RECIPROCAL SUMS OF SEQUENCES INVOLVING BALANCING AND LUCAS-BALANCING NUMBERS

GOPAL KRISHNA PANDA, TAKAO KOMATSU and RAVI KUMAR DAVALA

Communicated by Alexandru Zaharescu

Many authors studied bounds for reciprocal sums involving terms from Fibonacci and other related sequences. The purpose of this paper is to provide bounds for reciprocal sums with terms from balancing and Lucas-balancing sequences.

AMS 2010 Subject Classification: 11B39, 11B83.

Key words: balancing and Lucas balancing numbers, recurrence relation, Binet form.

1. INTRODUCTION

A positive integer x is called balancing number if

$$1 + 2 + \cdots + (x - 1) = (x + 1) + \cdots + (y - 1)$$

holds for some integer $y \geq x + 2$. The problem of determining all balancing numbers leads to the Pell equation $y^2 - 8x^2 = 1$, whose solutions in x can be described by the recurrence $B_n = 6B_{n-1} - B_{n-2}$ $(n \geq 2)$ with $B_0 = 0$ and $B_1 = 1$ (see (*e.g.* [1, 2, 12]). Balancing numbers have been extensively studied by many authors. Karaatli *et. al.* [11] expressed the positive integral solutions of a Diophantine equation in terms of balancing numbers. Liptai [8] proved that there is no Fibonacci balancing number except 1. In [13], the period rank and order of the sequence of balancing numbers are studied. One of the most general extensions of the defining equation of balancing numbers is the Diophantine equation

$$1^k + 2^k + \cdots + (x - 1)^k = (x + 1)^l + \cdots + (y - 1)^l,$$

where the exponents k and l are given positive integers. In [9] effective and non-effective finiteness theorems for the above equation are proved. In [6] a balancing problem of ordinary binomial coefficients is studied, and effective and non-effective finiteness theorems are given.

The numbers $C_n = \sqrt{8B_n^2 + 1}$ is called the n^{th} Lucas-balancing number [12], and these numbers satisfy the recurrence relation $C_n = 6C_{n-1} - C_{n-2}$ with initial values $C_0 = 1, C_1 = 3$. The Binet forms for B_n and C_n are respectively

$$B_n = \frac{\alpha^n - \beta^n}{4\sqrt{2}}, \quad C_n = \frac{\alpha^n + \beta^n}{2}$$

where $\alpha = 3 + 2\sqrt{2}$ and $\beta = 3 - 2\sqrt{2}$. The balancing and Lucas-balancing numbers satisfy the identities

$$B_{n-r}B_{n+r} = B_n^2 - B_r^2, \quad C_{n-r}C_{n+r} = C_n^2 + C_r^2 - 1$$

for $n \geq r$ respectively. In particular, for $n \geq 1$ we have

$$B_{n-1}B_{n+1} = B_n^2 - 1 \text{ and } C_{n-1}C_{n+1} = C_n^2 + 8.$$

The identity

$$B_1 + B_3 + \cdots + B_{2n-1} = B_n^2$$

gives

$$B_{2n-1} = B_n^2 - B_{n-1}^2.$$

The proofs of the above identities are available in [14]. In the subsequent sections, we shall use the above identities without further reference.

The intention of this paper is to develop certain interesting bounds for reciprocal sums with terms involving balancing and Lucas-balancing numbers in some combinations.

The reciprocal of partial infinite sums of reciprocal Fibonacci numbers has been extensively studied by many authors (*e.g.*, see [4, 5, 7, 10, 15]). In [3], the following identities are shown for generalized Fibonacci numbers G_n, defined by

$$G_n = aG_{n-1} + G_{n-2} \quad (n \geq 2), \quad G_0 = 0, \quad G_1 = 1,$$

where a is a positive integer. If $a = 1$, then G_n's are equal to the Fibonacci numbers. Throughout this paper, integer part of a number is denoted by $\lfloor \cdot \rfloor$.

Proposition 1.1.

