Many authors studied bounds for reciprocal sums involving terms from Fibonacci and other related sequences. The purpose of this paper is to provide bounds for reciprocal sums with terms from balancing and Lucas-balancing sequences.

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1. INTRODUCTION

A positive integer $x$ is called balancing number if

$$1 + 2 + \cdots + (x - 1) = (x + 1) + \cdots + (y - 1)$$

holds for some integer $y \geq x + 2$. The problem of determining all balancing numbers leads to the Pell equation $y^2 - 8x^2 = 1$, whose solutions in $x$ can be described by the recurrence $B_n = 6B_{n-1} - B_{n-2}$ ($n \geq 2$) with $B_0 = 0$ and $B_1 = 1$ (see (e.g. [1, 2, 12]). Balancing numbers have been extensively studied by many authors. Karaatli et. al. [11] expressed the positive integral solutions of a Diophantine equation in terms of balancing numbers. Liptai [8] proved that there is no Fibonacci balancing number except 1. In [13], the period rank and order of the sequence of balancing numbers are studied. One of the most general extensions of the defining equation of balancing numbers is the Diophantine equation

$$1^k + 2^k + \cdots + (x - 1)^k = (x + 1)^l + \cdots + (y - 1)^l,$$

where the exponents $k$ and $l$ are given positive integers. In [9] effective and non-effective finiteness theorems for the above equation are proved. In [6] a balancing problem of ordinary binomial coefficients is studied, and effective and non-effective finiteness theorems are given.

The numbers $C_n = \sqrt{8B_n^2 + 1}$ is called the $n^{th}$ Lucas-balancing number [12], and these numbers satisfy the recurrence relation $C_n = 6C_{n-1} - C_{n-2}$ with
initial values $C_0 = 1, C_1 = 3$. The Binet forms for $B_n$ and $C_n$ are respectively

$$B_n = \frac{\alpha^n - \beta^n}{4\sqrt{2}}, \quad C_n = \frac{\alpha^n + \beta^n}{2}$$

where $\alpha = 3 + 2\sqrt{2}$ and $\beta = 3 - 2\sqrt{2}$. The balancing and Lucas-balancing numbers satisfy the identities

$$B_{n-r}B_{n+r} = B_n^2 - B_r^2, \quad C_{n-r}C_{n+r} = C_n^2 + C_r^2 - 1$$

for $n \geq r$ respectively. In particular, for $n \geq 1$ we have

$$B_{n-1}B_{n+1} = B_n^2 - 1 \quad \text{and} \quad C_{n-1}C_{n+1} = C_n^2 + 8.$$

The identity

$$B_1 + B_3 + \cdots + B_{2n-1} = B_n^2$$

gives

$$B_{2n-1} = B_n^2 - B_{n-1}^2.$$

The proofs of the above identities are available in [14]. In the subsequent sections, we shall use the above identities without further reference.

The intention of this paper is to develop certain interesting bounds for reciprocal sums with terms involving balancing and Lucas-balancing numbers in some combinations.

The reciprocal of partial infinite sums of reciprocal Fibonacci numbers has been extensively studied by many authors (e.g., see [4, 5, 7, 10, 15]). In [3], the following identities are shown for generalized Fibonacci numbers $G_n$, defined by

$$G_n = aG_{n-1} + G_{n-2} \quad (n \geq 2), \quad G_0 = 0, \quad G_1 = 1,$$

where $a$ is a positive integer. If $a = 1$, then $G_n$'s are equal to the Fibonacci numbers. Throughout this paper, integer part of a number is denoted by $\lfloor \cdot \rfloor$.

**Proposition 1.1.**

(1) $$\left[ \left( \sum_{k=n}^{\infty} \frac{1}{G_k} \right)^{-1} \right] = \begin{cases} G_n - G_{n-1} & \text{if } n \text{ is even and } n \geq 2; \\ G_n - G_{n-1} - 1 & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

(2) $$\left[ \left( \sum_{k=n}^{\infty} \frac{1}{G_k^2} \right)^{-1} \right] = \begin{cases} aG_{n-1}G_n - 1 & \text{if } n \text{ is even and } n \geq 2; \\ aG_{n-1}G_n & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

(3) $$\left[ \left( \sum_{k=n}^{\infty} \frac{1}{G_{2k}} \right)^{-1} \right] = G_{2n} - G_{2n-2} - 1 \quad (n \geq 1)$$

(4) $$\left[ \left( \sum_{k=n}^{\infty} \frac{1}{G_{2k-1}} \right)^{-1} \right] = G_{2n-1} - G_{2n-3} \quad (n \geq 2)$$
In this paper, we shall show some analogous results for the sequences of balancing and Lucas-balancing numbers.

