COMPLETE HOMOMORPHISMS BETWEEN THE LATTICES OF RADICAL SUBMODULES

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Let R be a commutative ring and $\mathcal{R}(M)$ be the lattice of radical submodules of an R-module M. In this paper, we examine the properties of the mapping $\sigma : \mathcal{R}(M) \to \mathcal{R}(R)$ defined by $\sigma(N) = (N : M)$ and the mapping $\rho : \mathcal{R}(R) \to \mathcal{R}(M)$ defined by $\rho(I) = \mathrm{rad}(IM)$, in particular considering when these are complete homomorphisms of the lattices. It is shown that a finitely generated module M is a multiplication module if and only if σ is a lattice homomorphism if and only if σ is a complete lattice homomorphism. It is also proved that for modules over an Artinian ring, finitely generated faithful multiplication modules and projective modules, ρ is a complete lattice homomorphism.

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1. INTRODUCTION

A lattice L is called *complete* provided every non-empty subset S has a least upper bound $\vee S$ and a greatest lower bound $\wedge S$. Given complete lattices L and L' we say that a mapping $\varphi : L \to L'$ is a *complete homomorphism* provided

$$\varphi(\vee S) = \vee\{\varphi(x) : x \in S\} \text{ and } \varphi(\wedge S) = \wedge\{\varphi(x) : x \in S\},$$

for every non-empty subset S of L. A complete homomorphism which is a bijection (respectively, injection, surjection) will be called a *complete isomorphism* (respectively, *complete monomorphism*, *complete epimorphism*).

Throughout this paper all rings are commutative with identity and all modules are unital. Let R be a ring and M be any R-module. Let $\mathcal{L}({}_RM)$ (or simply $\mathcal{L}(M)$ if no ambiguity can arise) denote the complete lattice of all submodules of M with respect to the following definitions:

$$\wedge\mathcal{T} = \bigcap_{N\in\mathcal{T}} N \text{ and } \vee\mathcal{T} = \sum_{N\in\mathcal{T}} N$$

for every non-empty collection $\mathcal{T}$ of submodules of M. As in [17], we consider the mapping $\lambda : \mathcal{L}(_RR) \to \mathcal{L}(_RM)$ given by $\lambda(I) = IM$, and the mapping $\mu : \mathcal{L}(_RM) \to \mathcal{L}(_RR)$ given by $\mu(N) = (N : M)$, where $(N : M) = \{r \in R : rM \subseteq N\}$. It is easily seen that

$$\lambda(\vee\mathcal{S}) = \vee\{\lambda(I) : I \in \mathcal{S}\} \text{ and } \mu(\wedge\mathcal{T}) = \wedge\{\mu(N) : N \in \mathcal{T}\}.$$

for every non-empty collection $\mathcal{S}$ of ideals of R and every non-empty collection $\mathcal{T}$ of submodules of M. An R-module M is called a *λ-module* (resp. *μ-module*), if λ (resp. μ) is a lattice homomorphism [17]. Also, M is called *λ-complete* (resp. *μ-complete*) if the above mapping λ (resp. μ) is a complete homomorphism [18]. P. F. Smith investigated the mappings λ and μ in [17] and [18], and examined when these mappings are lattice or complete lattice homomorphisms. In [12], the authors have considered the lattices of radical ideals and radical submodules, and explore the properties of certain mappings between these lattices. They have also studied the relationships between these mappings and the mappings λ and μ. In this work, we turn our attention to the case in which these mappings are complete homomorphisms.

Firstly, we refresh our memory about prime and radical submodules, and fix some notation. Let R be a ring. A proper submodule N of an R-module M is said to be a *prime submodule* of M if whenever $rm \in N$ for $r \in R$ and $m \in M$, then either $r \in (N : M)$ or $m \in N$. This notion of prime submodule was first introduced and systematically studied in [3] and recently has received a good deal of attention from several authors; see for example [5,6,10,13]. Let N be a submodule of an R-module M. Then the *radical* of N is the intersection of all prime submodules of M containing N and denoted by $\operatorname{rad}_M N$ (or simply $\operatorname{rad} N$ if no ambiguity can arise). If there is no prime submodule containing N, we define $\operatorname{rad} N = M$; in particular, $\operatorname{rad} M = M$ [7]. A submodule N of M is called a *radical submodule* if $\operatorname{rad} N = N$. For an ideal I of a ring R, we assume throughout that $\sqrt{I}$ denotes the radical of I.

Let R be a ring and M be any R-module. The set of radical submodules of M forms a complete lattice with respect to the following definitions:

$$\wedge\mathcal{T} = \bigcap_{N\in\mathcal{T}} N \text{ and } \vee\mathcal{T} = \operatorname{rad}(\sum_{N\in\mathcal{T}} N)$$

for every non-empty collection $\mathcal{T}$ of radical submodules of M. We denote this complete lattice by $\mathcal{R}(_RM)$ (or simply $\mathcal{R}(M)$ if no ambiguity can arise), and in the case that $M = R$ by $\mathcal{R}(R)$. In general $\mathcal{R}(_RM)$ is not a (complete) sublattice of $\mathcal{L}(_RM)$ [12, p. 36]. In [12], we investigate the mapping $\rho : \mathcal{R}(R) \to \mathcal{R}(M)$ which is defined by $\rho(I) = \operatorname{rad}(\lambda(I)) = \operatorname{rad}(IM)$ for every ideal I of R and the mapping $\sigma : \mathcal{R}(M) \to \mathcal{R}(R)$ which is defined by $\sigma(N) = \mu(N) = (N : M)$ for every submodule N of M. Although ρ is a lattice homomorphism by [12, p. 37], it is not necessarily a complete lattice homomorphism as the following example shows.