(1) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_k} \right)^{-1} \right\rfloor = \begin{cases} G_n - G_{n-1} & \text{if } n \text{ is even and } n \geq 2; \\ G_n - G_{n-1} - 1 & \text{if } n \text{ is odd and } n \geq 1\,. \end{cases}$$

(2) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} aG_{n-1}G_n - 1 & \text{if } n \text{ is even and } n \geq 2; \\ aG_{n-1}G_n & \text{if } n \text{ is odd and } n \geq 1\,. \end{cases}$$

(3) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_{2k}} \right)^{-1} \right\rfloor = G_{2n} - G_{2n-2} - 1 \qquad (n \geq 1)$$

(4) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_{2k-1}} \right)^{-1} \right\rfloor = G_{2n-1} - G_{2n-3} \qquad (n \geq 2)$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_{2k-1}G_{2k+1}} \right)^{-1} \right\rfloor = G_{4n-1} - G_{4n-3} \qquad (n \geq 1) \tag{5}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_{2k}G_{2k+2}} \right)^{-1} \right\rfloor = G_{4n+1} - G_{4n-1} - 1 \qquad (n \geq 1) \tag{6}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_{2k}^2} \right)^{-1} \right\rfloor = G_{4n-1} - G_{4n-3} - 1 \qquad (n \geq 1) \tag{7}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_{2k-1}^2} \right)^{-1} \right\rfloor = G_{4n-3} - G_{4n-5} \qquad (n \geq 2) \tag{8}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_{2k-1}G_{2k}} \right)^{-1} \right\rfloor = G_{4n-2} - G_{4n-4} \qquad (n \geq 1) \tag{9}$$

In this paper, we shall show some analogous results for the sequences of balancing and Luca-balancing numbers.

Proposition 1.2.

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_k} \right)^{-1} \right\rfloor = B_n - B_{n-1} - 1 \qquad (n \geq 1) \tag{1}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_k^2} \right)^{-1} \right\rfloor = B_n^2 - B_{n-1}^2 - 1 = B_{2n-1} - 1 \qquad (n \geq 1) \tag{2}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_kB_{k+1}} \right)^{-1} \right\rfloor = B_nB_{n+1} - B_{n-1}B_n - 1 \qquad (n \geq 1) \tag{3}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_k} \right)^{-1} \right\rfloor = C_n - C_{n-1} \qquad (n \geq 2) \tag{4}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_k^2} \right)^{-1} \right\rfloor = C_n^2 - C_{n-1}^2 \qquad (n \geq 1) \tag{5}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_kC_{k+1}} \right)^{-1} \right\rfloor = C_nC_{n+1} - C_{n-1}C_n + 1 \qquad (n \geq 1) \tag{6}$$

2. RECIPROCAL SUMS INVOLVING BALANCING NUMBERS

In this section, we establish bounds for several reciprocal sums involving balancing numbers. By using the bounds, we obtain the results in Proposition 1.2 (1), (2) and (3).

The following theorem gives sharp bounds for reciprocal sums of balancing numbers.

THEOREM 2.1. *For any positive integer n,*

$$\frac{1}{B_n - B_{n-1}} < \sum_{k=n}^{\infty} \frac{1}{B_k} < \frac{1}{B_n - B_{n-1} - 1}.$$

Proof. For any positive integer n,

$$\begin{aligned}\frac{1}{B_n - B_{n-1}} - \frac{1}{B_n} &= \frac{B_{n-1}}{{B_n}^2 - B_n B_{n-1}} \\ &= \frac{B_{n-1}}{B_{n-1}(B_{n+1} - B_n) + 1} < \frac{1}{B_{n+1} - B_n}.\end{aligned}$$

Thus,

$$\frac{1}{B_n - B_{n-1}} < \frac{1}{B_n} + \frac{1}{B_{n+1} - B_n}.$$

Repeating the above steps, we can show that

$$\frac{1}{B_{n+1} - B_n} < \frac{1}{B_{n+1}} + \frac{1}{B_{n+2} - B_{n+1}}.$$

Combining the above two inequalities, we get

$$\frac{1}{B_n - B_{n-1}} < \frac{1}{B_n} + \frac{1}{B_{n+1}} + \frac{1}{B_{n+2} - B_{n+1}}.$$