**Proposition 1.2.**

1. \[
\left(\sum_{k=n}^{\infty} \frac{1}{B_k}\right)^{-1} = B_n - B_{n-1} - 1 \quad (n \geq 1)
\]

2. \[
\left(\sum_{k=n}^{\infty} \frac{1}{B_k^2}\right)^{-1} = B_n^2 - B_{n-1}^2 - 1 = B_{2n-1} - 1 \quad (n \geq 1)
\]

3. \[
\left(\sum_{k=n}^{\infty} \frac{1}{B_kB_{k+1}}\right)^{-1} = B_nB_{n+1} - B_{n-1}B_n - 1 \quad (n \geq 1)
\]

4. \[
\left(\sum_{k=n}^{\infty} \frac{1}{C_k}\right)^{-1} = C_n - C_{n-1} \quad (n \geq 2)
\]

5. \[
\left(\sum_{k=n}^{\infty} \frac{1}{C_k^2}\right)^{-1} = C_n^2 - C_{n-1}^2 \quad (n \geq 1)
\]

6. \[
\left(\sum_{k=n}^{\infty} \frac{1}{C_kC_{k+1}}\right)^{-1} = C_nC_{n+1} - C_{n-1}C_n + 1 \quad (n \geq 1)
\]
2. RECIPROCAL SUMS INVOLVING BALANCING NUMBERS

In this section, we establish bounds for several reciprocal sums involving balancing numbers. By using the bounds, we obtain the results in Proposition 1.2 (1), (2) and (3).

The following theorem gives sharp bounds for reciprocal sums of balancing numbers.

**Theorem 2.1.** For any positive integer $n$,

\[
\frac{1}{B_n - B_{n-1}} < \sum_{k=n}^{\infty} \frac{1}{B_k} < \frac{1}{B_n - B_{n-1} - 1}.
\]

**Proof.** For any positive integer $n$,

\[
\frac{1}{B_n - B_{n-1}} - \frac{1}{B_n} = \frac{B_{n-1}}{B_n^2 - B_n B_{n-1}}.
\]

Thus,

\[
\frac{1}{B_n - B_{n-1}} < \frac{1}{B_n} + \frac{1}{B_{n+1} - B_n}.
\]

Repeating the above steps, we can show that

\[
\frac{1}{B_{n+1} - B_n} < \frac{1}{B_{n+1}} + \frac{1}{B_{n+2} - B_{n+1}}.
\]

Combining the above two inequalities, we get

\[
\frac{1}{B_n - B_{n-1}} < \frac{1}{B_n} + \frac{1}{B_{n+1}} + \frac{1}{B_{n+2} - B_{n+1}}.
\]

Continuing in this manner, one can arrive at the inequality

\[
(2.1) \quad \frac{1}{B_n - B_{n-1}} < \sum_{k=n}^{\infty} \frac{1}{B_k}.
\]

On the other hand,

\[
\frac{1}{B_n - B_{n-1} - 1} - \frac{1}{B_n} = \frac{B_{n-1} + 1}{B_n^2 - B_n B_{n-1} - B_n}.
\]

Thus,

\[
\frac{1}{B_n - B_{n-1} - 1} - \frac{1}{B_n} < \frac{B_{n-1} + 1}{B_n - B_{n-1} - 1}.
\]

On the other hand,
\[
\frac{1}{B_{n+1} - B_n - 1},
\]
which gives
\[
\frac{1}{B_n - B_{n-1} - 1} > \frac{1}{B_n} + \frac{1}{B_{n+1} - B_n - 1}.
\]
Continuing in a similar fashion, we finally obtain
\[
(2.2) \quad \frac{1}{B_n - B_{n-1} - 1} > \sum_{k=n}^{\infty} \frac{1}{B_k}.
\]
Combining (2.1) and (2.2), we get the desired inequality as stated in the theorem. □

The following theorem provides sharp bounds for the reciprocal sum of squares of balancing numbers.

