THE PROBLEM OF DENSITY ON COMMUTATIVE STRONG HYPERGROUPS

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In this paper, we give some sufficient conditions for the set $R(\{\pm\langle x, .\rangle f : x \in K\})$ to be dense in $L^2(\hat{K}, \pi)$, where K is a commutative strong hypergroup.

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1. INTRODUCTION AND PRELIMINARIES

Let X be a Banach space and $M \subseteq X$. The problem of finding necessary or sufficient conditions on M and X for the additive subsemigroup of X generated by M, that is $R(M) := \left\{\sum_{j=1}^{n} x_j : n \in \mathbb{N}, x_j \in M\right\}$, to be dense in X has been studied in several papers (see [2], [3], and [10]). In [9], we consider sets of the type

$$R(f) = \{\sum_j \epsilon_j f_{x_j} : x_j \in G, \epsilon_j \in \{\pm 1\}\} \subseteq L^2(G)$$

obtained from a single function f by translations, and give criteria for the density of such subsets in $L^2(G)$, where G is a locally compact abelian group. In this paper, as an application of the extended Wiener's theorem [9], we develop the main results of [9] to locally compact hypergroups, and we give some sufficient conditions for the set $R(\{\pm\langle x, .\rangle f : x \in K\})$ to be dense in $L^2(\hat{K}, \pi)$, where K is a commutative strong hypergroup. Our main motivation for this generalization is the Corollary 2 in [3] in which P. Borodin gives some sufficient conditions for the additive subgroup generated by the translations of a single function to be dense in $L^2(\mathbb{R})$. Locally compact hypergroups, as extensions of locally compact groups, were introduced in a series of papers by R.I. Jewett [6], C.F. Dunkl [4], and R. Spector [7] in the 70's. Examples include locally compact groups, double-coset hypergroups, G_H hypergroups, polynomial hypergroups, etc. (see [1] and [6] for more details).

Let K be a locally compact Hausdorff space. We denote by $M(K)$ the space of all regular complex Borel measures on K, and by δ_x the Dirac measure at the point x. The support of a measure $\mu \in M(K)$ is denoted by $\mathrm{supp}(\mu)$.

Definition 1.1. Suppose that K is a locally compact Hausdorff space, $(\mu, \nu) \mapsto \mu * \nu$ is a bilinear positive-continuous mapping from $M(K) \times M(K)$ into $M(K)$ (called convolution), and $x \mapsto x^-$ is an involutive homeomorphism on K (called involution) with the following properties:

(i) $M(K)$ with $*$ is a complex associative algebra;

(ii) if $x, y \in K$, then $\delta_x * \delta_y$ is a probability measure with compact support;

(iii) the mapping $(x, y) \mapsto \mathrm{supp}(\delta_x * \delta_y)$ from $K \times K$ into $\mathbf{C}(K)$ is continuous, where $\mathbf{C}(K)$ is the set of all non-empty compact subsets of K equipped with Michael topology;

(iv) there exists a (necessarily unique) element $e \in K$ (called identity) such that for all $x \in K$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$;

(v) for all $x, y \in K$, $e \in \mathrm{supp}(\delta_x * \delta_y)$ if and only if $x = y^-$;

Then $K \equiv (K, *, ^-, e)$ is called a locally compact hypergroup.

Note that in a hypergroup K, $\delta_x * \delta_y$ is not necessarily a Dirac measure. A hypergroup K is called commutative if for all $x, y \in K$, $\delta_x * \delta_y = \delta_y * \delta_x$. Every commutative hypergroup carries a nonzero nonnegative regular Borel measure m, such that for each $x \in K$, $\delta_x * m = m$ [8]; such measure m is called (left) Haar measure. Let f and g be complex-valued Borel functions on K. For any $x, y \in K$ we define

$$L_x f(y) = f(x * y) := \int_K f \, \mathrm{d}(\delta_x * \delta_y) \quad \text{and} \quad (f * g)(x) := \int_K f(x * y) g(y^-) \, \mathrm{d}m(y).$$