Continuing in this manner, one can arrive at the inequality

$$\frac{1}{B_n - B_{n-1}} < \sum_{k=n}^{\infty} \frac{1}{B_k}. \tag{2.1}$$

On the other hand,

$$\begin{aligned}\frac{1}{B_n - B_{n-1} - 1} - \frac{1}{B_n} &= \frac{B_{n-1} + 1}{{B_n}^2 - B_n B_{n-1} - B_n} \\ &= \frac{B_{n-1} + 1}{B_{n-1}(B_{n+1} - B_n) - B_n + 1} \\ &> \frac{B_{n-1}}{B_{n-1}(B_{n+1} - B_n) - B_n} \\ &> \frac{B_{n-1}}{B_{n-1}(B_{n+1} - B_n) - B_{n-1}}\end{aligned}$$

$$=\frac{1}{B_{n+1}-B_n-1},$$

which gives

$$\frac{1}{B_n-B_{n-1}-1}>\frac{1}{B_n}+\frac{1}{B_{n+1}-B_n-1}.$$

Continuing in a similar fashion, we finally obtain

$$\frac{1}{B_n-B_{n-1}-1}>\sum_{k=n}^{\infty}\frac{1}{B_k}. \tag{2.2}$$

Combining (2.1) and (2.2), we get the desired inequality as stated in the theorem. □

The following theorem provides sharp bounds for the reciprocal sum of squares of balancing numbers.

Theorem 2.2. *For any positive integer n,*

$$\frac{1}{B_{2n-1}}<\sum_{k=n}^{\infty}\frac{1}{B_k^2}<\frac{1}{B_{2n-1}-1}.$$

Proof. Since

$$\frac{B_n}{B_{n-1}}-\frac{B_{n+1}}{B_n}=\frac{1}{B_nB_{n-1}}>0,$$

for each $n>1$, we have $\frac{B_n}{B_{n-1}}>\frac{B_{n+1}}{B_n}$. Thus, for each $n>1$,

$$\begin{aligned}\frac{1}{B_{2n-1}}-\frac{1}{B_n^2}&=\frac{1}{B_n^2-B_{n-1}^2}-\frac{1}{B_n^2}\\&=\frac{B_{n-1}^2}{B_n^2(B_n^2-B_{n-1}^2)}\\&=\frac{1}{B_n^2(\frac{B_n^2}{B_{n-1}^2}-1)}\\&<\frac{1}{B_n^2(\frac{B_{n+1}^2}{B_n^2}-1)}\\&=\frac{1}{B_{n+1}^2-B_n^2}=\frac{1}{B_{2n+1}},\end{aligned}$$

which yields

$$\frac{1}{B_{2n-1}}<\frac{1}{B_n^2}+\frac{1}{B_{2n+1}}.$$

This inequality is also valid for $n = 1$. Recursive iteration of the last inequality gives

$$\frac{1}{B_{2n-1}} < \sum_{k=n}^{\infty} \frac{1}{B_k^2}. \tag{2.3}$$

On the other hand,

$$\begin{aligned}\frac{1}{B_{2n-1}-1} - \frac{1}{B_n^2} &= \frac{1}{B_n^2 - B_{n-1}^2 - 1} - \frac{1}{B_n^2}\\ &= \frac{B_{n-1}^2+1}{B_n^2(B_n^2 - B_{n-1}^2 - 1)}.\end{aligned}$$

For $n \geq 1$,

$$\begin{aligned}&(B_{2n+1}-1)(B_{n-1}^2+1) - B_n^2(B_n^2 - B_{n-1}^2 - 1)\\ &= (B_{n+1}^2 - B_n^2 - 1)(B_{n-1}^2+1) - B_n^2(B_n^2 - B_{n-1}^2 - 1)\\ &= B_{n+1}^2 B_{n-1}^2 + B_{n+1}^2 - B_n^4 - B_{n-1}^2 - 1\\ &= (B_n^2-1)^2 + (6B_n - B_{n-1})^2 - B_n^4 - B_{n-1}^2 - 1\\ &= 34B_n^2 - 12B_nB_{n-1} > 0.\end{aligned}$$

Thus, we have

$$\frac{1}{B_{2n-1}-1} > \frac{1}{B_n^2} + \frac{1}{B_{2n+1}-1}.$$

Iterating recursively, we get

$$\frac{1}{B_{2n-1}-1} > \sum_{k=n}^{\infty} \frac{1}{B_k^2}. \tag{2.4}$$

Combining inequalities (2.3) and (2.4), we get what has been claimed. □

The reciprocal sum of products of two consecutive balancing numbers has analogous bounds. The following theorem is important in this regard.