Example 1.1. Let $R = \mathbb{Z}$ be the ring of integers, and $M = \mathbb{Q}$ be the field of rational numbers. Let $\mathcal{S} = \{p\mathbb{Z} : p \in \Omega\}$ where Ω is the set of all prime integers p. Then $\rho(\wedge\mathcal{S}) = \rho(\bigcap_{p\in\Omega} p\mathbb{Z}) = \rho(0) = \mathrm{rad}(0\mathbb{Q}) = 0$ and $\wedge\rho(\mathcal{S}) = \wedge\{\rho(p\mathbb{Z}) : p \in \Omega\} = \bigcap_{p\in\Omega} \mathrm{rad}(p\mathbb{Q}) = \mathrm{rad}(\mathbb{Q}) = \mathbb{Q}$. Thus $\mathbb{Q}$ is not ρ-complete.

For a ring R, an R-module M will be called *ρ-complete* in case the above mapping $\rho : \mathcal{R}(R) \to \mathcal{R}(M)$ is a complete homomorphism. Unlike ρ, the mapping σ need not be a lattice homomorphism and thus σ is not necessarily a complete lattice homomorphism. For example, let V be a two-dimensional vector space over a field F with the basis $\{e_1, e_2\}$. Then $\sigma(Fe_1) \vee \sigma(Fe_2) = \sqrt{(Fe_1 : V) + (Fe_2 : V)} = 0$ while $\sigma(Fe_1 \vee Fe_2) = (\mathrm{rad}(Fe_1 + Fe_2) : V) = F$. The module M is called *σ-module* in case the mapping $\sigma : \mathcal{R}(M) \to \mathcal{R}(R)$ is a homomorphism [12], and *σ-complete* if $\sigma : \mathcal{R}(M) \to \mathcal{R}(R)$ is a complete homomorphism [18].

In what follows, we continue the investigation in [12] of the mappings $\rho : \mathcal{R}(R) \to \mathcal{R}(M)$ and $\sigma : \mathcal{R}(M) \to \mathcal{R}(R)$ and we will be mostly interested in conditions under which these mappings are complete homomorphisms.

In Section 2, we investigate σ-complete modules. We show that if M is a finitely generated R-module, then M is σ-complete if and only if M is μ-complete if and only if M is a multiplication module (*i.e.*, λ is a surjective map) (Theorem 2.3).

In Section 3, we explore some properties of ρ-complete modules, and provide a considerable amount of examples of ρ-complete modules. For example, we show that projective modules and finitely generated faithful multiplication modules are ρ-complete (Theorem 3.4 and Theorem 3.7).

Section 4 is devoted to studying the relationships between ρ-complete and λ-complete modules. We show that every module over an Artinian ring is ρ-complete (Theorem 4.5). Moreover, if R is a Noetherian ring and every R-module is ρ-complete, then R is Artinian (Theorem 4.10). Using this fact, we conclude that if R is a Noetherian ring, then every R-module is ρ-complete if and only if every R-module is λ-complete (Corollary 4.11).

2. σ-COMPLETE MODULES

In this section, we shall investigate σ-complete modules. We first note the following simple fact.

Lemma 2.1. *Let R be a ring and M be an R-module. Then M is a σ-complete module if and only if* $(\mathrm{rad}(\sum_{N\in\mathcal{T}} N) : M) = \sqrt{\sum_{N\in\mathcal{T}}(N : M)}$ *for every non-empty collection $\mathcal{T}$ of radical submodules of M.*

Proof. Let $\mathcal{T}$ be a non-empty subset of radical submodules of M. Then $\sigma(\wedge\mathcal{T}) = \sigma(\bigcap_{N\in\mathcal{T}} N) = (\bigcap_{N\in\mathcal{T}} N : M) = \bigcap_{N\in\mathcal{T}}(N : M) = \wedge\sigma(\mathcal{T})$. Now, from

$$\sigma(\vee\mathcal{T}) = \sigma(\mathrm{rad}(\sum_{N\in\mathcal{T}} N)) = (\mathrm{rad}(\sum_{N\in\mathcal{T}} N) : M)$$

and

$$\vee\sigma(\mathcal{T}) = \bigvee_{N\in\mathcal{T}}\{\sigma(N) : N \in \mathcal{S}\} = \sqrt{\sum_{N\in\mathcal{T}}(N : M)}$$

the result follows. □

As it is illustrated in [18, p. 18], a μ-module is not necessarily μ-complete. Now, we show that these are equivalent for finitely generated modules.

Lemma 2.2. *Let R be a ring and M be a finitely generated R-module. Then M is a μ-module if and only if it is a μ-complete module.*

Proof. ($\Leftarrow$) Obviously every μ-complete module is μ-module. ($\Rightarrow$) Let $M = Rm_1 + \cdots + Rm_l$ be a μ-module and $\mathcal{T}$ be a collection of submodules of M. Clearly $\sum_{N\in\mathcal{T}}(N : M) \subseteq (\sum_{N\in\mathcal{T}} N : M)$. Now, let $r \in (\sum_{N\in\mathcal{T}} N : M)$. Then $rm_i = n_{i_1} + \cdots + n_{i_{k_i}}$ $(i = 1, \cdots, l)$. Thus $rM \subseteq N_1 + \cdots + N_t$ for some submodules $N_1, \ldots, N_t \in \mathcal{T}$, that is $r \in (N_1 + \cdots + N_t : M) = (N_1 : M) + \cdots + (N_t : M) \subseteq \sum_{N\in\mathcal{T}}(N : M)$. Therefore $\sum_{N\in\mathcal{T}}(N : M) = (\sum_{N\in\mathcal{T}} N : M)$ and hence by [17, Lemma 2.1], M is μ-complete. □

Let M be a μ-complete R-module. Then, since $R = (M : M) = (\sum_{m\in M} Rm : M) = \sum_{m\in M}(Rm : M)$, by [16, Corollary 1 to Theorem 2], M is a multiplication R-module. In particular, by [18, Theorem 2.2] or [16, by Corollary 2(ii) to Theorem 2], M is a μ-complete R-module if and only if M is a finitely generated multiplication R-module. This result can be extended to the context of σ-complete modules as follows.

