbb{C} \setminus \{0\}$. Then
\[
N(r, f(z + c)) \leq N(r, f) + S(r, f),
\]
\[
T(r, f(z + c)) \leq T(r, f) + S(r, f).
\]

Proof. We have
\[
\delta := \limsup_{r \to \infty} \frac{\log \log N(r, f)}{\log r} \leq \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r} = \varsigma(f) < 1.
\]
Hence, by Lemma 2.1, we get
\[
N(r + |c|, f) = N(r, f) + o\left(\frac{N(r, f)}{r^{1-\delta-\varepsilon}}\right)
\]
\[
\leq N(r, f) + o\left(\frac{T(r, f)}{r^{1-\varsigma-\varepsilon}}\right)
\]
\[
= N(r, f) + S(r, f).
\]
Therefore, by Lemma 2.2, we have
\[
T(r, f(z + c)) = m(r, f(z + c)) + N(r, f(z + c))
\]
\[
= m(r, \frac{f(z + c)}{f(z)} \cdot f(z)) + N(r, f(z + c))
\]
\[
\leq m(r, \frac{f(z + c)}{f(z)}) + m(r, f) + N(r + |c|, f)
\]
\[
\leq m(r, \frac{f(z + c)}{f(z)}) + m(r, f) + N(r, f) + S(r, f)
\]
We have completed the proof of Lemma 2.3. □

From the basic properties of characteristic function, we get easily that:

**Lemma 2.4.** Let \( f_1, \ldots, f_n \) be distinct meromorphic functions. Then

\[
T(r, \sum_{\lambda \in I} a_{\lambda} f_1^{l_{\lambda,1}} \cdots f_n^{l_{\lambda,n}}) \leq \sum_{j=1}^{n} \ell_j T(r, f_j) + \sum_{\lambda \in I} O(T(r, a_{\lambda})) ,
\]

where \( I = \{ \lambda : \lambda = (l_{\lambda,1}, \ldots, l_{\lambda,n}) | l_{\lambda,j} \in \mathbb{N}, j = 1, \ldots, n \} \) is a finite index, \( \ell_j = \max_{\lambda \in I} l_{\lambda,j}, j = 1, \ldots, n \) and \( a_{\lambda} (\lambda \in I) \) are meromorphic functions.

### 3. PROOF OF THEOREMS

**Proof of Theorem 1.** Take \( \phi(z) = f^n(z) - P(z, f) - s(z) \).

If \( \phi(z) \equiv c \), where \( c \) is a constant, then \( f^n(z) = P(z, f) + s(z) + c \). Then by Lemma 2.3 and Lemma 2.4, we have

\[
nT(r, f) \leq \ell T(r, f) + S(r, f).
\]

This is in contradiction with \( n \geq \ell + m + 3 \). Hence, \( \phi(z) \) is nonconstant.

It is easy to see that

\[
\frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)} \neq 0.
\]

Indeed, otherwise,

\[
\frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)} \equiv 0,
\]

then \( \phi(z) = bf^n(z) \). This implies that \((b - 1)f^n(z) = P(z, f) + s(z)\).

- If \( b = 1 \), then \( P(z, f) + s(z) \equiv 0 \). This is in contradiction with hypothesis.
- If \( b \neq 1 \), then \( nT(r, f) \leq \ell T(r, f) + S(r, f) \). This is in contradiction with \( n \geq \ell + m + 3 \).

We have

\[
-f^n(z) = \frac{\phi'(z)}{\phi(z)}[P(z, f) + s(z)] - \frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)}.
\]

Then

\[
\frac{\phi'(z)}{\phi(z)}[P(z, f) + s(z)] - \frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)}(3.1)
\]

\[
nT(r, f) = T(r, -f^n(z)) = T(r, \frac{\phi'(z)}{\phi(z)}[P(z, f) + s(z)] - \frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)}).
\]
It is clear that

$$\leq m(r, P(z, f) + s(z))$$

$$+ N(r, \frac{\phi'(z)}{\phi(z)} [P(z, f) + s(z)] - [P(z, f) + s(z)]')$$

$$+ m(r, \frac{\phi'(z)}{\phi(z)} - \frac{(P(z, f) + s(z))'}{P(z, f) + s(z)} + m(r, \frac{f^n(z)'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)})$$

$$+ N(r, \frac{f^n(z)'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)}) + S(r, f).$$