Theorem 2.3. *For any positive integer n,*

$$\frac{1}{B_nB_{n+1} - B_{n-1}B_n} < \sum_{k=n}^{\infty} \frac{1}{B_kB_{k+1}} < \frac{1}{B_nB_{n+1} - B_{n-1}B_n - 1}.$$

Proof. Using the fact $\frac{B_{n-1}}{B_n} < \frac{B_n}{B_{n+1}}$ $n \geq 1$, we have

$$\begin{aligned}\frac{1}{B_nB_{n+1} - B_{n-1}B_n} - \frac{1}{B_nB_{n+1}} &= \frac{B_{n-1}}{B_n}\left(\frac{1}{B_{n+1}^2 - B_{n-1}B_{n+1}}\right)\\ &< \frac{B_n}{B_{n+1}}\left(\frac{1}{B_nB_{n+2} - B_n^2 + 2}\right)\end{aligned}$$

$$< \frac{1}{B_{n+1}B_{n+2}-B_nB_{n+1}}.$$

Thus,

$$\frac{1}{B_nB_{n+1}-B_{n-1}B_n} < \frac{1}{B_nB_{n+1}} + \frac{1}{B_{n+1}B_{n+2}-B_nB_{n+1}}.$$

Iterating recursively, we get

$$\sum_{k=n}^{\infty} \frac{1}{B_kB_{k+1}} > \frac{1}{B_nB_{n+1}-B_{n-1}B_n}. \tag{2.5}$$

On the other hand,

$$\begin{aligned} \frac{1}{B_nB_{n+1}-B_{n-1}B_n-1} - \frac{1}{B_nB_{n+1}} &= \frac{B_{n-1}B_n+1}{B_nB_{n+1}(B_nB_{n+1}-B_{n-1}B_n-1)} \\ &= \left[B_nB_{n+1}\left(\frac{B_nB_{n+1}}{B_{n-1}B_n+1}-1\right)\right]^{-1}. \end{aligned}$$

Since

$$\begin{aligned} &\frac{B_{n+1}B_{n+2}}{B_nB_{n+1}+1} - \frac{B_nB_{n+1}}{B_{n-1}B_n+1} \\ &\quad = \frac{(B_n^2-1)(B_{n+1}^2-1)+B_{n+1}B_{n+2}-B_n^2B_{n+1}^2-B_nB_{n+1}}{(B_{n-1}B_n+1)(B_nB_{n+1}+1)} \\ &\quad = \frac{5B_{n+1}^2-2B_nB_{n+1}-B_n^2+1}{(B_{n-1}B_n+1)(B_nB_{n+1}+1)} > 0, \end{aligned}$$

we have

$$\begin{aligned} \frac{1}{B_nB_{n+1}-B_{n-1}B_n-1} - \frac{1}{B_nB_{n+1}} &> \left[B_nB_{n+1}\left(\frac{B_{n+1}B_{n+2}}{B_nB_{n+1}+1}-1\right)\right]^{-1} \\ &= \frac{B_nB_{n+1}+1}{B_nB_{n+1}(B_{n+2}B_{n+1}-B_nB_{n+1}-1)} \\ &> \frac{1}{B_{n+2}B_{n+1}-B_nB_{n+1}-1}. \end{aligned}$$

The last inequality can be rearranged as

$$\frac{1}{B_nB_{n+1}-B_{n-1}B_n-1} > \frac{1}{B_nB_{n+1}} + \frac{1}{B_{n+2}B_{n+1}-B_nB_{n+1}-1}.$$

Repeated iteration gives

$$\sum_{k=n}^{\infty} \frac{1}{B_kB_{k+1}} < \frac{1}{B_nB_{n+1}-B_{n-1}B_n-1}. \tag{2.6}$$

Combining the inequalities (2.5) and (2.6) together, we get

$$\frac{1}{B_nB_{n+1}-B_{n-1}B_n} < \sum_{k=n}^{\infty}\frac{1}{B_kB_{k+1}} < \frac{1}{B_nB_{n+1}-B_{n-1}B_n-1}.$$

This ends the proof. □

The following theorem can be proved in a similar fashion. However, the bounds are not as sharp as those in the previous theorems.

