CLASSIFICATION OF PAIR OF NILPOTENT LIE ALGEBRAS BY THEIR SCHUR MULTIPLIERS

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Let L be a finite dimensional nilpotent Lie algebra and N, K be ideals of L such that $L = N \oplus K$, with $dimN = n$ and $dimK = m$. We denote $t = \frac{1}{2}n(n+2m-1) - dimM(L,N)$, where $M(L,N)$ is the Schur multiplier of a pair (L,N). In the present paper, we characterize the pair (L,N) for which $t = 0, 1, ..., 6$. Also we prove $dimM(L,N) \leq \frac{1}{2}(n-1)(n-2)+1+(n-1)m$, when N is a non-abelian ideal of L and classify the pair (L,N) for $s' = 0,1,2$, where $s' = \frac{1}{2}(n-1)(n-2)+1+(n-1)m - dimM(L,N)$..

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1. INTRODUCTION AND PRELIMINARIES

Let L be a Lie algebra over a field Ω with characteristic different from 2 and a free presentation $0 \to R \to F \to L \to 0$, where F is a free Lie algebra. Then the Schur multiplier of L, denoted by $M(L)$, is defined to be the factor Lie algebra $\frac{(R \cap F^2)}{[R,F]}$. Batten (1993) showed that if L is finite dimensional, then its Schur multiplier is isomorphic to $H^2(L,F)$, the second cohomology of L. The classification of finite dimensional nilpotent Lie algebras has interested the works of several authors both in topology and in algebra, as we can note from [1,6]. Moneyhum [7] obtained the following upper bound for Schur multiplier of Lie algebras.

Let L be a finite dimensional Lie algebra of dimension n, then

$$dimM(L) \leqslant \frac{1}{2}n(n-1).$$

The above upper bound implies that $dimM(L) = \frac{1}{2}n(n-1) - t(L)$, where $t(L)$ is non-negaive integer. The n-dimensional nilpotent Lie algebras were characterized for $t(L) = 0, 1, ..., 8$ as follows:

Theorem 1.1 ([3–5]). *Let L be a n-dimensional nilpotent Lie algebra. Then*

(a) $t(L) = 0$ *if and only if* L *be an abelian Lie algebra ;*
(b) $t(L) = 1$ *if and only if* $L \cong H(1)$*;*
(c) $t(L) = 2$ *if and only if* $L \cong H(1) \oplus A(1)$*;*
(d) $t(L) = 3$ *if and only if* $L \cong H(1) \oplus A(2)$*;*
(e) $t(L) = 4$ *if and only if* $L \cong H(1) \oplus A(3), L(3,4,1,4)$ *or* $L(4,5,2,4)$*;*
(f) $t(L) = 5$ *if and only if* $L \cong H(1) \oplus A(4)$ *or* $H(2)$*;*
(g) $t(L) = 6$ *if and only if* $L \cong L(3,4,1,4) \oplus A(1), L(4,5,1,6), H(2) \oplus A(1), H(1) \oplus A(5)$ *or* $L(4,5,2,4)$*;*
(h) $t(L) = 7$ *if and only if* $L \cong H(1) \oplus A(6), H(2) \oplus A(2), H(3), L(7,5,2,7), L(7,5,1,7), L'(7,5,1,7), L(7,6,2,7),$ *or* $L(7,6,2,7,\beta_1,\beta_2)$.
(i) $t(L) = 8$ *if and only if* $L \cong H(1) \oplus A(7), H(2) \oplus A(3), H(3) \oplus A(1), L(3,4,1,4) \oplus A(2), L(4,5,1,6) \oplus A(1)$ *or* $L(4,5,2,4) \oplus A(2)$.

Here $H(n)$ denotes the Heisenberg algebra of dimension $2n+1$, $A(n)$ is an n-dimensional abelian Lie algebra and $L(a,b,c,d)$ will denote the algebra discovered during $t(L) = a$ case in [5], where $b = \dim L$, $c = \dim Z(L)$ and $d = t(L)$, the Lie brackets of them are described at the end of next section.

Niroomand *et al.* [8] proved the better upper bound for non-abelian nilpotent Lie algebras as follows:

Theorem 1.2 ([8]). *Let L be a nilpotent Lie algebra of* $\dim(L) = n$ *and* $\dim(L^2) = m (m \geq 1)$*. Then*

$$\dim M(L) \leq \frac{1}{2}(n+m-2)(n-m-1)+1$$

Moreover, if $m = 1$, then the equality holds if and only if $L \cong H(1) \oplus A$, where A is an abelian Lie algebra of $\dim(A) = n-3$.

