ON BIPARTITE CAYLEY GRAPHS OF SMALL DIAMETER

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The degree-diameter problem is the problem of finding the largest number of vertices in graphs of given maximum degree and diameter. We use semidirect products of groups to study bipartite Cayley graphs of small diameter. A bipartite graph is a graph whose vertices can be divided into two disjoint sets such that no two vertices within the same set are adjacent. A Cayley graph $C(G, X)$ is specified by a group G and an identity-free generating set X for this group such that $X = X^{-1}$. The vertices of $C(G, X)$ are the elements of G and there is an edge between two vertices u and v if and only if there is a generator $a \in X$ such that $v = ua$. We present the largest known constructions of bipartite Cayley graphs of diameters 4, 5 and 6 for an infinite set of degrees.

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1. INTRODUCTION

We consider undirected graphs without loops and multiple edges. The diameter of a graph is the greatest of the distances between all pairs of vertices in a graph. The degree of a vertex v is the number of edges incident to v. The degree-diameter problem is to determine or bound the largest possible number of vertices (the largest order) in graphs of given maximum degree and diameter.

Suppose that one wants to set up a network in which each node has just a limited number of direct connections to other nodes, and one requires that any two nodes can communicate by a route of limited length. What is the maximum number of nodes one can have under the two constraints? It is clear that this question can be translated into the language of graph theory. The problem is to find the largest possible number of vertices in a graph of given maximum degree and diameter. Vertices of a graph represent nodes of a network, while edges represent connections.

Various modifications and a number of subproblems of the main problem have been studied for decades. Hypergraph version of the degree-diameter problem was studied in [4], directed Cayley graphs were considered in [8] and Cayley graphs of diameter 2 were investigated by Abas [1, 2] and Ždímalová [7]. Cayley graphs are useful, because constructions of Cayley graphs yield symmetric networks and it is easier to set up such a network and check its properties.

We study the problem for bipartite Cayley graphs. A bipartite graph is a graph whose vertices can be divided into two disjoint sets such that no two vertices within the same set are adjacent. A Cayley graph $C(G, X)$ is given by a group G and an identity-free generating set X for this group, where $X = X^{-1}$. The vertices of $C(G, X)$ are the elements of G and there is an edge between two vertices u and v in $C(G, X)$ if and only if there is a generator $a \in X$ such that $v = ua$.

Let $BC_{d,k}$ be the largest number of vertices in a bipartite Cayley graph of degree d and diameter k. Biggs [3] showed that the number of vertices in a bipartite graph of degree $d \geq 3$ and diameter k cannot exceed the bipartite Moore bound $\frac{2(d-1)^k-2}{d-2}$, hence we have $BC_{d,k} \leq \frac{2(d-1)^k-2}{d-2}$. Improvements of this bound were given by Pineda-Villavicencio [5], who showed that $BC_{d,k} \leq \frac{2(d-1)^k-2}{d-2} - 4$ for any $d \geq 3$ and $k \geq 5$ with $k \neq 6$. For bipartite Cayley graphs of degree 2 or diameter 2 we have exact values of $BC_{d,k}$. For any $d, k \geq 2$ we have $BC_{d,2} = 2d$ and $BC_{2,k} = 2k$.

The largest known bipartite Cayley graphs for large degree d and diameter k were given in [6], where it is proved that for any $d \geq 6$ we have $BC_{d,k} \geq 2(k-1)(\frac{d-4}{3})^{k-1}$ if $k \geq 4$ is even, and $BC_{d,k} \geq (k-1)(\frac{d-2}{3})^{k-1}$ if $k \geq 7$ such that $k \equiv 3 \pmod 4$. In this paper we improve the bound $2(k-1)(\frac{d-4}{3})^{k-1}$ for $k = 4$ and 6, and we also present a construction of bipartite Cayley graphs of diameter 5 and degree $d \geq 8$, where d is a multiple of 4.

