ON THE VALUE DISTRIBUTION OF DIFFERENCE POLYNOMIALS

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We investigate the value distribution of difference polynomials

$$f^n - \sum_{\lambda\in I} a_\lambda(z)\prod_{\nu=1}^{m} f(z+c_\nu)^{l_{\lambda,\nu}} - s(z)$$

and

$$f^n - \sum_{\lambda\in I} a_\lambda(z)\prod_{\nu=1}^{m} (\Delta^\nu f)^{l_{\lambda,\nu}} - s(z),$$

where $s(z)$ and the coefficients $a_\lambda(z)(\lambda \in I)$ are small functions relative of f, $I = \{\lambda : \lambda = (l_{\lambda,1},\dots,l_{\lambda,m})|l_{\lambda,\nu} \in \mathbb{N}\}$ is a finite index set, $c_\nu, \nu = 1,\dots,m$ are distinct complex numbers, and Δ is a difference operator.

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1. INTRODUCTION

We use standard notations from Nevanlinna theory. We denote by $\sigma(f)$ the order of growth of the meromorphic function f on the complex plane $\mathbb{C}$, and also use the notation $\varsigma(f)$ to denote the hyper-order of f,

$$\sigma(f) = \limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},\ \ \varsigma(f) = \limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the characteristic function of f.

A well known result of Hayman [2] is stated that:

THEOREM A. *Let f be a transcendental meromorphic function and $a \neq 0, b$ be finite complex constants. Then $f^n + af' - b$ has infinitely many zeros for $n \geq 5$. If f is transcendental entire, this holds for $n \geq 3$, resp. $n \geq 2$, if $b = 0$.*

Recently, Halburd-Korhonen [3,4], and Chiang-Feng [1] have established the Nevanlinna theory for difference operators. As an application of this theory, Liu and Lain [6,7] gave the following difference counterpart of Theorem A.

Theorem B. *Let f be a transcendental meromorphic function of finite order $\rho(f)=\rho$, not of period c, a be a non-zero complex constant. Then the difference polynomial $f^n(z)+a(f(z+c)-f(z))-s(z)$ has infinitely many zeros in the complex plane, provided $n\geq 8$, where $s(z)$ is small function respective of f. If f is transcendental entire function with finite order and $s(z)$ is entire function which is small function of f, then statement holds with $n\geq 3$.*

A number of results on the value distribution of some difference polynomials were given by Qi-Ding-Yuan [10], Zheng-Chen [11], Liu-Yi [8]. However, these results have been still restricted to the case of meromorphic functions f having finite order, and hence, hyper-order $\varsigma(f)=0$. The purpose of this paper is to examine this problem in the case where meromorphic function f having hyper-order $\varsigma(f)<1$, and in the general case of difference polynomials.

For each meromorphic function f on complex plane, by a difference product, we mean a difference monomial and its shifts, that is, an expression of type

$$\prod_{\nu=1}^{m} f(z+c_\nu)^{l_\nu},$$

where $c_1,\dots,c_m$ are distinct complex numbers and $l_1,\dots,l_m$ are natural numbers. The difference polynomial of f is given by

$$P(z,f)=\sum_{\lambda\in I} a_\lambda(z)\prod_{\nu=1}^{m} f(z+c_\nu)^{l_{\lambda,\nu}},$$

where the coefficients $a_\lambda(z)(\lambda\in I)$ are small functions with respect to f and $I=\{\lambda:\lambda=(l_{\lambda,1},\dots,l_{\lambda,m})|l_{\lambda,\nu}\in\mathbb{N}\}$ is a finite index set and $c_\nu,\nu=1,\dots,m$ are distinct complex numbers. We denote $\ell_\nu=\max_{\lambda\in I} l_{\lambda,\nu}$ and $\ell=\sum_{\nu=1}^{m}\ell_\nu$.

Our results are stated as follows:

Theorem 1. *Let f be a transcendental meromorphic function with hyper-order $\varsigma:=\varsigma(f)<1$ and $s(z)$ be a small function with respect to f. Assume that $n\geq \ell+m+3$ and $P(z,f)+s(z)\not\equiv 0$. Then the polynomial difference $f^n(z)-P(z,f)-s(z)$ has infinitely many zeros.*

If in Theorem 1 we take $\ell=m$ and $P(z,f)=\sum_{j=1}^{m} a_j(z)f(z+c_j)$, then we obtain the following corollary.