THEOREM 2.4. *For positive integers n and $r \geq 3$,*

$$\frac{1}{B_n^r-B_{n-1}^r} < \sum_{k=n}^{\infty}\frac{1}{B_k^r} < \frac{1}{B_n^r-(B_{n-1}-1)^r}.$$

3. RECIPROCAL SUMS INVOLVING LUCAS-BALANCING NUMBERS

In this section, we shall establish certain bounds for the reciprocal sums involving Lucas-balancing numbers. By using these bounds, we obtain the results in Proposition 1.2 (4), (5) and (6).

The following theorem provides sharp bounds for reciprocal sums of the Lucas-balancing numbers.

THEOREM 3.1. *For any positive integer $n \geq 2$,*

$$\frac{1}{C_n-C_{n-1}+1} < \sum_{k=n}^{\infty}\frac{1}{C_k} < \frac{1}{C_n-C_{n-1}}.$$

Proof. For any positive integer $n \geq 2$,

$$\frac{1}{C_n-C_{n-1}}-\frac{1}{C_n}=\frac{C_{n-1}}{{C_n}^2-C_nC_{n-1}}=\frac{C_{n-1}}{C_{n-1}(C_{n+1}-C_n)-8}>\frac{1}{C_{n+1}-C_n}.$$

Thus,

$$\frac{1}{C_n-C_{n-1}} > \frac{1}{C_n}+\frac{1}{C_{n+1}-C_n}.$$

Repeating the above steps, we can obtain

$$\frac{1}{C_{n+1}-C_n} > \frac{1}{C_{n+1}}+\frac{1}{C_{n+2}-C_{n+1}}.$$

Combining the above two inequalities, we get

$$\frac{1}{C_n-C_{n-1}} > \frac{1}{C_n}+\frac{1}{C_{n+1}}+\frac{1}{C_{n+2}-C_{n+1}}.$$

Continuing in this manner, one can arrive at the inequality

$$\frac{1}{C_n - C_{n-1}} > \sum_{k=n}^{\infty} \frac{1}{C_k}. \tag{3.1}$$

On the other hand, since

$$\begin{aligned}
\frac{1}{C_n - C_{n-1} + 1} - \frac{1}{C_n} &= \frac{C_{n-1} - 1}{C_n^2 - C_n C_{n-1} + C_n} \\
&= \frac{C_{n-1} - 1}{C_{n-1}(C_{n+1} - C_n) + C_n - 8} \\
&< \frac{C_{n-1}}{C_{n-1}(C_{n+1} - C_n) + C_{n-1}} \\
&= \frac{1}{C_{n+1} - C_n + 1},
\end{aligned}$$

we have

$$\frac{1}{C_n - C_{n-1} + 1} < \frac{1}{C_n} + \frac{1}{C_{n+1} - C_n + 1}.$$

Continuing in the same way, one can obtain

$$\frac{1}{C_n - C_{n-1} + 1} < \sum_{k=n}^{\infty} \frac{1}{C_k}. \tag{3.2}$$

Combining inequalities (3.1) and (3.2), we get the desired inequality as stated in the theorem. □

The following theorem provides sharp bounds for the reciprocal sum of squares of Lucas-balancing numbers.

Theorem 3.2. *For any positive integer n,*

$$\frac{1}{C_n^2 - C_{n-1}^2 + 1} < \sum_{k=n}^{\infty} \frac{1}{C_k^2} < \frac{1}{C_n^2 - C_{n-1}^2}$$

Proof. For each $n \geq 1$

$$\frac{C_n}{C_{n-1}} - \frac{C_{n+1}}{C_n} = \frac{-8}{C_{n-1} C_n} < 0,$$

we have $\frac{C_n}{C_{n-1}} < \frac{C_{n+1}}{C_n}$. Thus for each $n \geq 1$,

$$\begin{aligned}
\frac{1}{C_n^2 - C_{n-1}^2} - \frac{1}{C_n^2} &= \frac{C_{n-1}^2}{C_n^2(C_n^2 - C_{n-1}^2)} \\
&= \frac{1}{C_n^2\left(\frac{C_n^2}{C_{n-1}^2} - 1\right)}
\end{aligned}$$

$$
\begin{aligned}
&> \frac{1}{C_n^2\left(\frac{C_{n+1}^2}{C_n^2} - 1\right)} \\
&= \frac{1}{C_{n+1}^2 - C_n^2}.
\end{aligned}
$$