The above upper bound implies that

$$\dim M(L) = \frac{1}{2}(n-1)(n-2)+1-s(L)$$

where $s(L) \geq 0$. Niroomand *et al.* in [8, 9] classified the structure of L when $s(L) = 0, 1, 2$ as follows:

Theorem 1.3. *Let L be a non-abelian n-dimensional nilpotent Lie algebra. Then*

(i) $s(L) = 0$ *if and only if* $L \cong H(1) \oplus A(n-3)$*;*
(ii) $s(L) = 1$ *if and only if* $L \cong L(4,5,2,4)$*;*
(iii) $s(L) = 2$ *if and only if* $L \cong L(3,4,1,4), L(4,5,2,4) \oplus A(1), H(m) \oplus A(n-2m-1) (m \geq 2)$.

Let (L, N) be a pair of Lie algebras, where N is an ideal in L. In 2011 Saeedi *et al.* [10] defined the Schur multiplier of the pair (L, N) to be the abelian Lie algebra $M(L, N)$ appearing in the following natural exact sequence of Lie algebras

$$H_3(L) \to H_3(\frac{L}{N}) \to M(L, N) \to M(L)$$
$$\to M(\frac{L}{N}) \to \frac{L}{[L, N]} \to \frac{L}{L^2} \to \frac{L}{L^2 + N} \to 0$$

where $M(-)$ and $H_3(-)$ denote the Schur multiplier and the third homology of a Lie algebra, respectively. In this case, for each free presentation $0 \to R \to F \to L \to 0$ of L, $M(L, N)$ is isomorphic to the factor Lie algebra $\dfrac{R \cap [S, F]}{[R, F]}$, where S is an ideal of F such that $\dfrac{S}{R} \simeq N$. In particular, if $N = L$, then the Schur multiplier of (L, N) will be $M(L) = \dfrac{(R \cap F^2)}{[R, F]}$ (see [3, 11, 12]). In this paper, we use the above results and the following lemmas to classify the pair of nilpotent Lie algebras by their Schur multipliers.

Lemma 1.4 ([3]). *Let L be a finite algebra and N, K be ideals of L such that $L = N \oplus K$. Then*

$$\dim M(L, N) = \dim(N) + \dim(N^{ab} \otimes K^{ab}),$$

where $N^{ab} = \frac{N}{[N,N]}$ and $K^{ab} = \frac{K}{[K,K]}$.

Lemma 1.5 ([8]). *Let L be an n-dimensional Lie algebra and* $\dim L^2 = 1$. *Then for some $m \geq 1$*

$$L \cong H(m) \oplus A(n - 2m - 1).$$

We can use the Schur multiplier for classifying the pairs of nilpotent Lie algebras. In fact Saeedi *et al.* [10], obtained the following upper bound for Schur multiplier pair of nilpotent Lie algebras.

Theorem 1.6 ([10]). *Let (L, N) be a pair of finite dimensional nilpotent Lie algebras and K be the complement of N in L. Assume N and K are of dimension n and m respectively. Then*

$$\dim M(L, N) + \dim[L, N] \leq \frac{1}{2}n(n + 2m - 1).$$

The above result implies that there exists a non-negative integer $t(L, N)$ such that $\dim M(L, N) = \dfrac{1}{2}n(n + 2m - 1) - t(L, N)$. Arabyani *et al.* [2], charactrized all the pairs (L, N), with $t = t(L, N) = 0, 1, 2, 3, 4$ under some conditions. In the present paper, we classify all the pairs (L, N), when $L = N \oplus K$ and $t \leq 6$.

THEOREM 1.7. *Let L be a finite finite dimensional nilpotent Lie algebras and N, K be ideals of L such that $L = N \oplus K$. Let* $\dim N = n$, $\dim K = m$. *and* $\dim N^2 = d \geq 1$. *Then* $\dim M(L, N) \leq \frac{1}{2}(n+d-2)(n-d-1)+1+(n-d)m.$

Proof. We can obtain the result from Theorem 1.2 and Lemma 1.4. □

We can see

$$\frac{1}{2}(n+d-2)(n-d-1)+1+(n-d)m \leq \frac{1}{2}(n-1)(n-2)+1+(n-1)m.$$

This upper bound is better than the one obtained in Theorem 1.6. We take $s' = s(L, N) = \frac{1}{2}(n-1)(n-2)+1+(n-1)m - \dim M(L, N)$, and then characterize the pairs (L, N) for $s' = 0, 1, 2$.