2. RESULTS

We construct the largest known bipartite Cayley graphs of diameter 4, 5 and 6. Let H be a group of order $m \geq 2$ with identity element e. Let H^{k-1} be the product $H \times H \times \cdots \times H$, where H appears $k-1$ times, and let α be the automorphism of H^{k-1} such that

$$\alpha(x_1, x_2, \ldots, x_{k-1}) = (x_{k-1}, x_1, x_2, \ldots, x_{k-2}).$$

We denote the cyclic group of order p by Z_p. We study the semidirect products $G = H^{k-1} \rtimes Z_p$, where p is a multiple of $k-1$, with multiplication given by

$$(x, y)(x', y') = (x\alpha^y(x'), y + y'), \tag{1}$$

where α^y is the composition of α with itself y times, $x, x' \in H^{k-1}$ and $y, y' \in Z_p$. Elements of G will be written in the form $(x_1, x_2, \ldots, x_{k-1}; y)$, where $x_1, x_2, \ldots, x_{k-1} \in H$ and $y \in Z_p$. Semidirect products of this type were used in [6].

We consider generating sets X, which consist of classes of elements of the form $(x_1, x_2, \ldots, x_{k-1}; y)$, where x_i, $1 \leq i \leq k-1$, is either e or g for any $g \in H$. In [6] bipartite Cayley graphs were found by use of $G = H^{k-1} \rtimes Z_p$, where $p = k-1$ or $2(k-1)$, and generators with at most one non-identity entry among the first $k-1$ coordinates. In this paper, we also use generators with two non-identity entries among the first $k-1$ coordinates; increasing this number did not yield better graphs. Note that in order to construct large bipartite Cayley graphs, our limitation is that we can use only the groups $H^{k-1} \rtimes Z_p$, where p is even, and generators which have an odd last coordinate.

Let us present our results.

THEOREM 2.1. *Let $d \geq 8$ be a multiple of* 4. *Then* $BC_{d,4} \geq \frac{3d^3}{8}$.

Proof. We use the group G with multiplication (1) defined above. Let $G = H^3 \rtimes Z_{24}$, $a_g = (g, e, e; 1)$, $\bar{a}_{g'} = (e, e, g'; -1)$, $b_h = (h, e, e; 5)$ and $\bar{b}_{h'} = (e, h', e; -5)$. Let

$$X = \{a_g, \bar{a}_{g'}, b_h, \bar{b}_{h'} \mid \text{for any } g, g', h, h' \in H\}$$

be the generating set for G. We have $a_g^{-1} = \bar{a}_{g^{-1}}$ and $b_h^{-1} = \bar{b}_{h^{-1}}$, hence $X = X^{-1}$. The Cayley graph $C(G, X)$ is of degree $d = |X| = 4m$ where $m \geq 2$, and order $|G| = 24m^3 = 24(\frac{d}{4})^3 = \frac{3d^3}{8}$.

We show that the diameter of $C(G, X)$ is at most 4, which is equivalent to showing that each element of G can be expressed as a product of at most 4 elements of X. For any $x_1, x_2, x_3 \in H$ we have

$$\begin{aligned}
(x_1, x_2, x_3; 1) &= b_{x_1} a_{x_3} \bar{b}_{x_2} = (x_1, e, e; 5)(x_3, e, e; 1)(e, x_2, e; -5), \\
(x_1, x_2, x_3; 3) &= a_{x_1} a_{x_2} a_{x_3} = (x_1, e, e; 1)(x_2, e, e; 1)(x_3, e, e; 1), \\
(x_1, x_2, x_3; 5) &= a_{x_1} b_{x_2} \bar{a}_{x_3} = (x_1, e, e; 1)(x_2, e, e; 5)(e, e, x_3; -1), \\
(x_1, x_2, x_3; 7) &= a_{x_1} a_{x_2} b_{x_3} = (x_1, e, e; 1)(x_2, e, e; 1)(x_3, e, e; 5), \\
(x_1, x_2, x_3; 9) &= \bar{b}_{x_2} \bar{b}_{x_3} \bar{b}_{x_1} = (e, x_2, e; -5)(e, x_3, e; -5)(e, x_1, e; -5), \\
(x_1, x_2, x_3; 11) &= b_{x_1} b_{x_3} a_{x_2} = (x_1, e, e; 5)(x_3, e, e; 5)(x_2, e, e; 1).
\end{aligned}$$