The above inequality yields

$$\frac{1}{C_n^2 - C_{n-1}^2} > \frac{1}{C_n^2} + \frac{1}{C_{n+1}^2 - C_n^2}$$

and iterating recursively, we get

$$\frac{1}{C_n^2 - C_{n-1}^2} > \sum_{k=n}^{\infty} \frac{1}{C_k^2}. \tag{3.3}$$

On the other hand, for $n \geq 1$,

$$\frac{1}{C_n^2 - C_{n-1}^2 + 1} - \frac{1}{C_n^2} = \frac{C_{n-1}^2 - 1}{C_n^2(C_n^2 - C_{n-1}^2 + 1)} = C_n^{-2}\left(\frac{C_n^2}{C_{n-1}^2 - 1} - 1\right)^{-1}.$$

If we set $x_n = \frac{C_n^2}{C_{n-1}^2 - 1}$, then

$$x_{n+1} - x_n = \frac{C_{n+1}^2}{C_n^2 - 1} - \frac{C_n^2}{C_{n-1}^2 - 1} = \frac{17C_n^2 - C_{n+1}^2 + 64}{(C_n^2 - 1)(C_{n-1}^2 - 1)} < 0.$$

Hence,

$$
\begin{aligned}
C_n^{-2}\left(\frac{C_n^2}{C_{n-1}^2 - 1} - 1\right)^{-1} &< C_n^{-2}\left(\frac{C_{n+1}^2}{C_n^2 - 1} - 1\right)^{-1} \\
&= \frac{C_n^2 - 1}{C_n^2(C_{n+1}^2 - C_n^2 + 1)} < \frac{1}{C_{n+1}^2 - C_n^2 + 1}.
\end{aligned}
$$

Thus, we have

$$\frac{1}{C_n^2 - C_{n-1}^2 + 1} < \frac{1}{C_n^2} + \frac{1}{C_{n+1}^2 - C_n^2 + 1}.$$

Iterating recursively, we get

$$\frac{1}{{C_n}^2 - {C_{n-1}}^2 + 1} < \sum_{k=n}^{\infty} \frac{1}{{C_k}^2}. \tag{3.4}$$

Combining inequalities (3.3) and (3.4), we get what has been claimed. □

The reciprocal sum of products of two consecutive Lucas-balancing numbers has analogous bounds. The following theorem is important in this regard.

THEOREM 3.3. *For any positive integer n,*

$$\frac{1}{C_nC_{n+1}-C_{n-1}C_n+2}<\sum_{k=n}^{\infty}\frac{1}{C_kC_{k+1}}<\frac{1}{C_nC_{n+1}-C_{n-1}C_n+1}.$$

Proof. Since for each positive integer n,

$$\begin{aligned}&(C_{n-1}C_n-1)(C_{n+1}C_{n+2}-C_nC_{n+1}+1)-C_nC_{n+1}(C_nC_{n+1}-C_{n-1}C_n+1)\\&=C_{n-1}C_nC_{n+1}C_{n+2}-C_n^2C_{n+1}^2-C_{n+1}C_{n+2}+C_{n-1}C_n-1\\&=(C_n^2+8)(C_{n+1}^2+8)-C_n^2C_{n+1}^2-C_{n+1}(6C_{n+1}-C_n)+C_{n-1}C_n-1\\&=2C_{n+1}^2+8C_n^2+C_nC_{n+1}+C_{n-1}C_n+63>0,\end{aligned}$$

we have

$$\begin{aligned}\frac{1}{C_nC_{n+1}-C_{n-1}C_n+1}-\frac{1}{C_nC_{n+1}}&=\frac{C_{n-1}C_n-1}{C_nC_{n+1}(C_nC_{n+1}-C_{n-1}C_n+1)}\\&>\frac{1}{C_{n+1}C_{n+2}-C_nC_{n+1}+1}.\end{aligned}$$

Thus,

$$\frac{1}{C_nC_{n+1}-C_{n-1}C_n+1}>\frac{1}{C_nC_{n+1}}+\frac{1}{C_{n+1}C_{n+2}-C_nC_{n+1}+1}.$$