**Theorem 2.2.** For any positive integer \( n \),
\[
\frac{1}{B_{2n-1}} < \sum_{k=n}^{\infty} \frac{1}{B^2_k} < \frac{1}{B_{2n-1} - 1}.
\]

**Proof.** Since
\[
\frac{B_n}{B_{n-1}} - \frac{B_{n+1}}{B_n} = \frac{1}{B_n B_{n-1}} > 0,
\]
for each \( n > 1 \), we have \( \frac{B_n}{B_{n-1}} > \frac{B_{n+1}}{B_n} \). Thus, for each \( n > 1 \),
\[
\frac{1}{B_{2n-1}} - \frac{1}{B^2_n} = \frac{1}{B^2_n - B^2_{n-1}} - \frac{1}{B^2_n} = \frac{B^2_{n-1}}{B^2_n (B^2_n - B^2_{n-1})} = \frac{1}{B^2_n (\frac{B^2_n}{B^2_{n-1}} - 1)} < \frac{1}{B^2_n (\frac{B^2_{n+1}}{B^2_n} - 1)} = \frac{1}{B^2_{n+1} - B^2_n} = \frac{1}{B_{2n+1}},
\]
which yields
\[
\frac{1}{B_{2n-1}} < \frac{1}{B^2_n} + \frac{1}{B_{2n+1}}.
\]
This inequality is also valid for $n = 1$. Recursive iteration of the last inequality gives

\begin{equation}
\frac{1}{B_{2n-1}} < \sum_{k=n}^{\infty} \frac{1}{B_k^2}.
\end{equation}

On the other hand,

\[
\frac{1}{B_{2n-1} - 1} - \frac{1}{B_n^2} = \frac{1}{B_n^2 - B_{n-1}^2 - 1} - \frac{1}{B_n^2} = \frac{B_n^2 + 1}{B_n^2(B_n^2 - B_{n-1}^2 - 1)}.
\]

For $n \geq 1$,

\[
(B_{2n+1} - 1)(B_{n-1}^2 + 1) - B_n^2(B_n^2 - B_{n-1}^2 - 1)
= (B_{n+1}^2 - B_n^2 - 1)(B_{n-1}^2 + 1) - B_n^2(B_n^2 - B_{n-1}^2 - 1)
= B_{n+1}^2B_{n-1}^2 + B_{n+1}^2 - B_n^4 - B_{n-1}^2 - 1
= (B_n^2 - 1)^2 + (6B_n - B_{n-1})^2 - B_n^4 - B_{n-1}^2 - 1
= 34B_n^2 - 12B_nB_{n-1} > 0.
\]

Thus, we have

\[
\frac{1}{B_{2n-1} - 1} > \frac{1}{B_n^2} + \frac{1}{B_{2n+1} - 1}.
\]

Iterating recursively, we get

\begin{equation}
\frac{1}{B_{2n-1} - 1} > \sum_{k=n}^{\infty} \frac{1}{B_k^2}.
\end{equation}

Combining inequalities (2.3) and (2.4), we get what has been claimed. \(\square\)

The reciprocal sum of products of two consecutive balancing numbers has analogous bounds. The following theorem is important in this regard.

**Theorem 2.3.** For any positive integer $n$,

\[
\frac{1}{B_nB_{n+1} - B_{n-1}B_n} < \sum_{k=n}^{\infty} \frac{1}{B_kB_{k+1}} < \frac{1}{B_nB_{n+1} - B_{n-1}B_n - 1}.
\]