Set $\varphi_1 = \frac{\phi'(z)}{\phi(z)} [P(z, f) + s(z)] - [P(z, f) + s(z)]'$. It is clear that

$$(3.2) \quad N(r, \varphi_1) \leq \overline{N}(r, \frac{1}{\varphi_1}) + \overline{N}(r, f) + N(r, P(z, f) + s(z)) + S(r, f).$$

(note that $\text{Pole}(\phi) \subset \text{Pole}(f)$).

Next, we prove that:

$$(3.3) \quad N(r, \frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)}) \leq \overline{N}(r, \frac{1}{\varphi_1}) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, P(z, f) + s(z)) + S(r, f).$$

It is clear that all pole of $\frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)}$ are simple and $\text{Pole}(\frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)}) \subset \text{Zero}(f) \cup \text{Pole}(f) \cup \text{Zero}(\phi)$. From this fact, for (3.3), it suffices to prove that:

If $z_0$ is a pole of $f$ but it is not a pole of $P(z, f) + s(z)$, then $\frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)}$ is holomorphic at $z_0$.

Indeed, we write $f(z) = \frac{\varphi(z)}{(z-z_0)^k}$ in a neighborhood of $z_0$, where $\varphi(z)$ is a holomorphic function and $\varphi(z_0) \neq 0$ and $k \geq 1$.

We get easily that

$$\frac{f^n(z)'}{f^n(z)} = \frac{(z - z_0)(\varphi^n)' - nk\varphi^n}{(z - z_0)\varphi^n}$$

and

$$\phi = \frac{\varphi - (P(z, f) + s(z))(z - z_0)^n}{(z - z_0)^n}.$$ 

Hence

$$\frac{\phi'}{\phi} = \frac{(z - z_0)[\varphi^n - g(z)(z - z_0)^n]'}{(z - z_0)[\varphi^n - g(z)(z - z_0)^n]}.$$

This implies that

$$\frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)} = \frac{(z - z_0)(\varphi^n)' - nk\varphi^n}{(z - z_0)\varphi^n}.$$
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\[
\frac{(z - z_0)[\varphi^n - g(z)(z - z_0)^nk]' - nk[\varphi^n - g(z)(z - z_0)^nk]}{(z - z_0)[\varphi^n - g(z)(z - z_0)^nk]} = \frac{1}{z - z_0} \cdot \frac{\theta(z)}{\varphi^n(\varphi^n - g(z)(z - z_0)^nk)},
\]

where

\[
\theta(z) = [(z - z_0)(\varphi^n)' - nk\varphi^n][\varphi^n - g(z)(z - z_0)^nk] - [(z - z_0)[\varphi^n - g(z)(z - z_0)^nk]' - nk[\varphi^n - g(z)(z - z_0)^nk]]\varphi^n.
\]

On the other hand,

\[
\left. \frac{\theta(z)}{\varphi^n(\varphi^n - g(z)(z - z_0)^nk)} \right|_{z = z_0} = \frac{-nk\varphi^{2n}(z_0) + nk\varphi^{2n}(z_0)}{\varphi^{2n}(z_0)} = 0.
\]

Therefore, \(z_0\) is not a pole of \(\frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)}\).