Theorem 2.3. *Let R be a ring and M be a finitely generated R-module. Then the following are equivalent:*

(1) *M is μ-module.*
(2) *M is σ-module.*
(3) *M is μ-complete module.*
(4) *M is σ-complete module.*

(5) *M is a multiplication module.*
Moreover, in this case σ and μ are monomorphisms.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (5) by [12, Theorem 2.11]. (1) $\Leftrightarrow$ (3) by Lemma 2.2. (3) $\Rightarrow$ (4) Let $\mathcal{T}$ be a non-empty collection of radical submodules of M. By Lemma 2.1, it suffices to show that $(\mathrm{rad}(\sum_{N\in\mathcal{T}} N) : M) = \sqrt{\sum_{N\in\mathcal{T}}(N : M)}$. Since M is finitely generated and μ-complete, we have $\sqrt{\sum_{N\in\mathcal{T}}(N:M)} = \sqrt{(\sum_{N\in\mathcal{T}} N:M)}$ $= (\mathrm{rad}(\sum_{N\in\mathcal{T}} N) : M)$. (4) $\Rightarrow$ (2) Clear.

The "Moreover" part is clear by [18, Theorem 2.2]. $\square$

Remark 2.4. By [18, Theorem 2.2], every μ-complete module is finitely generated. Thus (3) in Theorem 2.3 implies the other statements. In particular, it implies that M is σ-complete. However, if M is not finitely generated, the converse need not be true. For example, if $R = \mathbb{Z}$, then the prüfer p-group $M = \mathbb{Z}(p^\infty)$ is a σ-complete R-module, since M has no prime submodule. But M is not μ-complete, by [18, p. 18].

Corollary 2.5. *Let R be a ring and M be a finitely generated R-module. Then the following statements are equivalent:*

(1) $(\sum_{N\in\mathcal{T}} N : M) = \sum_{N\in\mathcal{T}}(N : M)$ *for every non-empty collection $\mathcal{T}$ of submodules of M.*

(2) $(\mathrm{rad}(\sum_{N\in\mathcal{T}} N) : M) = \sqrt{\sum_{N\in\mathcal{T}}(N : M)}$ *for evey non-empty collection $\mathcal{T}$ of radical submodules of M.*

Proof. By Theorem 2.3, (3) $\Leftrightarrow$ (4). $\square$

It is worthwhile mentioning that Theorem 2.3 presents the following results for σ-complete modules analogous to the case for σ-modules found in [12].

Let R be a ring, M an R-module and S a multiplicatively closed subset of R. In the following, R_S is the ring of fractions of R and M_S the R_S-module of fractions of M with respect to S. Also, if P is a prime ideal of R, we put $S = R \setminus P$ and write R_P and M_P instead of R_S and M_S, respectively.

Corollary 2.6. *Let R be a ring and M be a finitely generated μ-module (σ-module) over R. Also, let S be a multiplicatively closed subset of R. Then M_S is a μ-complete module (σ-complete module) over R_S.*

Proof. By Theorem 2.3 and [12, Lemma 2.18]. $\square$

Corollary 2.7. *Let R be a ring and M be a finitely generated R-module. Then the following are equivalent.*

(1) *M is a σ-complete R-module;*
(2) *M_P is a σ-complete R_P-module for all prime ideals P of R;*
(3) *$M_{\mathfrak{m}}$ is a σ-complete $R_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$ of R.*

Proof. By Theorem 2.3 and [12, Theorem 2.19]. □

PROPOSITION 2.8. *Every homomorphic image of a σ-complete module is a σ-complete module.*

Proof. Similar to the proof of [12, Proposition 2.20]. □

COROLLARY 2.9. *Let R be a ring. Then every cyclic R-module M is σ-complete. The converse is true when M is finitely generated and R is local.*

Proof. Since R is a σ-complete module over R, it is clear that every cyclic R-module is a σ-complete module by Proposition 2.8. The second part is obtained by Theorem 2.3 and [12, Corollary 2.21]. □

Although every cyclic R-module is σ-complete, there is no ring for which every finitely generated module is σ-complete. For example, by [12, Corollary 2.13], the R-module $R \oplus R$ is not a σ-module and so is not σ-complete.

3. ρ-COMPLETE MODULES

Given a ring R, in this section we shall consider the mapping ρ from the complete lattice $\mathcal{R}(R)$ of radical ideals of R to the complete lattice $\mathcal{R}(M)$ of radical submodules of M defined by $\rho(I) = \mathrm{rad}(IM)$. Recall that a module M is a ρ-complete in case the mapping ρ is a complete homomorphism. In [12, p. 37], it is shown that every R-module is a ρ-module, *i.e.*, ρ is a lattice homomorphism. However, as Example 1.1 shows, there is a module which is not ρ-complete. First we begin with some properties of radical of submodules which are used in the rest of paper.

LEMMA 3.1 (See [7], Proposition 2 and [8], Corollary 1 to Proposition 1). *Let R be a ring and M an R-module. Let N and L be submodules of M and $\mathcal{T}$ be any collection of submodules of M. Then*
(1) $N \subseteq \mathrm{rad}\, N$;
(2) $\mathrm{rad}(\mathrm{rad}\, N) = \mathrm{rad}\, N$;
(3) $\mathrm{rad}(N \cap L) \subseteq \mathrm{rad}\, N \cap \mathrm{rad}\, L$;
(4) $\mathrm{rad}(N + L) = \mathrm{rad}(\mathrm{rad}\, N + \mathrm{rad}\, L)$;
(5) $\mathrm{rad}(IM) = \mathrm{rad}(\sqrt{I}M)$, *for every ideal I of R;*
(6) $\sqrt{(N : M)} \subseteq (\mathrm{rad}\, N : M)$;
(7) $\mathrm{rad}(\sum_{N\in\mathcal{T}} N) = \mathrm{rad}(\sum_{N\in\mathcal{T}} \mathrm{rad}\, N)$;

(8) $\operatorname{rad}(\bigcap_{N\in\mathcal{T}} N) \subseteq \bigcap_{N\in\mathcal{T}} \operatorname{rad}(N), = \operatorname{rad}(\bigcap_{N\in\mathcal{T}} \operatorname{rad}(N))$.