2. MAIN RESULTS

In this section, first we characterize all finite dimensional pairs of nilpotent Lie algebras with $t = 0, 1, 2, ..., 6$ and then determine the structure of all finite dimensional pairs (L, N) with $s' = 0, 1, 2$.

THEOREM 2.1. *Let L be a finite dimensional nilpotent Lie algebra and N, K be ideals of L such that $L = N \oplus K$. Let* $\dim N = n$, $\dim K = m$ *and* $\dim M(L, N) = \frac{1}{2}n(n+2m-1) - t$, *where $t \geq 0$. Then:*

(a) *If N is an abelian Lie algebra, then $(L, N) \cong (A(n) \oplus K_r, A(n))$, where $r \geq 1$ and $t = nr$ and K_r is a m-dimensinal Lie algebra with* $\dim(K'_r) = r$.

(b) *If N is a non-abelian Lie algebra, then:*

(i) *$t = 1$ if and only if $(L, N) \cong (H(1), H(1))$.*

(ii) *$t = 2$ if and only if $(L, N) \cong (H(1) \oplus A(1), H(1))$ or $(H(1) \oplus A(1), H(1) \oplus A(1))$.*

(iii) *$t = 3$ if and only if (L, N) be isomorphic with one of the following pairs of Lie algebras:*
$(H(1) \oplus A(2), H(1) \oplus A(2)), (H(1) \oplus A(2), H(1))$ or $(H(1) \oplus A(2), H(1) \oplus A(1))$.

(iv) *$t = 4$ if and only if (L, N) be isomorphic with one of the following pairs of Lie algebras:*
$(H(1) \oplus A(3), H(1) \oplus A(2)), (H(1) \oplus A(3), H(1) \oplus A(1)), (H(1) \oplus A(3), H(1)),$ $(H(1) \oplus A(3), H(1) \oplus A(3)), (L(3,4,1,4), L(3,4,1,4))$ or $(L(4,5,2,4), L(4,5,2,4))$.

(v) $t = 5$ *if and only if* (L, N) *be isomorphic with one of the following pairs of Lie algebras:*
$(H(1) \oplus A(4), H(1) \oplus A(4)), (H(1) \oplus A(4), H(1) \oplus A(3)), (H(1) \oplus A(4), H(1) \oplus A(2)), (H(1) \oplus A(4), H(1)), (H(1) \oplus A(4), H(1) \oplus A(1))$ *or* $(H(2), H(2))$.

(vi) $t = 6$ *if and only if* (L, N) *be isomorphic with one of the following pairs of Lie algebras:*
$(H(1) \oplus A(5), H(1) \oplus A(5)), (H(1) \oplus A(5), H(1) \oplus A(4)), (H(1) \oplus A(5), H(1) \oplus A(3)), (H(1) \oplus A(5), H(1) \oplus A(2)), (H(1) \oplus A(5), H(1) \oplus A(1)), (H(2) \oplus A(1), H(2) \oplus A(1)), (H(1) \oplus A(5), H(1)), (H(2) \oplus A(1), H(2)), (L(4,5,2,4) \oplus A(1), L(4,5,2,4) \oplus A(1))$ *or* $(L(3,4,1,4) \oplus A(1), L(3,4,1,4) \oplus A(1))$.

Proof. Let $\dim M(N) = \frac{1}{2}n(n-1) - l$ where l is a non-negative integer. By using Lemma 1.4, we have $\frac{1}{2}n(n+2m-1) - t = \frac{1}{2}n(n-1) - l + \dim(N^{ab} \otimes K^{ab})$. Therefore

$$mn = (t - l) + \dim(N^{ab})\dim(K^{ab}). \tag{1}$$

Also $\dim(K^{ab}) \leq m$ and $\dim(N^{ab}) = n - \dim N'$, Hence

$$m.\dim(N') \leq t - l \tag{2}$$

This implies that $t \geq l$.

(a) Let N be an abelian Lie algebra, then $l = 0$ and by using (1), we have

$$mn = (t - l) + n(\dim(K) - \dim(K')) = (t - l) + n(m - \dim(K')).$$

So $n\dim(K') = t - l = t$. If N is abelian then $(L, N) \cong (A(n) \oplus K_r, A(n))$, where K_r is a m-dimension Lie algebra and $\dim(K_r') = r$, $t = nr$.