Let K be a commutative hypergroup. A bounded continuous function $\xi : K \to \mathbb{C}$ is called a character if for all $x, y \in K$, $\xi(x^-) = \overline{\xi(x)}$ and $\xi(x * y) = \xi(x)\xi(y)$. The set of all characters of K equipped with uniform convergence topology on compact subsets of K is called dual of K and is denoted by $\hat{K}$. In general, $\hat{K}$ is not necessarily a hypergroup. A hypergroup K is called strong if its dual $\hat{K}$ is also a hypergroup with complex conjugation as involution, pointwise product as convolution, that is

$$\xi(x)\eta(x) = \int_{\hat{K}} \chi(x) \, \mathrm{d}(\delta_\xi * \delta_\eta)(\chi)$$

where $\xi, \eta \in \hat{K}$ and $x \in K$, and constant function 1 as the identity element. Throughout this paper, K is a commutative strong hypergroup with Haar measure m. By [1], there is a Plancherel measure π on $\hat{K}$ such that for each $f \in L^1(K, m) \cap L^2(K, m)$, $\int_K |f|^2 \, \mathrm{d}m = \int_{\hat{K}} |\hat{f}|^2 \, d\pi$, where $\hat{f}(\xi) :=$

$\int_K f(x)\overline{\xi(x)}\,\mathrm{d}m(x)$ $(\xi \in \hat{K})$ is the Fourier transform of f. The Fourier transform can be extend to an isometric isomorphism from $L^2(K,m)$ onto $L^2(\hat{K},\pi)$.

2. MAIN RESULTS

The following extension of the Wiener's classical theorem is a key tool in the proof of our main theorem.

THEOREM 2.1. *Let K be a locally compact commutative strong hypergroup with Haar measure m and associated Plancherel measure π, $\epsilon > 0$, and $A := \{\sum_{j=1}^{n} \lambda_j \langle x_j, \cdot\rangle : n \in \mathbb{N}, \lambda_j \in \mathbb{C}, x_j \in K\ (j = 1,\dots,n)\}$, where for each $x \in K$ and $\xi \in \hat{K}$, $\langle x, \xi\rangle := \xi(x)$. If $k_1 \in L^2(\hat{K},\pi)$ has a null zeros set with respect to Plancherel measure π, then for each $k_2 \in L^2(\hat{K},\pi)$ there exists an element $\psi \in A$ such that $\|k_2 - \psi k_1\|_2 < \epsilon$.*

Proof. See [9]. □

Definition 2.2. A Banach space X with unit sphere $S(X)$ is said to be uniformly smooth if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x \in S(X)$ and $\|y\| < \delta$ then $\|x + y\| + \|x - y\| \le 2 + \epsilon\|y\|$.

Remark 2.3. A Banach space X is uniformly smooth if and only if its modulus of smoothness

$$s(\tau) := \sup\left\{\left\|\frac{x+y}{2}\right\| + \left\|\frac{x-y}{2}\right\| - 1 : \|x\| = 1, \|y\| = \tau\right\}, \quad \tau \ge 0,$$

satisfies $s(\tau) = o(\tau)$ as $\tau \to 0$. Every Hilbert space is uniformly smooth with modulus of smoothness $s(\tau) = \sqrt{1+\tau^2} - 1$. [5]

Here we recall the following lemma from [3].

LEMMA 2.4. *Let F be a closed additive subgroup in a uniformly smooth Banach space X with modulus of smoothness $s(\tau)$, $\tau \ge 0$. Suppose that $a, b \in F$ and for every $\epsilon > 0$ there are $x_0, x_1, \dots, x_n \in F$ such that $x_0 = a$, $x_n = b$ and $\sum_{k=1}^{n} s(\|x_k - x_{k-1}\|) < \epsilon$. Then $[a,b] \subseteq F$, where $[a,b] = \{(1-t)a + tb : 0 \le t \le 1\}$.*

Definition 2.5. Suppose that K is a locally compact commutative hypergroup, $f \in L^2(\hat{K})$ and U is a neighborhood of e in K. Then we define

$$\omega_2(f,U) := \sup\left\{\int_{\hat{K}} |f(\xi)|^2 |\xi(a) - \xi(b)|^2 \,\mathrm{d}\pi(\xi) : a, b \in K \text{ and } \{a\} * \{b^-\} \subseteq U\right\}.$$

We denote $\omega_2(f,U) = o(U)$ as $U \to \{e\}$ if $\lim_{U\to\{e\}} \frac{\omega_2(f,U)}{m(U)} = 0$.