Now, we prove an elementary result characterizing ρ-complete modules.

LEMMA 3.2. *Let R be a ring and M be an R-module. Then M is a ρ-complete module if and only if* $\operatorname{rad}((\bigcap_{I\in\mathcal{S}} I)M) = \bigcap_{I\in\mathcal{S}} \operatorname{rad}(IM)$ *for every non-empty collection $\mathcal{S}$ of radical ideals of R.*

Proof. Let $\mathcal{S}$ be a collection of radical ideals of R. Then, by Lemma 3.1, we have $\rho(\vee\mathcal{S}) = \rho(\sqrt{\sum_{I\in\mathcal{S}} I}) = \operatorname{rad}(\sqrt{\sum_{I\in\mathcal{S}} IM}) = \operatorname{rad}((\sum_{I\in\mathcal{S}} I)M) = \operatorname{rad}(\sum_{I\in\mathcal{S}} IM)$ and $\vee\rho(\mathcal{S}) = \vee\{\rho(I) : I \in \mathcal{S}\} = \vee\{\operatorname{rad}(IM) : I \in \mathcal{S}\} = \operatorname{rad}(\sum_{I\in\mathcal{S}} \operatorname{rad}(IM)) = \operatorname{rad}(\sum_{I\in\mathcal{S}} IM)$. Hence $\rho(\vee\mathcal{S}) = \vee\rho(\mathcal{S})$. Now, since

$$\rho(\wedge\mathcal{S}) = \rho(\bigcap_{I\in\mathcal{S}} I) = \operatorname{rad}((\bigcap_{I\in\mathcal{S}} I)M)$$

and

$$\wedge\rho(\mathcal{S}) = \wedge\{\operatorname{rad}(IM) : I \in \mathcal{S}\} = \bigcap_{I\in\mathcal{S}} \operatorname{rad}(IM),$$

the assertion follows. □

LEMMA 3.3. *Let R be any ring. Then*

(1) *Every direct summand of a ρ-complete module is ρ-complete.*

(2) *Every direct sum of ρ-complete modules is also ρ-complete.*

Proof. (1) Let K be a direct summand of a ρ-complete module M. Let $\mathcal{S}$ be any non-empty collection of radical ideals of R. Then, using [21, Lemma 2.1], we have

$$\operatorname{rad}_M((\bigcap_{I\in\mathcal{S}} I)M) = \operatorname{rad}_M((\bigcap_{I\in\mathcal{S}} I)K \oplus (\bigcap_{I\in\mathcal{S}} I)L) = \operatorname{rad}_K((\bigcap_{I\in\mathcal{S}} I)K) \oplus \operatorname{rad}_L((\bigcap_{I\in\mathcal{S}} I)L)$$

and

$$\begin{aligned}\bigcap_{I\in\mathcal{S}} \operatorname{rad}_M(IM) &= \bigcap_{I\in\mathcal{S}} \operatorname{rad}_M(IK \oplus IL) = \bigcap_{I\in\mathcal{S}} (\operatorname{rad}_K(IK) \oplus \operatorname{rad}_L(IL)) \\ &= (\bigcap_{I\in\mathcal{S}} \operatorname{rad}_K(IK)) \oplus (\bigcap_{I\in\mathcal{S}} \operatorname{rad}_L(IL)).\end{aligned}$$

Since M is ρ-complete, we have $\operatorname{rad}_M((\bigcap_{I\in\mathcal{S}} I)M) = \bigcap_{I\in\mathcal{S}} \operatorname{rad}_M(IM)$ and thus $\operatorname{rad}_K((\bigcap_{I\in\mathcal{S}} I)K) \oplus \operatorname{rad}_L((\bigcap_{I\in\mathcal{S}} I)L) = (\bigcap_{I\in\mathcal{S}} \operatorname{rad}_K(IK)) \oplus (\bigcap_{I\in\mathcal{S}} \operatorname{rad}_L(IL))$. Hence $\operatorname{rad}_K((\bigcap_{I\in\mathcal{S}} I)K) = \bigcap_{I\in\mathcal{S}} \operatorname{rad}_K(IK)$ which shows that K is ρ-complete by Lemma 3.2.

(2) Let $\mathcal{T}$ be any collection of ρ-complete modules and $M = \bigoplus_{N\in\mathcal{T}} N$. Let $\mathcal{S}$ be a non-empty collection of radical ideals of R. Again, using [21, Lemma 2.1],

we have

$$\begin{aligned}\operatorname{rad}_M((\bigcap_{I\in\mathcal{S}} I)M) &= \operatorname{rad}_M((\bigcap_{I\in\mathcal{S}} I)(\bigoplus_{N\in\mathcal{T}} N)) = \operatorname{rad}_M(\bigoplus_{N\in\mathcal{T}}(\bigcap_{I\in\mathcal{S}} I)N) \\ &= \bigoplus_{N\in\mathcal{T}} \operatorname{rad}_M((\bigcap_{I\in\mathcal{S}} I)N) = \bigoplus_{N\in\mathcal{T}}(\bigcap_{I\in\mathcal{S}} \operatorname{rad}_M IN) \\ &= \bigcap_{I\in\mathcal{S}}(\bigoplus_{N\in\mathcal{T}} \operatorname{rad}_M IN) = \bigcap_{I\in\mathcal{S}} \operatorname{rad}_M(\bigoplus_{N\in\mathcal{T}} IN) \\ &= \bigcap_{I\in\mathcal{S}}(\operatorname{rad}_M I(\bigoplus_{N\in\mathcal{T}} N)) = \bigcap_{I\in\mathcal{S}}(\operatorname{rad}_M IN).\end{aligned}$$