It is easy to see that if $(x_1, x_2, x_3; y) = abc$, where $a, b, c \in X$, then

$$(x^{-1}_{y \pmod 3)+1}, x^{-1}_{y+1 \pmod 3)+1}, x^{-1}_{y+2 \pmod 3)+1}, -y) = c^{-1} b^{-1} a^{-1},$$

hence we can obtain the elements of G with the last coordinate y, where y is odd and $-11 \leq y \leq -1$, as a product of at most 3 elements of X too. Any element $(x_1, x_2, x_3; y+1)$, where y is odd, can be expressed as

$$(x_1, x_2, x_3; y+1) = (x_1, x_2, x_3; y)a_e,$$

which means that the diameter of $C(G, X)$ is at most 4. Note that if $x_i \neq e$, $i = 1, 2, 3$, then $(x_1, x_2, x_3; y)$ cannot be obtained as a product of less than 3 elements of X and $(x_1, x_2, x_3; y+1)$ cannot be obtained as a product of less than 4 elements of X. Thus the diameter of $C(G, X)$ is exactly 4.

It can be seen that the graph $C(G, X)$ is bipartite, because the last coordinate of any element in the generating set X is odd, which means that no two different vertices $(x_1, x_2, x_3; y)$ and $(x'_1, x'_2, x'_3; y')$ of $C(G, X)$ are adjacent if either both y, y' are even or both y, y' are odd. Hence we obtain the bound $BC_{d,4} \geq \frac{3d^3}{8}$. □

THEOREM 2.2. *Let $d \geq 8$ be a multiple of 4. Then $BC_{d,5} \geq \frac{d^4}{8}$.*

Proof. Let $G = H^4 \rtimes Z_{32}$ and $X = \{a_g, \bar{a}_{g'}, b_h, \bar{b}_{h'} \mid g, g', h, h' \in H\}$, where $a_g = (g, e, e, e; 1)$, $\bar{a}_{g'} = (e, e, e, g'; -1)$, $b_h = (e, e, h, h; 5)$ and $\bar{b}_{h'} = (e, h', h', e; -5)$. We have $a_g^{-1} = \bar{a}_{g^{-1}}$ and $b_h^{-1} = \bar{b}_{h^{-1}}$, therefore $X = X^{-1}$. The Cayley graph $C(G, X)$ is of degree $d = |X| = 4m$, $m \geq 2$, and order $|G| = 32m^4 = \frac{d^4}{8}$. Since the last coordinate of every element in X is odd, the graph $C(G, X)$ is bipartite.

We show that all elements $(x_1, x_2, x_3, x_4; y)$ of G, where y is even, can be expressed as a product of 4 elements of X. We have