(b) Let N be a non-abelian Lie algebra, in this case for different values of t, the pairs (L, N) are determined.

(i) Now assume that $t = 1$, then since N is non-abelian hence $l \neq 0$, and by using (2), $l = 1$ and $m.\dim N' = 0$. But $\dim N' \neq 0$, so $m = 0$. By Theorem 1.1, $N \cong H(1)$ and hence $(L, N) \cong (H(1), H(1))$.

(ii) Assume that $t = 2$, then the different values of (l, m) that satisfy in (2) will be $(2, 0), (1, 1)$ and $(1, 0)$. If $(l, m) = (1, 0)$ then $(L, N) \cong (H(1), H(1))$, which is a contradiction in case $t = 1$. Suppose that $(l, m) = (1, 1)$ then using Theorem 1.1, $N \cong H(1)$ we have $(L, N) \cong (H(1) \oplus A(1), H(1))$. In case $(l, m) = (2, 0)$, $K = 0$ and using Theorem 1.1, $(L, N) \cong (H(1) \oplus A(1), H(1) \oplus A(1))$.

(iii) Assume that $t = 3$, then $l = 1, 2, 3$. If $l = 1$, then by using Theorem1.1,

$N \cong H(1)$. Hence $n = 3$, $\dim(N') = 1$. By (2), we have $m \leq 2$. If $m = 0$, then $K = 0$ so (1) implies that $t = l$ which is a contradiction. If $m = 1$ then $3 = 2 + (3-1)(1-0)$ which is a contradiction. In the case $m = 2$, we have $6 = 2 + (3-1)(2 - \dim(K'))$, hence $\dim(K') = 0$ and K is a 2-dimension abelian Lie algebra, so $(L, N) \cong (H(1) \oplus A(2), H(1))$. Suppose that $l = 2$, then similar to the previous case we obtain $N \cong H(1) \oplus A(1)$, $m = 1$ and hence $(L, N) \cong (H(1) \oplus A(2), H(1) \oplus A(1))$. Assume that $l = 3$, then $m = 0$ and $(L, N) \cong (H(1) \oplus A(2), H(1) \oplus A(2))$.

(iv) Assume that $t = 4$. If $l = 1$ then by Theorem 1.1 $N \cong H(1)$, similar to the previous case $m = 0$ and $(L, N) \cong (H(1), H(1))$, which is a contradiction in case $t = 1$. If $m = 1$ or 2, then we can use (1) and obtain a contradiction. If $m = 3$ then $\dim(K') = 0$, so K is an abelian Lie algebra and $(L, N) \cong (H(1) \oplus A(3), H(1))$. Assume that $l = 2$, then by Theorem 1.1, $N \cong H(1) \oplus A(1)$ and $\dim N' = 1$. By using (2), the only acceptable value $m = 2$ and $\dim K' = 0$, so $(L, N) \cong (H(1) \oplus A(3), H(1) \oplus A(1))$. If $l = 3$ then $(L, N) \cong (H(1) \oplus A(3), H(1) \oplus A(2))$. Suppose that $l = 4$, then by Theorem 1.1, $N \cong H(1) \oplus A(3), L(3, 4, 1, 4)$ or $L(4, 5, 2, 4)$, by using (2), $K = 0$ and $(L, N) \cong (L(4, 5, 2, 4), L(4, 5, 2, 4))$, $(H(1) \oplus A(3), H(1) \oplus A(3))$, or $(L(3, 4, 1, 4), L(3, 4, 1, 4))$.

(v) Assume that $t = 5$, then $l = 1, 2, 3, 4, 5$. If $l = 1$ then $N \cong H(1)$. In this case the only acceptable value $m = 4$, $\dim(K') = 0$, hence $(L, N) \cong (H(1) \oplus A(4), H(1))$. The similar argument implies that if $l = 2$ then $(L, N) \cong (H(1) \oplus A(4), H(1) \oplus A(1))$. If $l = 3$ then $(L, N) \cong (H(1) \oplus A(4), H(1) \oplus A(2))$. If $l = 4$ then $(L, N) \cong (H(1) \oplus A(4), H(1) \oplus A(3))$. If $l = 5$ then $N \cong H(1) \oplus A(4)$ or $H(2)$ and in both cases $m = 0$ and $(L, N) \cong (H(1) \oplus A(4), H(1) \oplus A(4))$ or $(H(2), H(2))$.