Now Lemma 3.2 completes the proof. □

THEOREM 3.4. *Let R be a ring. Then every projective R-module is ρ-complete.*

Proof. Let $\mathcal{S}$ be a non-empty collection of radical ideals of R. Then

$$\bigcap_{I\in\mathcal{S}} I \subseteq \operatorname{rad}(\bigcap_{I\in\mathcal{S}} I) \subseteq \bigcap_{I\in\mathcal{S}} \operatorname{rad}(I) = \bigcap_{I\in\mathcal{S}} I.$$

Hence $\operatorname{rad}(\bigcap_{I\in\mathcal{S}} I) = \bigcap_{I\in\mathcal{S}} \operatorname{rad}(I)$. Thus R is ρ-complete, and so by Lemma 3.3 (2) every free R-module is ρ-complete. Now, since every projective module is a direct summand of a free module, Lemma 3.3 (1) gives the result. □

COROLLARY 3.5. *Let R be a ring. Then the following statements are equivalent:*

(1) *Every R-module is ρ-complete.*

(2) *Every homomorphic image of a ρ-complete module is ρ-complete.*

Proof. (1) ⇒ (2) Clear. (2) ⇒ (1) Let M be an R-module. There exist a free R-module F and a submodule K of F such that $M \cong F/K$. By Theorem 3.4, F is ρ-complete and so is M by (2). □

THEOREM 3.6. *Let R be a ring and M be a finitely generated flat R-module. Then M is a ρ-complete R-module in each of the following cases:*

(1) *R is a Noetherian ring.*

(2) *M is a faithful λ-complete R-module.*

Proof. (1) is a direct result of Theorem 3.4 and [15, Corollary 3.57]. (2) Let $\mathcal{S}$ be a collection of radical ideals of R. Since M is a finitely generated faithful flat λ-complete R-module, by combining [14, p.51] and [9, Theorem 2.2 and Theorem 5.5 (2)], we have $\operatorname{rad}(IM) = \sqrt{I}M$ for all ideals I of R. Again, since M is λ-complete, we have $\operatorname{rad}((\bigcap_{I\in\mathcal{S}} I)M) = \operatorname{rad}(\bigcap_{I\in\mathcal{S}}(IM)) \subseteq \bigcap_{I\in\mathcal{S}} \operatorname{rad}(IM) = \bigcap_{I\in\mathcal{S}} \sqrt{I}M = \bigcap_{I\in\mathcal{S}}(IM)$. It follows that $\operatorname{rad}((\bigcap_{I\in\mathcal{S}} I)M) = \bigcap_{I\in\mathcal{S}} \operatorname{rad}(IM)$, as required. □

THEOREM 3.7. *For a ring R, every finitely generated faithful multiplication R-module is ρ-complete.*

Proof. Let M be a finitely generated faithful multiplication R-module and I be a radical ideal of R. Since M is finitely generated faithful, $(IM : M) = I$ by [4, Theorem 3.1 (ii)], and since M is multiplication, $\mathrm{rad}(IM) = \sqrt{IM : M}M = \sqrt{I}M = IM$ by [4, Theorem 2.12]. Now, let $\mathcal{S}$ be a non-empty collection of radical ideals of R. Then by [4, Theorem 1.6], we have

$$(\bigcap_{I\in\mathcal{S}} I)M \subseteq \mathrm{rad}((\bigcap_{I\in\mathcal{S}} I)M) \subseteq \bigcap_{I\in\mathcal{S}} \mathrm{rad}(IM) = \bigcap_{I\in\mathcal{S}} IM = (\bigcap_{I\in\mathcal{S}} I)M.$$

Thus $\mathrm{rad}((\bigcap_{I\in\mathcal{S}} I)M) = \bigcap_{I\in\mathcal{S}} \mathrm{rad}(IM)$, and so M is ρ-complete. $\square$

LEMMA 3.8. *Let R be a ring and I be an ideal of R. Then the R-module R/I is ρ-complete if and only if $\sqrt{\bigcap_{J\in\mathcal{S}} J + I} = \bigcap_{J\in\mathcal{S}} \sqrt{(J+I)}$, for every non-empty collection $\mathcal{S}$ of radical ideals of R.*

Proof. Let $\mathcal{S}$ be a non-empty collection of ideals of R. Then

$$\rho(\wedge\mathcal{S}) = \rho(\bigcap_{J\in\mathcal{S}} J) = \mathrm{rad}((\bigcap_{J\in\mathcal{S}} J)\frac{R}{I}) = \mathrm{rad}(\frac{\bigcap_{J\in\mathcal{S}} J + I}{I}) = \frac{\sqrt{\bigcap_{J\in\mathcal{S}} J + I}}{I}$$

and

$$\wedge\rho(\mathcal{S}) = \wedge\{\mathrm{rad}(J\frac{R}{I}) : J \in \mathcal{S}\} = \bigcap_{J\in\mathcal{S}} \mathrm{rad}(J\frac{R}{I}) = \bigcap_{J\in\mathcal{S}} \mathrm{rad}(\frac{J+I}{I})$$
$$= \bigcap_{J\in\mathcal{S}} \frac{\mathrm{rad}(J+I)}{I} = \frac{\bigcap_{J\in\mathcal{S}} \sqrt{J+I}}{I}.$$

Now, Lemma 3.2 gives the result. $\square$

Let R be a ring and M be an R-module. We say that a submodule N of M has a *radical supplement* K in case K is a radical submodule of M minimal between the radical submodules of M with respect to the property that $M = N + K$.