**Proof.** Using the fact $\frac{B_{n-1}}{B_n} < \frac{B_n}{B_{n+1}}$ $n \geq 1$, we have

\[
\frac{1}{B_nB_{n+1} - B_{n-1}B_n} - \frac{1}{B_nB_{n+1}} = \frac{B_{n-1}}{B_n} \left( \frac{1}{B_{n+1} - B_{n-1}B_{n+1}} \right)
< \frac{B_n}{B_{n+1}} \left( \frac{1}{B_nB_{n+2} - B_n^2 + 2} \right).
\]
Thus,\[
\frac{1}{B_n B_{n+1} - B_{n-1} B_n} < \frac{1}{B_n B_{n+1}} + \frac{1}{B_{n+1} B_{n+2} - B_n B_{n+1}}.
\]
Iterating recursively, we get
\[
\sum_{k=n}^{\infty} \frac{1}{B_k B_{k+1}} > \frac{1}{B_n B_{n+1} - B_{n-1} B_n}.
\]
On the other hand,\[
\frac{1}{B_n B_{n+1} - B_{n-1} B_n - 1} - \frac{1}{B_n B_{n+1}} = \frac{B_{n-1} B_n + 1}{B_n B_{n+1} (B_n B_{n+1} - B_{n-1} B_n - 1)}
\]
\[
= \frac{B_n B_{n+1} + 1}{B_n B_{n+1} (B_n B_{n+1} + 1)}
\]
\[
= \frac{B_n B_{n+1} + 1}{B_n B_{n+1} (B_n B_{n+1} + 1)}
\]
\[
= \left[ \frac{B_n B_{n+1}}{B_n B_{n+1} + 1} \right]^{-1}
\]
\[
\begin{align*}
\frac{B_n B_{n+1} - B_{n-1} B_n + 1}{B_n B_{n+1} + 1} &= \frac{(B_n^2 - 1)(B_{n+1}^2 - 1) + B_n B_{n+1} - B_{n-1} B_n + 1}{B_n B_{n+1} + 1} \\
&= \frac{5B_n^2 - 2B_n B_{n+1} - B_n^2 + 1}{(B_n B_{n+1} + 1)^2} > 0,
\end{align*}
\]
we have\[
\frac{1}{B_n B_{n+1} - B_{n-1} B_n - 1} - \frac{1}{B_n B_{n+1}} > \left[ \frac{B_n B_{n+1}}{B_n B_{n+1} + 1} \right]^{-1}
\]
\[
= \frac{B_{n+1} B_{n+2}}{B_{n+2} B_{n+1} + 1} \frac{B_n B_{n+1} + 1}{B_n B_{n+1} (B_{n+2} B_{n+1} - B_{n+1} B_{n+1} - 1)}
\]
\[
> \frac{1}{B_{n+2} B_{n+1} - B_n B_{n+1} - 1}.
\]
The last inequality can be rearranged as\[
\frac{1}{B_n B_{n+1} - B_{n-1} B_n - 1} > \frac{1}{B_n B_{n+1}} + \frac{1}{B_{n+2} B_{n+1} - B_n B_{n+1} - 1}.
\]
Repeated iteration gives
\[
\sum_{k=n}^{\infty} \frac{1}{B_k B_{k+1}} < \frac{1}{B_n B_{n+1} - B_{n-1} B_n - 1}.
\]
Combining the inequalities (2.5) and (2.6) together, we get
\[ \frac{1}{B_nB_{n+1} - B_{n-1}B_n} < \sum_{k=n}^{\infty} \frac{1}{B_kB_{k+1}} < \frac{1}{B_nB_{n+1} - B_{n-1}B_n - 1}. \]
This ends the proof. \(\square\)

The following theorem can be proved in a similar fashion. However, the bounds are not as sharp as those in the previous theorems.

**Theorem 2.4.** For positive integers \(n\) and \(r \geq 3\),
\[ \frac{1}{B^n - B^{n-1}} < \sum_{k=n}^{\infty} \frac{1}{B^k} < \frac{1}{B^n - (B^{n-1} - 1)^r}. \]

### 3. Reciprocal Sums Involving Lucas-Balancing Numbers

In this section, we shall establish certain bounds for the reciprocal sums involving Lucas-balancing numbers. By using these bounds, we obtain the results in Proposition 1.2 (4), (5) and (6).

The following theorem provides sharp bounds for reciprocal sums of the Lucas-balancing numbers.

**Theorem 3.1.** For any positive integer \(n \geq 2\),
\[ \frac{1}{C_n - C_{n-1} + 1} < \sum_{k=n}^{\infty} \frac{1}{C_k} < \frac{1}{C_n - C_{n-1}}. \]

*Proof.* For any positive integer \(n \geq 2\),
\[ \frac{1}{C_n - C_{n-1} + 1} - \frac{1}{C_n} = \frac{C_{n-1}}{C_n^2 - C_nC_{n-1}} = \frac{C_{n-1}}{C_{n-1}(C_{n+1} - C_n) - 8} > \frac{1}{C_{n+1} - C_n}. \]
Thus,
\[ \frac{1}{C_n - C_{n-1}} > \frac{1}{C_n} + \frac{1}{C_{n+1} - C_n}. \]
Repeating the above steps, we can obtain
\[ \frac{1}{C_{n+1} - C_n} > \frac{1}{C_{n+1}} + \frac{1}{C_{n+2} - C_{n+1}}. \]
Combining the above two inequalities, we get
\[ \frac{1}{C_n - C_{n-1}} > \frac{1}{C_n} + \frac{1}{C_{n+1}} + \frac{1}{C_{n+2} - C_{n+1}}. \]
Continuing in this manner, one can arrive at the inequality

\[ \frac{1}{C_n - C_{n-1}} > \sum_{k=n}^{\infty} \frac{1}{C_k}. \]