$$(x_1, x_2, x_3, x_4; 0) = a_{x_1 x_3 x_4^{-1}} b_{x_4 x_3^{-1}} \bar{a}_{x_2} \bar{b}_{x_3},$$
$$(x_1, x_2, x_3, x_4; 2) = b_{x_4} a_{x_2 x_1^{-1}} a_{x_4^{-1} x_3} \bar{b}_{x_1},$$
$$(x_1, x_2, x_3, x_4; 4) = a_{x_1} a_{x_2} a_{x_3} a_{x_4},$$
$$(x_1, x_2, x_3, x_4; 6) = \bar{a}_{x_4} b_{x_3} a_{x_1} a_{x_3^{-1} x_2},$$
$$(x_1, x_2, x_3, x_4; 8) = b_{x_3} b_{x_3^{-1} x_4} \bar{a}_{x_2} \bar{a}_{x_4^{-1} x_3 x_1},$$
$$(x_1, x_2, x_3, x_4; 10) = a_{x_1 x_2^{-1} x_4^{-1}} b_{x_4} b_{x_2} \bar{a}_{x_3},$$
$$(x_1, x_2, x_3, x_4; 12) = a_{x_1 x_4^{-1}} b_{x_4} a_{x_3 x_2^{-1}} b_{x_2},$$
$$(x_1, x_2, x_3, x_4; 14) = \bar{a}_{x_4 x_1^{-1} x_3^{-1} x_2} b_{x_2} b_{x_2^{-1} x_3} b_{x_1},$$
$$(x_1, x_2, x_3, x_4; 16) = b_{x_3} b_{x_1 x_2^{-1}} b_{x_2} a_{x_2 x_1^{-1} x_3^{-1} x_4}.$$

Elements of G with the last coordinate y, where $y \in \{18, 20, \ldots, 30\}$, can be obtained by use of inverses of the above generators, and $(x_1, x_2, x_3, x_4; y+1) = (x_1, x_2, x_3, x_4; y)a_e$, therefore the diameter of $C(G, X)$ is at most 5. It is easy to check that elements of G with the last coordinate 13 cannot be obtained as a product of at most 4 elements of X, hence the diameter of $C(G, X)$ cannot be less than 5. It follows that $BC_{d,5} \geq \frac{d^4}{8}$. □

THEOREM 2.3. *Let $d \geq 8$ be a multiple of 4. Then $BC_{d,6} \geq \frac{25d^5}{512}$.*

Proof. Let $G = H^5 \rtimes Z_{50}$ and $X = \{a_g, \bar{a}_{g'}, b_h, \bar{b}_{h'} \mid \text{for any } g, g', h, h' \in H\}$ where $a_g = (g, e, e, e, e; 1)$, $\bar{a}_{g'} = (e, e, e, e, g'; -1)$, $b_h = (h, e, h, e, e; 9)$ and $\bar{b}_{h'} = (e, h', e; h', e; -9)$. The Cayley graph $C(G, X)$ is a bipartite graph of degree $d = 4m$ where $m \geq 2$, and order $|G| = 50m^5 = \frac{25d^5}{512}$.

Let us express any element $(x_1, x_2, x_3, x_4, x_5; y)$, where $y \in \{1, 3, \ldots, 25\}$, as a product of 5 elements of X. It can be checked that