(vi) Assume that $t = 6$. If $l = 1$ and $m = 0, 1, 2, 4$, we obtain a contradiction. Suppose that $m = 3$ and $\dim(K') = 1$, then Lemma 1.5 implies that $K \cong H(1)$, hence $(L, N) \cong (H(1) \oplus H(1), H(1))$. If $m = 5$ then $\dim K' = 0$ and $(L, N) \cong (H(1) \oplus A(5), H(1) \oplus A(5))$. For different values m, l with similar calculation we can obtain $(L, N) \cong (H(1) \oplus A(5), H(1) \oplus A(1)), (H(1) \oplus A(5), H(1) \oplus A(2)), (H(2) \oplus A(1), H(2))$, $H(1) \oplus A(5), H(1) \oplus A(3)), (H(1) \oplus A(5), H(1) \oplus A(4))$, $(H(2) \oplus A(1), H(2) \oplus A(1))$, $(L(3, 4, 1, 4) \oplus A(1), L(3, 4, 1, 4) \oplus A(1))$, $L(4, 5, 2, 4) \oplus A(1), L(4, 5, 2, 4) \oplus A(1))$, $(H(1) \oplus A(5), H(1) \oplus A(5))$. □

THEOREM 2.2. *Let L be a finite dimensional nilpotent Lie algebra and N, K be ideals of L. Assume that* $\dim N = n$, $\dim K = m$, $\dim N^2 \geq 1$ *and* $\dim M(L, N) = \frac{1}{2}(n-1)(n-2) + 1 + (n-1)m - s'$. *Then*

(i) $s' = 0$ *if and only if* $(L, N) \cong (H(1) \oplus A(n + m - 3), H(1) \oplus A(n - 3))$.

(ii) $s' = 1$ *if and only if* $(L, N) \cong (L(4,5,2,4), L(4,5,2,4))$.

(iii) $s' = 2$ *if and only if* (L, N) *is isomorphic with one of the following pairs of Lie algebras:*
$(H(1) \oplus H(k) \oplus A(m-2k-1), H(1)), (L(4,5,2,4) \oplus A(1), L(4,5,2,4)),$
$(H(k) \oplus A(m+n-2k-1), H(k) \oplus A(n-2k-1)), (L(3,4,1,4), L(3,4,1,4))$
or $(L(4,5,2,4) \oplus A(1), (L(4,5,2,4) \oplus A(1))$.

Proof. Let $\dim M(N) = \frac{1}{2}(n-1)(n-2) + 1 - s(L)$, by Lemma 1.4 we have $\frac{1}{2}(n-1)(n-2) + 1 + (n-1)m - s' = \frac{1}{2}(n-1)(n-2) + 1 - s + \dim N^{ab}.K^{ab}$, hence

$$nm - m = (s' - s) + \dim N^{ab}.\dim K^{ab} \tag{3}$$

Since $\dim K^{ab} \leq m$ and $\dim N^{ab} = n - \dim N^2$, hence

$$nm - m \leq (s' - s) + (n - \dim N^2)m \Rightarrow m(-1 + \dim N^2) \leq s' - s$$

Since $\dim N^2 \geq 1$, therefore $s' \geq s$.
(i) Assume that $s' = 0$, then $s = 0$ and by Theorem 1.2, we have $N \cong H(1) \oplus A(n-3)$. Also by using (3), we have

$$nm - m = (n-1)(m - \dim K^2) \Rightarrow \dim K^2 = 0.$$

Hence K is a m-dimensional abelian Lie algebra and $(L, N) \cong (H(1) \oplus A(m + n - 3), H(1) \oplus A(n-3))$.
(ii) If $s' = 0$, then $s = 0$ or 1. Let $s = 0$, by Theorem 1.2, $N \cong H(1) \oplus A(n-3)$, hence by using 3, $n = 2$, which is a contradiction. Assume that $s = 1$, then by Theorem 1.2 and (3), $m = 0$ and $(L, N) \cong (L(4,5,2,4), L(4,5,2,4))$.
(iii) Assume that $s' = 2$, then $s = 0, 1$ or 2. Let $s = 1$, then $N \cong H(1) \oplus A(n-3)$ and $\dim N^2 = 1$, by using (3), we have $n = 3$ and $\dim K^2 = 1$. By Lemma 1.5, there exists $K \geq 1$ such that $K \cong H(K) \oplus A(m-2k-1)$ and so $(L, N) \cong (H(1) \oplus H(k) \oplus A(m-2k-1), H(1))$. If $s = 1$ then $N \cong L(4,5,2,4)$, by using (3), we have $m = 1$, therefore $(L, N) \cong (L(4,5,2,4) \oplus A(1), L(4,5,2,4))$. Assume that $s = 2$, then by Theorem 1.3, $N \cong L(3,4,1,4)$ or $L(4,5,2,4) \oplus A(1)$ and $\dim N^2 = 2$ or $H(k) \oplus A(n-2k-1)$ $(k \geq 2)$ and $\dim N^2 = 1$. If $\dim N^2 = 1$ then by using (3), $\dim K^2 = 0$ hence $(L, N) \cong (H(k) \oplus A(m + n - 2k - 1), H(k) \oplus A(n-2k-1))$. If $\dim N^2 = 2$ then $m = 0$, $(L, N) \cong (L(4,5,2,4) \oplus A(1), L(4,5,2,4) \oplus A(1))$ or $(L(3,4,1,4), L(3,4,1,4))$. $\square$