Combining (3.1), (3.2) and (3.3), we have

\[
nT(r, f) \leq m(r, P(z, f) + s(z)) + N(r, P(z, f) + s(z)) + 2N(r, \frac{1}{\phi}) + N(r, \frac{1}{f}) + N(r, P(z, f) + s(z)) + S(r, f)
\]

\[
= T(r, P(z, f)) + 2N(r, \frac{1}{\phi}) + N(r, f) + N(r, \frac{1}{f})
\]

\[
+ N(r, P(z, f) + s(z)) + S(r, f).
\]

From the definition of \(P(z, f)\) and Lemma 2.3, we have

\[
N(r, P(z, f) + s(z)) \leq \sum_{\nu=1}^{m} N(r, f(z + c_\nu)) \leq \sum_{\nu=1}^{m} T(r+, f(z + c_\nu)) + S(r, f)
\]

\[
(3.5)
\]

By Lemma 2.3 and Lemma 2.4, we have

\[
T(r, P(z, f)) \leq \ell T(r, f) + S(r, f).
\]

From (3.4), (3.5), and (3.6), we have

\[
nT(r, f) \leq (\ell + m + 2)T(r, f) + 2N(r, \frac{1}{\phi}) + S(r, f).
\]

On the other hand \(n \geq \ell + m + 3\). Hence, \(\phi\) has infinitely many zeros. This completes the proof of Theorem 1. \(\square\)
Proof of Theorem 2. By an argument similar to the proof of Theorem 1, we obtain that
\[
nT(r, f) \leq m(r, Q(z, f) + s(z)) + N(r, Q(z, f) + s(z))
\]
\[+ 2N(r, \frac{1}{\phi}) + \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, Q(z, f) + s(z)) + S(r, f)\]
\[= T(r, Q(z, f)) + 2N(r, \frac{1}{\phi}) + \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, Q(z, f) + s(z)) + S(r, f).
\]
(3.7)
\[
\Delta^\nu f(z) = \sum_{i=0}^{\nu} (-1)^i C^i_\nu f(z + (\nu - i)c),
\]
for all \(\nu = 0, \ldots, m\).

By Lemma 2.3, we have
\[
T(r, \Delta^\nu f(z)) = T(r, \sum_{i=0}^{\nu} (-1)^i C^i_\nu f(z + (\nu - i)c))
\]
\[= m(r, \sum_{i=0}^{\nu} (-1)^i C^i_\nu f(z + (\nu - i)c))
\]
\[+ N(r, \sum_{i=0}^{\nu} (-1)^i C^i_\nu f(z + (\nu - i)c))
\]
\[= m(r, \frac{\sum_{i=0}^{\nu} (-1)^i C^i_\nu f(z + (\nu - i)c)}{f}.f)
\]
\[+ N(r, \sum_{i=0}^{\nu} (-1)^i C^i_\nu f(z + (\nu - i)c))
\]
\[\leq \sum_{i=0}^{\nu} m(r, \frac{f(z + (\nu - i)c)}{f}) + m(r, f)
\]
\[+ N(r, \sum_{i=0}^{\nu} (-1)^i C^i_\nu f(z + (\nu - i)c))
\]
\[
\leq (\nu + 1)T(r, f) + S(r, f).
\]
(3.8)

From the definition of \(Q(z, f)\) and by Lemma 2.3, we have
\[
\overline{N}(r, Q(z, f) + s(z)) \leq \sum_{i=0}^{m} \overline{N}(r, f(z + (m - i)c))
\]
\[\leq (m + 1)T(r, f) + S(r, f).
\]
(3.9)
By (3.8) and Lemma 2.3 and Lemma 2.4, we have

$$T(r, Q(z, f)) \leq \sum_{\nu=1}^{m} \ell_{\nu} T(r, \Delta^{\nu} f) + S(r, f)$$

(3.10)

$$\leq \sum_{\nu=1}^{m} \ell_{\nu}(\nu + 1) T(r, f) + S(r, f).$$

By (3.7) and (3.10), we have

$$nT(r, f) \leq \left[ \sum_{\nu=1}^{m} \ell_{\nu}(\nu + 1) + m + 3 \right] T(r, f) + 2N(r, \frac{1}{\phi}) + S(r, f).$$

On the other hand, $n \geq \sum_{\nu=1}^{m} \ell_{\nu}(\nu + 1) + m + 4$. Hence, $f^n - Q(z, f) - s(z)$ has infinitely zeros. We have completed the proof of Theorem 2.  \(\square\)

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