$$(x_1, x_2, x_3, x_4, x_5; 1) = b_{x_3}\bar{a}_{x_4}\bar{b}_{x_2}a_{x_2^{-1}x_5}a_{x_3^{-1}x_1},$$
$$(x_1, x_2, x_3, x_4, x_5; 3) = a_{x_1}\bar{b}_{x_5x_2^{-1}}a_{x_2x_5^{-1}x_3}a_{x_4}b_{x_2},$$
$$(x_1, x_2, x_3, x_4, x_5; 5) = a_{x_1}a_{x_2}a_{x_3}a_{x_4}a_{x_5},$$
$$(x_1, x_2, x_3, x_4, x_5; 7) = a_{x_1}b_{x_2}\bar{a}_{x_5}\bar{a}_{x_2^{-1}x_4}\bar{a}_{x_3},$$
$$(x_1, x_2, x_3, x_4, x_5; 9) = \bar{b}_{x_4}a_{x_4^{-1}x_2}b_{x_5}\bar{a}_{x_1x_3^{-1}x_5}b_{x_5^{-1}x_3},$$
$$(x_1, x_2, x_3, x_4, x_5; 11) = \bar{b}_{x_4x_1^{-1}}a_{x_1x_4^{-1}x_2}a_{x_3x_5^{-1}}b_{x_1}b_{x_5},$$
$$(x_1, x_2, x_3, x_4, x_5; 13) = a_{x_1}a_{x_2x_5^{-1}}a_{x_3}a_{x_4}b_{x_5},$$
$$(x_1, x_2, x_3, x_4, x_5; 15) = \bar{b}_{x_4x_1^{-1}}\bar{b}_{x_3}\bar{b}_{x_1}\bar{b}_{x_1x_4^{-1}x_2}a_{x_2^{-1}x_4x_1^{-1}x_3^{-1}x_5},$$
$$(x_1, x_2, x_3, x_4, x_5; 17) = a_{x_1}b_{x_4}\bar{a}_{x_5x_2^{-1}x_4}b_{x_4{}^{-1}x_2}\bar{a}_{x_3},$$
$$(x_1, x_2, x_3, x_4, x_5; 19) = \bar{a}_{x_5x_2^{-1}}b_{x_2}b_{x_1}a_{x_3}a_{x_1^{-1}x_4},$$
$$(x_1, x_2, x_3, x_4, x_5; 21) = b_{x_3}b_{x_2}a_{x_4}a_{x_2^{-1}x_5}a_{x_3^{-1}x_1},$$
$$(x_1, x_2, x_3, x_4, x_5; 23) = \bar{a}_{x_5}\bar{b}_{x_3}\bar{b}_{x_4x_1^{-1}x_3}a_{x_3^{-1}x_1x_4^{-1}x_2}\bar{b}_{x_3^{-1}x_1},$$
$$(x_1, x_2, x_3, x_4, x_5; 25) = b_{x_3}b_{x_5}b_{x_4}\bar{a}_{x_5^{-1}x_2}\bar{a}_{x_4^{-1}x_3^{-1}x_1}.$$

We can express the elements of G which have the last coordinate $-y$ with the help of inverses of the generators which were used to express $(x_1, x_2, x_3, x_4, x_5; y)$. Since $(x_1, x_2, x_3, x_4, x_5; y+1) = (x_1, x_2, x_3, x_4, x_5; y)a_e$, the diameter of $C(G, X)$ is at most 6. Moreover, no element of G with the last coordinate 22 can be expressed as a product of at most 4 elements of X, and no element with an even last coordinate can be obtained as a product of 5 elements of X, therefore the diameter of $C(G, X)$ cannot be less than 6. Hence $BC_{d,6} \geq \frac{25d^5}{512}$. □

We can modify construction given in Theorem 2.3 to obtain bipartite Cayley graphs for every $d \geq 8$. Clearly the group $G = H^5 \rtimes Z_{50}$ contains an element, say v, such that the last coordinate of v is odd, where v is not an involution and $v \notin X$. On the other hand, G also contains the involution $u = (e, e, e, e, e; 25)$. Let us consider new generating sets for G; $X_1 = X \cup \{u\}$, $X_2 = X \cup \{v, v^{-1}\}$ and $X_3 = X \cup \{u, v, v^{-1}\}$. Then the Cayley graph $C(G, X_i)$ is a bipartite graph of diameter at most 6, degree $d = |X_i| = 4m + i$ where

$m \geq 2$, and order $|G| = 50m^5 = \frac{25(d-i)^5}{512}$ for $i = 1, 2, 3$. Hence we get the following corollary:

Corollary 2.4. *Let $d \geq 8$ be any integer. Then $BC_{d,6} \geq \frac{25(d-3)^5}{512}$.*

Finally, let us note that constructions presented in Theorems 2.1 and 2.2 cannot be modified to obtain Cayley graphs for every $d \geq 8$, because the groups $H^3 \rtimes Z_{24}$ and $H^4 \rtimes Z_{32}$ do not contain involutions with an odd last coordinate. All involutions of $H^{k-1} \rtimes Z_p$ have the last coordinate either 0 or $\frac{p}{2}$.

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