Iterating recursively, we get

$$\sum_{k=n}^{\infty}\frac{1}{C_kC_{k+1}}<\frac{1}{C_nC_{n+1}-C_{n-1}C_n+1}. \tag{3.5}$$

On the other hand,

$$\begin{aligned}\frac{1}{C_nC_{n+1}-C_{n-1}C_n+2}-\frac{1}{C_nC_{n+1}}&=\frac{C_{n-1}C_n-2}{C_nC_{n+1}(C_nC_{n+1}-C_{n-1}C_n+2)}\\&=\left[C_nC_{n+1}\left(\frac{C_nC_{n+1}}{C_{n-1}C_n-2}-1\right)\right]^{-1}.\end{aligned}$$

Since

$$\begin{aligned}&\frac{C_{n+1}C_{n+2}}{C_nC_{n+1}-2}-\frac{C_nC_{n+1}}{C_{n-1}C_n-2}\\&\quad=\frac{(C_n^2+8)(C_{n+1}^2+8)-2C_{n+1}C_{n+2}-C_n^2C_{n+1}^2+2C_nC_{n+1}}{(C_{n-1}C_n-2)(C_nC_{n+1}-2)}\\&\quad=\frac{-4C_{n+1}^2+4C_nC_{n+1}+8C_n^2+64}{(C_{n-1}C_n-2)(C_nC_{n+1}-2)}<0,\end{aligned}$$

we obtain

$$\frac{1}{C_nC_{n+1}-C_{n-1}C_n+2}-\frac{1}{C_nC_{n+1}}<\left[C_nC_{n+1}\left(\frac{C_{n+1}C_{n+2}}{C_nC_{n+1}-2}-1\right)\right]^{-1}$$

$$= \frac{C_nC_{n+1}-2}{C_nC_{n+1}(C_{n+2}C_{n+1}-C_nC_{n+1}+2)}$$
$$< \frac{1}{C_{n+2}C_{n+1}-C_nC_{n+1}+2}.$$

The last inequality can be rearranged as

$$\frac{1}{C_nC_{n+1}-C_{n-1}C_n+2} < \frac{1}{C_nC_{n+1}} + \frac{1}{C_{n+2}C_{n+1}-C_nC_{n+1}+2},$$

which on repeated iteration gives

$$\sum_{k=n}^{\infty}\frac{1}{C_kC_{k+1}} > \frac{1}{C_nC_{n+1}-C_{n-1}C_n+2}. \tag{3.6}$$

Inequalities (3.5) and (3.6) combined together gives

$$\frac{1}{C_nC_{n+1}-C_{n-1}C_n+2} < \sum_{k=n}^{\infty}\frac{1}{C_kC_{k+1}} < \frac{1}{C_nC_{n+1}-C_{n-1}C_n+1}.$$

This ends the proof. □

The following theorem can be proved in a similar fashion. However, the bounds are not so sharp as compared to those in the previous theorems.

Theorem 3.4. *For positive integers n and $r \geq 3$*

$$\frac{1}{C_n^r-(C_{n-1}-1)^r} < \sum_{k=n}^{\infty}\frac{1}{C_k^r} < \frac{1}{C_n^r-C_{n-1}^r}.$$

4. ADDITIONAL RESULTS

Using the techniques of last two sections, one can establish the following results for balancing and Lucas-balancing numbers.

Proposition 4.1.

(1) $$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{B_{2k}}\right)^{-1}\right\rfloor = B_{2n}-B_{2n-2}-1 \qquad (n\geq 1)$$

(2) $$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{B_{2k-1}}\right)^{-1}\right\rfloor = B_{2n-1}-B_{2n-3}-1 \qquad (n\geq 2)$$

(3) $$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{B_{2k-1}B_{2k+1}}\right)^{-1}\right\rfloor = B_{2n}^2-B_{2n-2}^2-2 \qquad (n\geq 1)$$

(4) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_{2k}B_{2k+2}} \right)^{-1} \right\rfloor = B_{2n+1}^2 - B_{2n-1}^2 - 2 \qquad (n \geq 1)$$

(5) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_{2k}^2} \right)^{-1} \right\rfloor = B_{2n}^2 - B_{2n-2}^2 - 1 \qquad (n \geq 1)$$

(6) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_{2k-1}^2} \right)^{-1} \right\rfloor = B_{2n-1}^2 - B_{2n-3}^2 - 1 \qquad (n \geq 2)$$

Proposition 4.2.