On the other hand, since

\[
\frac{1}{C_n - C_{n-1} + 1} - \frac{1}{C_n} = \frac{C_n - 1}{C_n(C_n - C_{n-1} + 1) + C_n}
\]

\[
= \frac{C_n - 1}{C_n - C_{n-1}} + C_n
\]

\[
< \frac{1}{C_n - C_{n-1}} - 1
\]

\[
= \frac{1}{C_n - C_{n-1} + 1},
\]

we have

\[ \frac{1}{C_n - C_{n-1} + 1} < \frac{1}{C_n} + \frac{1}{C_n - C_{n-1} + 1}. \]

Continuing in the same way, one can obtain

\[ \frac{1}{C_n - C_{n-1} + 1} < \sum_{k=n}^{\infty} \frac{1}{C_k}. \]

Combining inequalities (3.1) and (3.2), we get the desired inequality as stated in the theorem. □

The following theorem provides sharp bounds for the reciprocal sum of squares of Lucas-balancing numbers.

**Theorem 3.2.** For any positive integer \( n \),

\[ \frac{1}{C_n^2 - C_{n-1}^2 + 1} < \sum_{k=n}^{\infty} \frac{1}{C_k^2} < \frac{1}{C_n^2 - C_{n-1}^2}. \]

**Proof.** For each \( n \geq 1 \)

\[ \frac{C_n}{C_{n-1}} - \frac{C_{n+1}}{C_n} = \frac{-8}{C_{n-1}C_n} < 0, \]

we have \( \frac{C_n}{C_{n-1}} < \frac{C_{n+1}}{C_n} \). Thus for each \( n \geq 1 \),

\[
\frac{1}{C_n^2 - C_{n-1}^2} - \frac{1}{C_n^2} = \frac{C_{n-1}^2}{C_n^2(C_n^2 - C_{n-1}^2)}
\]

\[
= \frac{1}{C_n^2 \left( \frac{C_n^2}{C_{n-1}^2} - 1 \right)}
\]
\[
\frac{1}{C_n} \left( \frac{C_{n+1}}{C_n} - 1 \right) > \frac{1}{C_{n+1}^2 - C_n^2} = \frac{1}{C_{n+1}^2 - C_n^2}.
\]

The above inequality yields
\[
\frac{1}{C_n} - \frac{1}{C_{n+1}^2 - C_n^2} > \frac{1}{C_{n+1}^2 - C_n^2} + \frac{1}{C_{n+1}^2 - C_n^2}
\]
and iterating recursively, we get
\[
(3.3) \quad \frac{1}{C_n} - \frac{1}{C_{n+1}^2 - C_n^2} > \sum_{k=n}^{\infty} \frac{1}{C_k^2}.
\]

On the other hand, for \( n \geq 1 \),
\[
\frac{1}{C_n^2 - C_{n-1}^2} + 1 - \frac{1}{C_n^2} = \frac{C_{n+1}^2 - 1}{C_n^2 (C_n^2 - C_{n-1}^2 + 1)} = C_n^{-2} \left( \frac{C_n^2}{C_{n-1}^2 - 1} - 1 \right)^{-1}.
\]

If we set \( x_n = \frac{C_n^2}{C_{n-1}^2 - 1} \), then
\[
x_{n+1} - x_n = \frac{C_{n+1}^2}{C_n^2 - 1} - \frac{C_n^2}{C_{n-1}^2 - 1} = \frac{17C_n^2 - C_{n+1}^2 + 64}{(C_n^2 - 1)(C_{n-1}^2 - 1)} < 0.
\]

Hence,
\[
C_n^{-2} \left( \frac{C_n^2}{C_{n-1}^2 - 1} - 1 \right)^{-1} < C_n^{-2} \left( \frac{C_{n+1}^2}{C_n^2 - 1} - 1 \right)^{-1}
\]
\[
= \frac{C_n^2 - 1}{C_n^2 (C_{n+1}^2 - C_n^2 + 1)} < \frac{1}{C_{n+1}^2 - C_n^2 + 1}.
\]