Remark 2.6. Let K be a locally compact commutative hypergroup with left Haar measure m, and $B>0$ be a real number. We say that K satisfies in B-condition if for all $a,b\in K$, and each compact neighborhood W of e in K, there exist compact symmetric neighborhoods $V_1,\ldots,V_n\subseteq W$ of e in K such that

(1) $\sum_{k=1}^{n} m(V_k)\leq B$,
(2) there are $x_0,\ldots,x_n\in K$ such that $x_0=a$, $x_n=b$, and for every $k=1,\ldots,n$, $\{x_k\}*\{x_{k-1}^{-}\}\subseteq V_k$.

THEOREM 2.7. *Suppose that $f\in L^2(\hat{K})$ and the integral modulus of continuity of f satisfies $\omega_2(f,U)=o(U)$ as $U\to\{e\}$. Let for a $B>0$, K satisfy in B-condition. Then for each $y\in K$ and $\beta\in\mathbb{R}$ we have $\beta\cdot(\langle y,.\rangle f-f)\in\overline{R(\{\pm\langle x,.\rangle f: x\in K\})}^{\|\cdot\|_2}$.*

Proof. Under the hypothesis, let F be the closure of the set $R(\{\pm\langle x,.\rangle f: x\in K\})$ in $L^2(\hat{K})$. Then F is a closed additive subgroup of $L^2(\hat{K})$. Since $L^2(\hat{K})$ is a Hilbert space, its modulus of smoothness is $s(\tau)=\sqrt{1+\tau^2}-1$, and so $L^2(\hat{K})$ is an uniformly smooth Banach space. Let $y\in K$ and $\lambda\in\mathbb{C}$. We show that $\beta\cdot(\lambda\cdot\langle y,.\rangle f-f)\in F$. Consider $f_1\in L^2(K)$ such that $\widehat{f_1}=f$. For any $x\in K$, we define the function $\phi: K\to L^2(\hat{K})$ as $\phi(t):=\widehat{L_xL_tf_1}$ $(t\in K)$. For each $a,b\in K$,

$$\begin{aligned}
s(\|\phi(a)-\phi(b)\|_2) &= s(\|L_xL_af_1-L_xL_bf_1\|_2)\\
&= s(\|L_x\left(L_af_1-L_bf_1\right)\|_2)\\
&\leq s(\|L_af_1-L_bf_1\|_2) \quad \text{(by ([6], 3.3B)}\\
&= s(\|\widehat{L_af_1}-\widehat{L_bf_1}\|_2) \quad \text{(by the Plancherel Theorem)}\\
&\leq \frac{1}{2}\,\|\widehat{L_af_1}-\widehat{L_bf_1}\|_2^2\\
&= \frac{1}{2}\int_{\hat{K}}|f(\xi)|^2\,|\xi(a)-\xi(b)|^2\,\mathrm{d}\pi(\xi).
\end{aligned}$$

Consider an arbitrary number $\epsilon>0$. Since $w_2(f,U)=o(U)$ as $U\to\{e\}$, there is a neighborhood W of e in K such that whenever $U\subseteq W$ is a neighborhood of e in K, $w_2(f,U)\leq\frac{\epsilon}{B}\cdot m(U)$. Let $s,t\in K$. By hypothesis, there are $V_1,\ldots,V_n\subseteq W$, compact symmetric neighborhoods of e in K, and $x_0,\ldots,x_n\in K$ such that $\sum_{k=1}^{n} m(V_k)\leq 2B$, $x_0=a$, $x_n=b$, and for each $k=1,\ldots,n$, $\{x_k\}*\{x_{k-1}^{-}\}\subseteq V_k$. So,