(1) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_{2k}} \right)^{-1} \right\rfloor = C_{2n} - C_{2n-2} \qquad (n \geq 1)$$

(2) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_{2k-1}} \right)^{-1} \right\rfloor = C_{2n-1} - C_{2n-3} \qquad (n \geq 2)$$

(3) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_{2k-1}C_{2k+1}} \right)^{-1} \right\rfloor = C_{2n}^2 - C_{2n-2}^2 + 8 \qquad (n \geq 1)$$

(4) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_{2k}C_{2k+2}} \right)^{-1} \right\rfloor = C_{2n+1}^2 - C_{2n-1}^2 + 8 \qquad (n \geq 1)$$

(5) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_{2k}^2} \right)^{-1} \right\rfloor = C_{2n}^2 - C_{2n-2}^2 \qquad (n \geq 1)$$

(6) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_{2k-1}^2} \right)^{-1} \right\rfloor = C_{2n-1}^2 - C_{2n-3}^2 \qquad (n \geq 2)$$

However, it seems difficult to get results involving higher power from Theorem 2.4 and Theorem 3.4.

Acknowledgments. The authors thank the anonymous referee for the careful reading of the manuscript.

REFERENCES

[1] A. Behera and G.K. Panda, *On the square roots of triangular numbers.* Fibonacci Quart. **37** (1999), 98–105.

[2] R. Finkelstein, *The house problem.* Amer. Math. Monthly **72** (1965), 1082–1088.

[3] S.H. Holliday and T. Komatsu, *On the sum of reciprocal generalized Fibonacci numbers.* Integers **11** (2011), 441–455.

[4] T. Komatsu and V. Laohakosol, *On the sum of reciprocals of numbers satisfying a recurrence relation of order s.* J. Integer Seq. **13** (2010), Article 10.5.8 (9 pages).

[5] T. Komatsu, *On the nearest integer of the sum of reciprocal Fibonacci numbers.* Aportaciones Mat. Investig. **20** (2011), 171–184.

[6] T. Komatsu and L. Szalay, *Balancing with binomial coefficients.* Int. J. Number Theory **10** (2014), 1729–1742.

[7] K. Kuhapatanakul, *On the sums of reciprocal generalized Fibonacci numbers.* J. Integer Seq. **16** (2013), Article 13.7.1 (8 pages).

[8] K. Liptai, *Fibonacci balancing numbers.* Fibonacci Quart. **42** (2004), 330–340.

[9] K. Liptai, F. Luca, Á. Pintér and L. Szalay, *Generalized balancing numbers.* Indag. Math. (N.S.) **20** (2009), 87–100.

[10] H. Ohtsuka and S. Nakamura, *On the sum of reciprocal Fibonacci numbers.* Fibonacci Quart. **46/47** (2008/2009), 153–159.

[11] O. Karaatli, R. Keskin and H. Zhu, *Infinitely many positive integer solutions of the quadratic Diophantine equation* $x^2 - 8B_nxy - 2y^2 = \pm 2r$. Irish Math. Soc. Bull. **73** (2014), 29–45.

[12] G.K. Panda, *Some fascinating properties of balancing numbers.* Congr. Numer. **194** (2009), 185–189.

[13] B.K. Patel and P.K. Ray, *The period, rank and order of the sequence of balancing numbers modulo m.* Math. Rep. (Bucur.) **18** (2016), *3*, 395–401.

[14] P.K. Ray, *Balancing and cobalancing numbers.* Ph.D. Thesis, National Institute of Technology, Rourkela, (2009), 100 p. Available at http://ethesis.nitrkl.ac.in/2750/1/Ph.D._Thesis_of_P.K._Ray..pdf.

[15] G.J. Zhang, *The infinite sum of reciprocal of Fibonacci numbers.* J. Math. Res. Exp. **31** (2011), 1030–1034.

Received 24 April 2016

National Institute of Technology,
Rourkela-769008, India
gkpanda_nit@rediffmail.com

Wuhan University,
School of Mathematics and Statistics,
Wuhan 430072, China
komatsu@whu.edu.cn

National Institute of Technology,
Rourkela-769008, India
davalaravikumar@gmail.com