THEOREM 3.9. *Let R be a ring and M be a simple R-module. Then* $\mathrm{ann}_R M$ *has a radical supplement in R if and only if M is a ρ-complete module.*

Proof. Since M is a simple R-module, M is isomorphic to $R/\mathfrak{m}$ as an R-module for some maximal ideal $\mathfrak{m}$ of R. It is clear that $\mathfrak{m} = \mathrm{ann}_R M$. $(\Rightarrow)$ Let I be a radical supplement of $\mathfrak{m}$ and $\mathcal{S}$ be a non-empty subset of radical ideals of R. By Lemma 3.8, it suffices to show that $\sqrt{\mathfrak{m} + \bigcap_{J\in\mathcal{S}} J} = \bigcap_{J\in\mathcal{S}} \sqrt{\mathfrak{m} + J}$. If $J \subseteq \mathfrak{m}$ for some $J \in \mathcal{S}$, then there is nothing to prove. Let $J \nsubseteq \mathfrak{m}$, for all $J \in \mathcal{S}$. Then $\mathfrak{m}+I = R = \mathfrak{m}+J$ which implies that $\mathfrak{m}+(I\cap J) = R$. Hence $I =$

$I \cap J \subseteq J$, for each $J \in \mathcal{S}$, since I is a radical supplement of $\mathfrak{m}$. It follows that $R = \mathfrak{m} + I \subseteq \mathfrak{m} + \bigcap_{J \in \mathcal{S}} J \subseteq \bigcap_{J \in \mathcal{S}} \sqrt{\mathfrak{m} + J} = R$. Thus $\sqrt{\mathfrak{m} + \bigcap_{J \in \mathcal{S}} J} = \bigcap_{J \in \mathcal{S}} \sqrt{\mathfrak{m} + J}$.

($\Leftarrow$) Suppose that M is ρ-complete. Let $\mathcal{S}$ denote the set of all radical ideals J of R such that $R = \mathfrak{m} + J$. Then, by Lemma 3.8, we have $R = \bigcap_{J \in \mathcal{S}} \sqrt{\mathfrak{m} + J} = \sqrt{\mathfrak{m} + \bigcap_{J \in \mathcal{S}} J}$ and hence $R = \mathfrak{m} + \bigcap_{J \in \mathcal{S}} J$. This shows that $\bigcap_{J \in \mathcal{S}} J$ is a radical supplement of $\mathfrak{m}$ in R. □

Corollary 3.10. *Let R be a semi-perfect ring and M be a semi-simple R-module. Then M is a ρ-complete R-module.*

Proof. Since M is semi-simple, M is a direct sum of simple modules M_i $(i \in I)$. By [20, Theorem 42.6], $\operatorname{ann}_R M_i$ has a supplement J in ${}_RR$. It follows that $\sqrt{J}$ is a radical supplement of $\operatorname{ann}_R M_i$ in ${}_RR$. In fact, if J' is a radical ideal of R such that $J' \subseteq \sqrt{J}$ and $\operatorname{ann}_R M_i + J' = R$, then $\operatorname{ann}_R M_i + J \cap J' = R$ and hence $J \subseteq J \cap J' \subseteq J'$. Taking radical gives that $J' = \sqrt{J}$. Therefore by Theorem 3.9, M_i is a ρ-complete module. Hence, by Lemma 3.3(2), M is ρ-complete. □

It is clear that, if M is a multiplication R-module, then the mapping ρ will be an epimorphism. (In fact, for any submodule $N \in \mathcal{R}(M)$, $I_N = (N : M)$ belongs to $\mathcal{R}(R)$ and we have $\rho(I_N) = \operatorname{rad}(I_N M) = N$). If M is also finitely generated faithful, then by Theorem 3.7, ρ will be a complete epimorphism. However, ρ need not be a monomorphism. For example, let I be a proper radical ideal of R which is generated by idempotent elements such that $\operatorname{ann}_R(I) = 0$. Then I is a finitely generated faithful multiplication R-module and the mapping $\rho : \mathcal{R}(R) \to \mathcal{R}(I)$ is a complete epimorphism which is not monomorphism, because $\rho(R) = RI = I = I^2 = \rho(I)$.

4. ρ-COMPLETE AND λ-COMPLETE MODULES

In this section, we investigate rings R such that every R-module is ρ-complete, and study conditions for which ρ-completeness and λ-completeness are equivalent.

Lemma 4.1. *Let R be a ring. Then R is a zero-dimensional ring if and only if $\sqrt{\bigcap_{\lambda \in \Lambda} I_\lambda} = \bigcap_{\lambda \in \Lambda} \sqrt{I_\lambda}$, for every family $\{I_\lambda\}_{\lambda \in \Lambda}$ of ideals of R.*

Proof. By [2, Theorem 1.3 and Theorem 2.4]. □

Theorem 4.2. *Let R be a zero-dimensional ring and M be an R-module such that $\mathcal{L}(M) = \mathcal{R}(M)$. Then M is λ-complete if and only if M is ρ-complete.*

Proof. (⇒) Let $\mathcal{S}$ be a non-empty subset of $\mathcal{R}(R)$. Then

$$\operatorname{rad}((\bigcap_{I\in\mathcal{S}} I)M) = (\bigcap_{I\in\mathcal{S}} I)M = \bigcap_{I\in\mathcal{S}} IM = \bigcap_{I\in\mathcal{S}} \operatorname{rad}(IM).$$

Thus, by Lemma 3.2, M is a ρ-complete R-module.