Corollary 1. *Let f be a transcendental meromorphic function with hyper-order $\varsigma(f)=\varsigma<1$, $s(z)$, $a_j(z)$, $j=1,\dots,m$ be small functions with respect to f, and c_j, $j=1,\dots,m$ be complex distinct constants. Assume that $n\geq 2m+3$ and $\sum_{j=1}^{m} a_j(z)f(z+c_j)+s(z)\not\equiv 0$. Then the polynomial difference $f^n(z)-\sum_{j=1}^{m} a_j(z)f(z+c_j)-s(z)$ has infinitely many zeros.*

If in Corollary 1 we take $m = 1$, $a_1(z) = \dfrac{1}{a}, a \neq 0, s(z) = -\dfrac{b}{a}, a, b \in \mathbb{C}$, we get the following corollary.

COROLLARY 2. *Let f be a transcendental meromorphic function with hyper-order $\varsigma(f) = \varsigma < 1$, and a, c be non-zero complex constants. Then for any integer $n \geq 5$,*

$$\Psi_1(z) = f(z + c) - af^n(z)$$

assume all finite $b \in \mathbb{C}$ infinitely often.

For each meromorphic function f, then $\Delta(f) = f(z+c)-f(z)$ $(c \in \mathbb{C}\backslash\{0\})$ is called a difference of $f(z)$. Set $\Delta^m f = \Delta(\Delta^{m-1} f)$, for each integer $m \geq 2$. By an easy computation, we have

$$\Delta^m f(z) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} f(z + (m - i)c).$$

Now we consider the difference polynomial

$$Q(z, f) = \sum_{\lambda \in I} a_\lambda(z) \prod_{\nu=1}^{m} (\Delta^\nu f)^{l_{\lambda,\nu}},$$

where the coefficients $a_\lambda(z)(\lambda \in I)$ are small functions with respect to f and $I = \{\lambda : \lambda = (l_{\lambda,1}, \ldots, l_{\lambda,m})|l_{\lambda,\nu} \in \mathbb{N}\}$ is a finite index set. We also denote $\ell_\nu = \max_{\lambda \in I} l_{\lambda,\nu}$ and $\ell = \sum_{\nu=1}^{m} \ell_\nu$.

THEOREM 2. *Let f be a transcendental meromorphic function with hyper-order $\varsigma(f) = \varsigma < 1$ and $s(z)$ be a small function with respect to f. Assume that $n \geq \sum_{\nu=1}^{m} \ell_\nu(\nu + 1) + m + 4$ and $Q(z, f) + s(z) \not\equiv 0$. Then the polynomial difference $f^n(z) - Q(z, f) - s(z)$ has infinitely many zeros.*

2. SOME LEMMAS

In order to prove the two theorems, we need the following lemmas, due to Halburd-Korhonen-Tohge [4].

LEMMA 2.1. *Let $T : [0, +\infty) \to [0, +\infty)$ be a non-decreasing continuous function, and let $s \in (0, +\infty)$. If the hyper-order of T is strictly less than one, i.e.,*

$$\limsup_{r\to\infty} \frac{\log \log T(r)}{\log r} = \varsigma < 1,$$

then

$$T(r + s) = T(r) + o(\frac{T(r)}{r^{1-\varsigma-\varepsilon}}),$$

where $\varepsilon > 0$ and $r \to \infty$ outside of a set of finite logarithmic measure.

Lemma 2.2. *Let f be a non-constant meromorphic function, $\varepsilon > 0$ and $c \in \mathbb{C}$.*

If f is of finite order, then there exists a set $E = E(f, \varepsilon)$ satisfying

$$\limsup_{r\to\infty} \frac{\int_{E\cap[1,r)} \mathrm{d}t/t}{\log r} \le \varepsilon$$

(i.e. of logarithmic density at most ε), such that

$$m\Big(r, \frac{f(z+c)}{f(z)}\Big) = O(\frac{\log r}{r} T(r,f))$$

for all r outside the set E.

If $\varsigma(f) = \varsigma < 1$ and $\varepsilon > 0$, then

$$m\Big(r, \frac{f(z+c)}{f(z)}\Big) = o(\frac{T(r,f)}{r^{1-\varsigma-\varepsilon}})$$

for all r outside of a set of finite logarithmic measure.