In the following table, we describe the nilpotent Lie algebras that we refer in Theorem 1.1 using [4, 5].

TABLE 1

t(L)	dim L	Non Zero Multiplication	Nilpotent Lie algebra
0			Abelian
1	3	$[x_1, x_2] = x_3$	$H(1)$
2	4	$[x_1, x_2] = x_3$	$H(1) \oplus A(1)$
3	5	$[x_1, x_2] = x_3$	$H(1) \oplus A(2)$
4	4	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_4$	$L(3, 4, 1, 4)$
4	5	$[x_1, x_2] = x_3,\ [x_1, x_4] = x_5$	$L(4, 5, 2, 4)$
4	6	$[x_1, x_2] = x_3$	$H(1) \oplus A(3)$
5	5	$[x_1, x_2] = x_5,\ [x_3, x_4] = x_5$	$H(2)$
5	7	$[x_1, x_2] = x_3$	$H(1) \oplus A(4)$
6	5	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_5$	$L(3, 4, 1, 4) \oplus A(1)$
6	5	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_5,\ [x_2, x_4] = x_5$	$L(4, 5, 1, 6)$
6	6	$[x_1, x_2] = x_5,\ [x_1, x_3] = x_5,\ [x_3, x_4] = x_5$	$H(2) \oplus A(1)$
6	6	$[x_1, x_2] = x_3,\ [x_1, x_4] = x_6$	$L(4, 5, 2, 4) \oplus A(1)$
6	8	$[x_1, x_2] = x_3$	$H(1) \oplus A(5)$
7	5	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_2, x_3] = x_5$	$L(7, 5, 2, 7)$
7	5	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_2, x_3] = x_5$	$L(7, 5, 1, 7)$
7	5	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_2, x_3] = x_5$	$L'(7, 5, 1, 7)$
7	6	$[x_1, x_2] = x_3,\ [x_1, x_4] = x_6,\ [x_2, x_5] = x_6$	$L(5, 6, 2, 7)$
7	6	$[x_1, x_2] = x_3,\ [x_4, x_5] = x_6$	$L'(5, 6, 2, 7)$
7	6	$[x_1, x_2] = x_5,\ [x_3, x_4] = x_6$	$L(7, 6, 2, 7)$
7	6	$[x_1, x_2] = x_5 + \beta_1 x_6,\ [x_3, x_4] = x_5$ $[x_1, x_4] = x_6,\ [x_3, x_2] = \beta_2 x_6$	$L(7, 6, 2, 7, \beta_1, \beta_2)$
7	7	$[x_1, x_2] = x_5,\ [x_3, x_4] = x_5$	$H(2) \oplus A(2)$
7	7	$[x_1, x_2] = x_7,\ [x_3, x_4] = x_7,\ [x_5, x_6] = x_7$	$H(3)$
8	6	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_6$	$L(3, 4, 1, 4) \oplus A(2)$
8	6	$[x_1, x_2] = x_3,\ [x_1, x_3] = x_6,\ [x_2, x_4] = x_6$	$L(4, 5, 1, 6) \oplus A(1)$
8	7	$[x_1, x_2] = x_3,\ [x_1, x_4] = x_7$	$L(4, 5, 2, 4) \oplus A(2)$
8	8	$[x_1, x_2] = x_5,\ [x_3, x_4] = x_5$	$H(2) \oplus A(3)$
8	8	$[x_1, x_2] = x_7,\ [x_3, x_4] = x_7,\ [x_5, x_6] = x_7$	$H(3) \oplus A(1)$
8	10	$[x_1, x_2] = x_3$	$H(1) \oplus A(7)$

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