$$\sum_{k=1}^{n} s(\|\phi(x_k)-\phi(x_{k-1})\|_2)\leq\frac{1}{2}\sum_{k=1}^{n}\int_{\hat{K}}|f(\xi)|^2\,|\xi(x_k)-\xi(x_{k-1})|^2\,\mathrm{d}\pi(\xi)$$

$$
\begin{aligned}
&\leq \frac{1}{2}\sum_{k=1}^{n} w_2(f, V_k) \\
&\leq \frac{\epsilon}{2B}\sum_{k=1}^{n} m(V_k) \\
&\leq \epsilon.
\end{aligned}
$$

Then by Lemma 2.4, for each $\beta \in \mathbb{R}$ and $x, y_1, y_2 \in K$ we have

$$\beta \cdot (\phi(y_1) - \phi(y_2)) = \beta \cdot \left(\widehat{L_x L_{y_1} f_1} - \widehat{L_x L_{y_2} f_1}\right) \in F.$$

So, if we put $x = y_2 = e$ and $y_1 = y \in K$, then we have

$$\beta \cdot (\langle y, .\rangle f - f) = \beta \cdot \left(\widehat{L_y f_1} - \widehat{f_1}\right) \in F. \quad \square$$

The following corollary extends the result [3, Corollary 2] on locally compact hypergroups.

Corollary 2.8. *Suppose that* $f \in L^2(\hat{K})$, $\pi(\{\xi \in \hat{G} : f(\xi) = 0\}) = 0$, *and the integral modulus of continuity of* f *satisfies* $\omega_2(f, U) = o(U)$ *as* $U \to \{e\}$. *Let for a* $B > 0$, K *satisfy in* B*-condition. Then* $R(\{\pm\langle x, .\rangle f : x \in K\})$ *is dense in* $L^2(G)$.

Proof. Let F be the closure of the set $R(\{\pm\langle x, .\rangle f : x \in K\})$ in $L^2(\hat{K})$. F is a closed additive subgroup of $L^2(\hat{K})$, and so by Corollary 2.7, we have

$$L := \{\sum_{j=1}^{n} \beta_j \cdot (\langle x_j, .\rangle f - f) : \beta_j \in \mathbb{R}, x_j \in K\} \subseteq F.$$

We claim that L is dense in $L^2(\hat{K})$. Let $h \in L^2(\hat{K})$ and $h \in L^{\perp}$. Specially, for each $x \in K$, we have $\langle\langle x, .\rangle f - f, h\rangle = 0$. By the Plancherel Theorem, there are $f_1, h_1 \in L^2(K)$ such that $\widehat{f_1} = f$ and $\widehat{h_1} = h$. So,

$$
\begin{aligned}
\langle f, h\rangle &= \langle\langle x, .\rangle f, h\rangle \\
&= \int_{\hat{K}} \eta(x)\hat{f}_1(\eta)\overline{\hat{h}_1(\eta)}\, \mathrm{d}\pi(\eta) \\
&= \int_{\hat{K}} \widehat{L_x f_1}(\eta)\overline{\hat{h}_1(\eta)}\, \mathrm{d}\pi(\eta) \\
&= \int_{K} L_x f_1(y)\overline{h_1(y)}\, \mathrm{d}m(y) \\
&= \int_{K} f_1(x * y)h_2(y^{-})\, \mathrm{d}m(y) \\
&= f_1 * h_2(x),
\end{aligned}
$$

where $h_2(t) := \overline{h_1(t^-)}$ $(t \in K)$. So, $f_2 * h_2$ is a constant function. Since $f_1, h_2 \in L^2(K)$, by ([6], 6.2F) we have $f_1 * h_2 \in C_0(K)$. Therefore, $f_1 * h_2 \equiv 0$. So, for all $\psi \in A$, we have $h \perp \psi f$. Then, by Theorem 2.1 we have $h \equiv 0$, and this completes the proof. $\square$

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