Lemma 2.3. *Let f be a non-constant meromorphic function with $\varsigma(f) = \varsigma < 1$, and $c \in \mathbb{C} \setminus \{0\}$. Then*

$$N(r, f(z+c)) \le N(r,f) + S(r,f),$$
$$T(r, f(z+c)) \le T(r,f) + S(r,f).$$

Proof. We have

$$\delta := \limsup_{r\to\infty} \frac{\log\log N(r,f)}{\log r} \le \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} = \varsigma(f) < 1.$$

Hence, by Lemma 2.1, we get

$$\begin{aligned} N(r+|c|, f) &= N(r,f) + o(\frac{N(r,f)}{r^{1-\delta-\varepsilon}}) \\ &\le N(r,f) + o(\frac{T(r,f)}{r^{1-\varsigma-\varepsilon}}) \\ &= N(r,f) + S(r,f). \end{aligned}$$

Therefore, by Lemma 2.2, we have

$$\begin{aligned} T(r, f(z+c)) &= m(r, f(z+c)) + N(r, f(z+c)) \\ &= m(r, \frac{f(z+c)}{f(z)}.f(z)) + N(r, f(z+c)) \\ &\le m(r, \frac{f(z+c)}{f(z)}) + m(r,f) + N(r+|c|, f) \\ &\le m(r, \frac{f(z+c)}{f(z)}) + m(r,f) + N(r,f) + S(r,f) \end{aligned}$$

$$= T(r, f) + S(r, f).$$

We have completed the proof of Lemma 2.3. □

From the basic properties of characteristic function, we get easily that:

LEMMA 2.4. *Let* $f_1, \dots, f_n$ *be distinct meromorphic functions. Then*

$$T(r, \sum_{\lambda \in I} a_\lambda f_1^{l_{\lambda,1}} \dots f_n^{l_{\lambda,n}}) \leq \sum_{j=1}^{n} \ell_j T(r, f_j) + \sum_{\lambda \in I} O(T(r, a_\lambda)),$$

where $I = \{\lambda : \lambda = (l_{\lambda,1}, \dots, l_{\lambda,n}) | \lambda_{l,j} \in \mathbb{N}, j = 1, \dots, n\}$ *is a finite index,* $\ell_j = \max_{\lambda \in I} l_{\lambda,j}$, $j = 1, \dots, n$ *and* $a_\lambda (\lambda \in I)$ *are meromorphic functions.*

3. PROOF OF THEOREMS

Proof of Theorem 1. Take $\phi(z) = f^n(z) - P(z, f) - s(z)$.
If $\phi(z) \equiv c$, where c is a constant, then $f^n(z) = P(z, f) + s(z) + c$. Then by Lemma 2.3 and Lemma 2.4, we have

$$nT(r, f) \leq \ell T(r, f) + S(r, f).$$

This is in contradiction with $n \geq \ell + m + 3$. Hence, $\phi(z)$ is nonconstant.

It is easy to see that

$$\frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)} \not\equiv 0.$$

Indeed, otherwise,

$$\frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)} \equiv 0,$$

then $\phi(z) = bf^n(z)$. This implies that $(b-1)f^n(z) = P(z, f) + s(z)$.
– If $b = 1$, then $P(z, f) + s(z) \equiv 0$. This is in contradiction with hypothesis.
– If $b \neq 1$, then $nT(r, f) \leq \ell T(r, f) + S(r, f)$. This is in contradiction with $n \geq \ell + m + 3$.

We have

$$-f^n(z) = \frac{\frac{\phi'(z)}{\phi(z)}[P(z, f) + s(z)] - [P(z, f) + s(z)]'}{\frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)}}.$$

Then

$$(3.1) \qquad nT(r, f) = T(r, \frac{\frac{\phi'(z)}{\phi(z)}[P(z, f) + s(z)] - [P(z, f) + s(z)]'}{\frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)}})$$

$$\leq m(r, P(z,f)+s(z))$$
$$+ N(r, \frac{\phi'(z)}{\phi(z)}[P(z,f)+s(z)] - [P(z,f)+s(z)]')$$
$$+ m(r, \frac{\phi'}{\phi} - \frac{(P(z,f)+s(z))'}{P(z,f)+s(z)}) + m(r, \frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)})$$
$$+ N(r, \frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)}) + S(r,f).$$

Set $\varphi_1 = \dfrac{\phi'(z)}{\phi(z)}[P(z,f)+s(z)] - [P(z,f)+s(z)]'$.
It is clear that

$$N(r,\varphi_1) \leq \overline{N}(r, \frac{1}{\phi}) + \overline{N}(r,f) + N(r, P(z,f)+s(z)) + S(r,f). \tag{3.2}$$

(note that $Pole(\phi) \subset Pole(f)$).