(⇐) Let $\mathcal{S}$ be a non-empty subset of $\mathcal{R}(R)$. Then, by Lemma 4.1 and hypothesis, we have

$$\begin{aligned}(\bigcap_{I\in\mathcal{S}} I)M &= \operatorname{rad}((\bigcap_{I\in\mathcal{S}} I)M) = \operatorname{rad}((\sqrt{\bigcap_{I\in\mathcal{S}} I})M) = \operatorname{rad}((\bigcap_{I\in\mathcal{S}} \sqrt{I})M)\\ &= \bigcap_{I\in\mathcal{S}} \operatorname{rad}(\sqrt{I}M) = \bigcap_{I\in\mathcal{S}} \operatorname{rad}(IM) = \bigcap_{I\in\mathcal{S}} IM.\end{aligned}$$

Hence, by [18, Lemma 3.1], M is a λ-complete R-module. □

LEMMA 4.3. *Let R be a ring, M be an R-module and N be a proper submodule of M such that* $\operatorname{rad}_{M_\mathfrak{m}}(N_\mathfrak{m}) = N_\mathfrak{m}$ *for all maximal ideals $\mathfrak{m}$ of R. Then* $\operatorname{rad}_M(N) = N$.

Proof. Straightforward. □

THEOREM 4.4. *Let R be an Artinian ring and M an R-module such that* $\operatorname{ann}_R M \in \mathcal{R}(R)$. *Then* $\mathcal{L}(M) = \mathcal{R}(M)$.

Proof. Let us denote the ring $R/\operatorname{ann}_R(M)$ by $\overline{R}$. Clearly $\dim \overline{R} = 0$ and since $\operatorname{ann}_R M \in \mathcal{R}(R)$, we have $\operatorname{Nil}(\overline{R}) = 0$. Thus by [1, Exercise 11, p. 44], $\overline{R}$ is absolutely flat and hence every principal ideal of $\overline{R}$ is idempotent, by [1, Exercise 27, p. 35]. According to the Lemma 4.3, without loss of generality, we can assume that $\overline{R}$ is a local ring with the unique maximal ideal $\mathfrak{m}$. Let N be a proper submodule of M, $r \in \mathfrak{m}$ and $m \in M$. Then $\overline{R}r\overline{R}m = \overline{R}r^2\overline{R}m$. Thus $rm = r_1r^2m$, for some $r_1 \in R$ and hence $rm = 0$, since $1 - r_1r$ is a unit in $\overline{R}$. So we have $\mathfrak{m}M = 0 \subseteq N$. This gives that $\mathfrak{m} = (N : M)$ and hence N is a prime submodule of M, that is every proper submodule of M is radical. □

THEOREM 4.5. *Let R be an Artinian ring. Then every R-module is ρ-complete.*

Proof. Let M be an R-module. Let $\mathcal{S}$ be any non-empty collection of radical ideals of R. Because R is Artinian, there exists a finite subset $\mathcal{S}'$ of $\mathcal{S}$ such that $\bigcap_{I\in\mathcal{S}} I = \bigcap_{I\in\mathcal{S}'} I$. Thus by [8, Corollary 2 to Proposition 1], we have

$$\bigcap_{I\in\mathcal{S}} \operatorname{rad}(IM) \subseteq \bigcap_{I\in\mathcal{S}'} \operatorname{rad}(IM) = \operatorname{rad}((\bigcap_{I\in\mathcal{S}'} I)M) = \operatorname{rad}((\bigcap_{I\in\mathcal{S}} I)M) \subseteq \bigcap_{I\in\mathcal{S}} \operatorname{rad}(IM).$$

Hence $\bigcap_{I\in\mathcal{S}} \operatorname{rad}(IM) = \operatorname{rad}((\bigcap_{I\in\mathcal{S}} I)M)$, and so by Lemma 3.2, M is ρ-complete. □

COROLLARY 4.6. *Let R be an Artinian ring and M be an R module such that* $\operatorname{ann}_R M \in \mathcal{R}(R)$. *Then M is λ-complete.*

Proof. By Theorem 4.2, Theorem 4.4 and Theorem 4.5. □

THEOREM 4.7. *Let R be a Boolean ring and M be an R-module. Then M is λ-complete if and only if M is ρ-complete.*

Proof. Every principal ideal of R is idempotent. The rest of the proof is similar to the proof of Theorem 4.4. □

LEMMA 4.8. *Let R be a ring and I be an ideal of R. If every R-module is a ρ-complete module, then every R/I-module is a ρ-complete R-module.*

Proof. Let M be an R/I-module and $\mathcal{S}$ be a non-empty subset of $\mathcal{R}(R/I)$. Every element of $\mathcal{S}$ has the form J/I for some radical ideal J of R. Let $\mathcal{S}' = \{J \in \mathcal{R}(R) : J/I \in \mathcal{S}\}$ and consider M as an R module in the usual way. Then we have

$$\begin{aligned}\mathrm{rad}((\bigcap_{K\in\mathcal{S}} K)M) = \mathrm{rad}((\bigcap_{J\in\mathcal{S}'} J/I)M) &= \mathrm{rad}(((\bigcap_{J\in\mathcal{S}'} J)/I)M) = \mathrm{rad}((\bigcap_{J\in\mathcal{S}'} J)M) \\ &= \bigcap_{J\in\mathcal{S}'} \mathrm{rad}(JM) = \bigcap_{J\in\mathcal{S}'} \mathrm{rad}((J/I)M) \\ &= \bigcap_{K\in\mathcal{S}} \mathrm{rad}(KM).\end{aligned}$$

Thus M is ρ-complete. □

LEMMA 4.9. *Let R be a domain with field of fractions F. Then the following are equivalent.*

(1) *R is a field.*
(2) *Every R-module is λ-complete.*
(3) *Every R-module is ρ-complete.*
(4) *The R-module F is ρ-complete.*

Proof. (1), (2) and (4) are equivalent by [18, Lemma 4.11].
(1) ⇒ (3) Since every proper submodule of a module over a field is prime, the result is clear.
(3) ⇒ (4) Clear. □