Thus, we have
\[
\frac{1}{C_n^2 - C_{n-1}^2 + 1} < \frac{1}{C_n^2} + \frac{1}{C_{n+1}^2 - C_n^2 + 1}.
\]

Iterating recursively, we get
\[
(3.4) \quad \frac{1}{C_n^2 - C_{n-1}^2 + 1} < \sum_{k=n}^{\infty} \frac{1}{C_k^2}.
\]

Combining inequalities (3.3) and (3.4), we get what has been claimed. \( \square \)

The reciprocal sum of products of two consecutive Lucas-balancing numbers has analogous bounds. The following theorem is important in this regard.
Theorem 3.3. For any positive integer \( n \),
\[
\frac{1}{C_nC_{n+1} - C_{n-1}C_n + 2} < \sum_{k=n}^{\infty} \frac{1}{C_kC_{k+1}} < \frac{1}{C_nC_{n+1} - C_{n-1}C_n + 1}.
\]

Proof. Since for each positive integer \( n \),
\[
(C_{n-1}C_n - 1)(C_{n+1}C_{n+2} - C_nC_{n+1} + 1) - C_nC_{n+1}(C_nC_{n+1} - C_{n-1}C_n + 1)
= C_nC_nC_{n+1}C_{n+2} - C_n^2C_{n+1}^2 - C_{n+1}^2C_n^2 + C_{n-1}C_n - 1
= (C_n^2 + 8)(C_{n+1}^2 + 8) - C_n^2C_{n+1}^2 - C_{n+1}(6C_n - C_n) + C_{n-1}C_n - 1
= 2C_{n+1}^2 + 8C_n^2 + C_nC_{n+1} + C_{n-1}C_n + 63 > 0,
\]
we have
\[
\frac{1}{C_nC_{n+1} - C_{n-1}C_n + 1} - \frac{1}{C_nC_{n+1}} = \frac{C_nC_{n-1} - 1}{C_nC_{n+1}(C_nC_{n+1} - C_{n-1}C_n + 1)} > \frac{1}{C_{n+1}C_{n+2} - C_nC_{n+1} + 1}.
\]
Thus,
\[
\frac{1}{C_nC_{n+1} - C_{n-1}C_n + 1} > \frac{1}{C_nC_{n+1}} + \frac{1}{C_{n+1}C_{n+2} - C_nC_{n+1} + 1}.
\]
Iterating recursively, we get
\[
\sum_{k=n}^{\infty} \frac{1}{C_kC_{k+1}} < \frac{1}{C_nC_{n+1} - C_{n-1}C_n + 1}.
\]

On the other hand,
\[
\frac{1}{C_nC_{n+1} - C_{n-1}C_n + 2} - \frac{1}{C_nC_{n+1}} = \frac{C_nC_{n-1} - 2}{C_nC_{n+1}(C_nC_{n+1} - C_{n-1}C_n + 2)} = \left[ \frac{C_nC_{n+1}}{C_{n-1}C_n - 2} \right]^{-1}.
\]
Since
\[
\frac{C_{n+1}C_{n+2}}{C_nC_{n+1} - 2} - \frac{C_nC_{n+1}}{C_{n-1}C_n - 2}
= \frac{(C_n^2 + 8)(C_{n+1}^2 + 8) - 2C_{n+1}^2C_n + 2C_n^2 + 2C_{n+1}}{(C_{n-1}C_n - 2)(C_nC_{n+1} - 2)} = \frac{-4C_{n+1}^2 + 4C_nC_{n+1} + 8C_n^2 + 64}{(C_{n-1}C_n - 2)(C_nC_{n+1} - 2)} < 0,
\]
we obtain
\[
\frac{1}{C_nC_{n+1} - C_{n-1}C_n + 2} - \frac{1}{C_nC_{n+1}} < \left[ \frac{C_nC_{n+1}}{C_{n-1}C_n - 2} \right]^{-1}.
\]
\[ \frac{C_n C_{n+1} - 2}{C_n C_{n+1} (C_{n+2} C_{n+1} - C_n C_{n+1} + 2)} < \frac{1}{C_{n+2} C_{n+1} - C_n C_{n+1} + 2} \]