Next, we prove that:

$$N(r, \frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)}) \leq \overline{N}(r, \frac{1}{\phi}) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, P(z,f)+s(z)) + S(r,f). \tag{3.3}$$

It is clear that all pole of $\dfrac{(f^n)'}{f^n} - \dfrac{\phi'}{\phi}$ are simple and $Pole(\dfrac{(f^n)'}{f^n} - \dfrac{\phi'}{\phi}) \subset Zero(f) \cup Pole(f) \cup Zero(\phi)$. From this fact, for (3.3), it suffices to prove that: If z_0 is a pole of f but it is not a pole of $P(z,f)+s(z)$, then $\dfrac{(f^n)'}{f^n} - \dfrac{\phi'}{\phi}$ is holomorphic at z_0.

Indeed, we write $f(z) = \dfrac{\varphi(z)}{(z-z_0)^k}$ in a neighborhood of z_0, where $\varphi(z)$ is a holomorphic function and $\varphi(z_0) \neq 0$ and $k \geq 1$.
We get easily that

$$\frac{f^n(z)'}{f^n(z)} = \frac{(z-z_0)(\varphi^n)' - nk\varphi^n}{(z-z_0)\varphi^n}$$

and

$$\phi = \frac{\varphi^n - (P(z,f)+s(z))(z-z_0)^{nk}}{(z-z_0)^{nk}}.$$

Hence

$$\frac{\phi'}{\phi} = \frac{(z-z_0)[\varphi^n - g(z)(z-z_0)^{nk}]' - nk[\varphi^n - g(z)(z-z_0)^{nk}]}{(z-z_0)[\varphi^n - g(z)(z-z_0)^{nk}]}.$$

This implies that

$$\frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)} = \frac{(z-z_0)(\varphi^n)' - nk\varphi^n}{(z-z_0)\varphi^n}$$

$$- \frac{(z-z_0)[\varphi^n - g(z)(z-z_0)^{nk}]' - nk[\varphi^n - g(z)(z-z_0)^{nk}]}{(z-z_0)[\varphi^n - g(z)(z-z_0)^{nk}]}$$
$$= \frac{1}{z-z_0} \cdot \frac{\theta(z)}{\varphi^n(\varphi^n - g(z)(z-z_0)^{nk})},$$

where

$$\theta(z) = [(z-z_0)(\varphi^n)' - nk\varphi^n][\varphi^n - g(z)(z-z_0)^{nk}]$$
$$- [(z-z_0)[\varphi^n - g(z)(z-z_0)^{nk}]' - nk[\varphi^n - g(z)(z-z_0)^{nk}]]\varphi^n.$$

On the other hand,

$$\frac{\theta(z)}{\varphi^n(\varphi^n - g(z)(z-z_0)^{nk})}\Big|_{z=z_0} = \frac{-nk\varphi^{2n}(z_0) + nk\varphi^{2n}(z_0)}{\varphi^{2n}(z_0)} = 0.$$

Therefore, z_0 is not a pole of $\frac{(f^n(z))'}{f^n(z)} - \frac{\phi'(z)}{\phi(z)}$.