THEOREM 4.10. *Let R be a Noetherian ring for which every R-module is ρ-complete. Then R is Artinian.*

Proof. Suppose that every R-module is ρ-complete. Let P be a prime ideal of R. By Lemma 4.8, every R/P-module is ρ-complete and hence the domain R/P is a field by Lemma 4.9. Thus every prime ideal of R is maximal. By [1, Theorem 8.5], the ring R is Artinian. □

COROLLARY 4.11. *Let R be a Noetherian ring. Then every R-module is ρ-complete if and only if every R-module is λ-complete.*

Proof. By Theorem 4.10 and [18, Theorem 4.12]. □

THEOREM 4.12. *Let R be a ring and M be an R-module. Then ρ is a complete isomorphism if and only if the mapping σ is a complete isomorphism. Moreover, in this case, if $0 \in \mathcal{R}(R)$, then M is a faithful R-module.*

Proof. By [12, Corollary 3.4] ρ is a bijection if and only if so is σ. So we have the result by [18, Lemma 1.1]. For the "moreover" part, assume that ρ is a complete isomorphism, and $A = \mathrm{ann}_R(M)$. Therefore $\rho(\sqrt{A}) = \mathrm{rad}(\sqrt{A}M) = \mathrm{rad}(AM) = \mathrm{rad}(0) = \rho(0)$. Now, since ρ is injective, we have $\sqrt{A} = 0$ and hence $A = 0$, *i.e.*, M is faithful. □

COROLLARY 4.13. *Let R be a ring and M be a finitely generated R-module. If the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}(M)$ is a complete isomorphism, then the mapping $\rho : \mathcal{R}(R) \to \mathcal{R}(M)$ is a complete isomorphism. Moreover, if R is a domain, then M is a ρ-complete R-module.*

Proof. Since λ is a complete isomorphism, by [18, Proposition 1.4], μ is also a complete isomorphism. So σ is injective, since σ is the restriction of μ to the set of radical submodules of M. If $I \in \mathcal{R}(R)$, then there exists a submodule N of M such that $\mu(N) = I$. Thus $\sigma(\mathrm{rad}\, N) = \mu(\mathrm{rad}\, N) = (\mathrm{rad}\, N : M) = \sqrt{(N : M)} = \sqrt{I} = I$, *i.e.*, σ is surjective. Hence by [18, Lemma 1.1], σ is a complete isomorphism and so is ρ by Theorem 4.12. For "Moreover" part, by Theorem 4.12, M is faithful, and since λ is surjective M is a multiplication module. Now, by Theorem 3.7, M is ρ-complete. □

REFERENCES

[1] M.F. Atiyah and I.G. MacDonald, *Introduction to Commutative Algebra.* Addison-Wesley, Reading, MA, 1969.

[2] J. Brewer and F. Richman, *Subrings of zero-dimensional rings.* In: J.W. Brewer, S. Glaz, W.J. Heinzer and B.M. Olberding (Eds.), *Multiplicative Ideal Theory in Commutative Algebra. A Tribute to the Work of Robert Gilmer.* Springer, pp. 73–88, 2006.

[3] J. Dauns, *Prime modules.* J. Reine Angew. Math. **298** (1978), 156–181.

[4] Z.A. El-Bast and F.P. Smith, *Multiplication modules.* Comm. Algebra **16** (1988), *4*, 755–799.

[5] J. Jenkins and P.F. Smith, *On the prime radical of a module over a commutative ring.* Comm. Algebra **20** (1992), *12*, 3593–3602.

[6] C.P. Lu, *Prime submodule of modules.* Comment. Math. Univ. St. Pauli **33** (1984), *1*, 61–69.

[7] C.P. Lu, *M-radical of submodules in modules.* Math. Japonica **34** (1989), *2*, 211–219.

[8] C.P. Lu, *M-radical of submodules in modules II.* Math. Japonica **35** (1990), *5*, 991–1001.

[9] C.P. Lu, *A module whose prime spectrum has the surjective natural map.* Houston J. Math. **33** (2007), *1*, 125–143.

[10] R.L. McCasland, M.E. Moore and P.F. Smith, *On the spectrum of a module over a commutative ring*. Comm. Algebra **25** (1997), *1*, 79–103.

[11] R.L. McCasland and P.F. Smith, *Prime submodules of Noetherian modules.* Rocky Mountain J. Math. **23** (1993), *3*, 1041–1062.

[12] H.F. Moghimi and J.B. Harehdashti, *Mappings between lattices of radical submodules.* Int. Electron. J. Algebra **19** (2016), 35–48.

[13] M.E. Moore and S.J. Smith, *Prime and radical submodules of modules over commutative rings*. Comm. Algebra **30** (2002), *10*, 5037–5064.

[14] J. Ohm and D.E. Rush, *Content modules and algebras*. Math. Scand. **31** (1972), 49–68.

[15] J.J. Rotman, *An Introduction to Homological Algebra. Second edition.* Universitext, Springer, New York, 2009.

[16] P.F. Smith, *Some remarks on multiplication modules*. Arch. Math. **50** (1988), 223–235.

[17] P.F. Smith, *Mappings between module lattices*. Int. Electron. J. Algebra **15** (2014), 173–195.

[18] P.F. Smith, *Complete homomorphism between module lattices*. Int. Electron. J. Algebra **16** (2014), 16–31.

[19] P.F. Smith, *Anti-homomorphisms between modules lattices.* J. Commut. Algebra **7** (2015), *4*, 567–592.

[20] R. Wisbauer, *Foundations of Module and Ring Theory.* Gordon and Breach, Reading, Philadelphia, 1991.

[21] D.P. Yilmaz and P.F. Smith, *Radicals of submodules of free modules*. Comm. Algebra **27** (1999), *5*, 2253–2266.

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