The last inequality can be rearranged as

\[ \frac{1}{C_n C_{n+1} - C_{n-1} C_n + 2} < \frac{1}{C_n C_{n+1}} + \frac{1}{C_{n+2} C_{n+1} - C_n C_{n+1} + 2} \]

which on repeated iteration gives

\[ \sum_{k=n}^{\infty} \frac{1}{C_k C_{k+1}} > \frac{1}{C_n C_{n+1} - C_{n-1} C_n + 2}. \]  

Inequalities (3.5) and (3.6) combined together gives

\[ \frac{1}{C_n C_{n+1} - C_{n-1} C_n + 2} < \sum_{k=n}^{\infty} \frac{1}{C_k C_{k+1}} < \frac{1}{C_n C_{n+1} - C_{n-1} C_n + 1}. \]

This ends the proof. \( \square \)

The following theorem can be proved in a similar fashion. However, the bounds are not so sharp as compared to those in the previous theorems.

**Theorem 3.4.** For positive integers \( n \) and \( r \geq 3 \)

\[ \frac{1}{C_n^r - (C_{n-1} - 1)^r} < \sum_{k=n}^{\infty} \frac{1}{C_k^r} < \frac{1}{C_n^r - C_{n-1}^r}. \]

### 4. ADDITIONAL RESULTS

Using the techniques of last two sections, one can establish the following results for balancing and Lucas-balancing numbers.

**Proposition 4.1.**

1. \[ \left( \sum_{k=n}^{\infty} \frac{1}{B_{2k}} \right)^{-1} = B_{2n} - B_{2n-2} - 1 \quad (n \geq 1) \]
2. \[ \left( \sum_{k=n}^{\infty} \frac{1}{B_{2k-1}} \right)^{-1} = B_{2n-1} - B_{2n-3} - 1 \quad (n \geq 2) \]
3. \[ \left( \sum_{k=n}^{\infty} \frac{1}{B_{2k-1} B_{2k+1}} \right)^{-1} = B_{2n}^2 - B_{2n-2}^2 - 2 \quad (n \geq 1) \]
\[
\begin{align*}
(4) \quad & \left( \sum_{k=n}^{\infty} \frac{1}{B_{2k}B_{2k+2}} \right)^{-1} = B_{2n+1}^2 - B_{2n-1}^2 - 2 \quad (n \geq 1) \\
(5) \quad & \left( \sum_{k=n}^{\infty} \frac{1}{B_{2k}^2} \right)^{-1} = B_{2n}^2 - B_{2n-2}^2 - 1 \quad (n \geq 1) \\
(6) \quad & \left( \sum_{k=n}^{\infty} \frac{1}{B_{2k-1}^2} \right)^{-1} = B_{2n-1}^2 - B_{2n-3}^2 - 1 \quad (n \geq 2)
\end{align*}
\]

**Proposition 4.2.**

\[
\begin{align*}
(1) \quad & \left( \sum_{k=n}^{\infty} \frac{1}{C_{2k}} \right)^{-1} = C_{2n} - C_{2n-2} \quad (n \geq 1) \\
(2) \quad & \left( \sum_{k=n}^{\infty} \frac{1}{C_{2k-1}} \right)^{-1} = C_{2n-1} - C_{2n-3} \quad (n \geq 2) \\
(3) \quad & \left( \sum_{k=n}^{\infty} \frac{1}{C_{2k-1}C_{2k+1}} \right)^{-1} = C_{2n}^2 - C_{2n-2}^2 + 8 \quad (n \geq 1) \\
(4) \quad & \left( \sum_{k=n}^{\infty} \frac{1}{C_{2k}C_{2k+2}} \right)^{-1} = C_{2n+1}^2 - C_{2n-1}^2 + 8 \quad (n \geq 1) \\
(5) \quad & \left( \sum_{k=n}^{\infty} \frac{1}{C_{2k}^2} \right)^{-1} = C_{2n}^2 - C_{2n-2}^2 \quad (n \geq 1) \\
(6) \quad & \left( \sum_{k=n}^{\infty} \frac{1}{C_{2k-1}^2} \right)^{-1} = C_{2n-1}^2 - C_{2n-3}^2 \quad (n \geq 2)
\end{align*}
\]

However, it seems difficult to get results involving higher power from Theorem 2.4 and Theorem 3.4.

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**REFERENCES**


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