Combining (3.1), (3.2) and (3.3), we have

$$\begin{aligned} nT(r,f) \leq & m(r, P(z,f)+s(z)) + N(r, P(z,f)+s(z)) \\ & + 2\overline{N}(r, \frac{1}{\phi}) + \overline{N}(r,f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, P(z,f)+s(z)) + S(r,f) \\ & = T(r, P(z,f)) + 2\overline{N}(r, \frac{1}{\phi}) + \overline{N}(r,f) + \overline{N}(r, \frac{1}{f}) \\ & \quad + \overline{N}(r, P(z,f)+s(z)) + S(r,f). \end{aligned} \tag{3.4}$$

From the definition of $P(z,f)$ and Lemma 2.3, we have

$$\begin{aligned} \overline{N}(r, P(z,f)+s(z)) & \leq \sum_{\nu=1}^{m} \overline{N}(r, f(z+c_\nu)) \\ & \leq \sum_{\nu=1}^{m} T(r+, f(z+c_\nu)) + S(r,f) \\ & \leq mT(r,f) + S(r,f). \end{aligned} \tag{3.5}$$

By Lemma 2.3 and Lemma 2.4, we have

$$T(r, P(z,f)) \leq \ell T(r,f) + S(r,f). \tag{3.6}$$

From (3.4), (3.5), and (3.6), we have

$$nT(r,f) \leq (\ell + m + 2)T(r,f) + 2\overline{N}(r, \frac{1}{\phi}) + S(r,f).$$

On the other hand $n \geq \ell + m + 3$. Hence, ϕ has infinitely many zeros. This completes the proof of Theorem 1. □

Proof of Theorem 2. By an argument similar to the proof of Theorem 1, we obtain that

$$\begin{aligned}
nT(r,f) \leq & m(r,Q(z,f)+s(z)) + N(r,Q(z,f)+s(z)) \\
& + 2\overline{N}(r,\frac{1}{\phi}) + \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + \overline{N}(r,Q(z,f)+s(z)) + S(r,f) \\
& = T(r,Q(z,f)) + 2\overline{N}(r,\frac{1}{\phi}) + \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) \\
& \quad + \overline{N}(r,Q(z,f)+s(z)) + S(r,f). \qquad (3.7)
\end{aligned}$$

Note that

$$\Delta^{\nu} f(z) = \sum_{i=0}^{\nu} (-1)^i C_{\nu}^i f(z+(\nu-i)c),$$

for all $\nu = 0, \ldots, m$.
By Lemma 2.3, we have

$$\begin{aligned}
T(r,\Delta^{\nu} f(z)) &= T(r,\sum_{i=0}^{\nu}(-1)^i C_{\nu}^i f(z+(\nu-i)c)) \\
&= m(r,\sum_{i=0}^{\nu}(-1)^i C_{\nu}^i f(z+(\nu-i)c)) \\
&\quad + N(r,\sum_{i=0}^{\nu}(-1)^i C_{\nu}^i f(z+(\nu-i)c)) \\
&= m(r,\frac{\sum_{i=0}^{\nu}(-1)^i C_{\nu}^i f(z+(\nu-i)c)}{f}.f) \\
&\quad + N(r,\sum_{i=0}^{\nu}(-1)^i C_{\nu}^i f(z+(\nu-i)c)) \\
&\leq \sum_{i=0}^{\nu} m(r,\frac{f(z+(\nu-i)c)}{f}) + m(r,f) \\
&\quad + N(r,\sum_{i=0}^{\nu}(-1)^i C_{\nu}^i f(z+(\nu-i)c)) \\
&\leq (\nu+1)T(r,f) + S(r,f). \qquad (3.8)
\end{aligned}$$

From the definition of $Q(z,f)$ and by Lemma 2.3, we have

$$\begin{aligned}
\overline{N}(r,Q(z,f)+s(z)) &\leq \sum_{i=0}^{m} \overline{N}(r,f(z+(m-i)c)) \\
&\leq (m+1)T(r,f) + S(r,f). \qquad (3.9)
\end{aligned}$$

By (3.8) and Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned} T(r, Q(z,f)) &\leq \sum_{\nu=1}^{m} \ell_\nu T(r, \Delta^\nu f) + S(r,f) \\ &\leq \sum_{\nu=1}^{m} \ell_\nu (\nu+1) T(r,f) + S(r,f). \end{aligned} \tag{3.10}$$

By (3.7) and (3.10), we have

$$nT(r,f) \leq [\sum_{\nu=1}^{m} \ell_\nu(\nu+1) + m + 3]T(r,f) + 2\overline{N}(r, \frac{1}{\phi}) + S(r,f).$$

On the other hand, $n \geq \sum_{\nu=1}^{m} \ell_\nu(\nu+1) + m + 4$. Hence, $f^n - Q(z,f) - s(z)$ has infinitely zeros. We have completed the proof of Theorem 2. □

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