AN INTERSECTION CONDITION FOR GRADED PRIME SUBMODULES IN Gr-MULTIPLICATION MODULES

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Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we will investigate gr-multiplication modules over commutative graded rings which satisfy the condition $(*)$. We say that a gr-multiplication module M over a commutative G-graded ring R satisfy the condition$(*)$ if P is a graded prime submodule of M and if $\{N_\alpha\}_{\alpha\in\Delta}$ is a family of graded submodules of M, then P contains $\cap_{\alpha\in\Delta}N_\alpha$ only if P contains some N_α. Furthermore, we introduce several results concerning gr-multiplication R-modules.

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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, all rings are commutative with identity and all modules are unitary. Graded prime ideals in a commutative graded ring have been introduced and studied in [3,19,20]. A proper graded ideal P of R is said to be a graded prime ideal of R if whenever r and s are homogeneous elements of R such that $rs \in P$, then either $r \in P$ or $s \in P$.

Graded prime submodules of graded modules over a graded commutative rings have been introduced and studied in [1,2,5,7,8,18]. A proper graded submodule N of a graded module M over G-graded ring R is said to be graded prime submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $r \in (N :_R M)$ or $m \in N$ (see [5]). Some graded modules have no graded prime submodules and call such modules G-primeless (see [7]). Also, graded multiplication modules (gr-multiplication modules) over a commutative graded rings have been introduced and studied in [4,6,9–11,13,14,21]. A graded R-module M over G-graded ring R is said to be graded multiplication module (gr-multiplication module) if for every graded submodule N of M there exists

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a graded ideal I of R such that $N = IM$. It is clear that M is gr-multiplication R-module if and only if $N = (N :_R M)M$ for every graded submodule N of M (see [11]). Note that for a graded R-module M, the set of graded prime submodules is non-empty precisely when M is a gr-multiplication module.

In [3], Al-Zoubi and Qarqaz studied a graded ring R with the following property; $(*)$ If P is a graded prime ideal of R and if $\{I_\alpha\}_{\alpha\in\Delta}$ is a family of graded ideals of R, then P contains $\cap_{\alpha\in\Delta} I_\alpha$ only if P contains some I_α.

In [7, Theorem 2.5] the authors proved the following property: If R is a G-graded ring, M a gr-multiplication module, $K_1, K_2, \ldots, K_n$ a finite number of graded R-submodules of M, and P a graded prime submodule of M such that $\cap_{i=1}^n K_i \subseteq P$, then $K_j \subseteq P$ for some $j \in \{1, 2, \ldots, n\}$. Note that this property is not valid for every graded module. For example, let's take a graded module over a graded G-ring M as in [15, Example 2.4]. Then $(\mathbb{Z}\times 0)\cap(0\times\mathbb{Z}) \subseteq (0\times 0)$, but $(\mathbb{Z} \times 0) \nsubseteq (0 \times 0)$ and $(0 \times \mathbb{Z}) \nsubseteq (0 \times 0)$.

In this paper, we generalize this property to infinite intersection. After this, we introduce several results concerning gr-multiplication R-modules.

2. AN INTERSECTION CONDITION FOR GRADED PRIME SUBMODULES

Definition 2.1. Let R be a G-graded ring. A gr-multiplication R-module M is said to satisfy the condition $(*)$ if P is a graded prime submodule of M and if $\{K_\alpha\}_{\alpha\in\Delta}$ is a family of graded submodules of M, then P contains $\cap_{\alpha\in\Delta} K_\alpha$ only if P contains some P_α.

The following lemma is known (see [5] and [18]).

LEMMA 2.2. *Let R be a G-graded ring and M a graded R-module. Then the following hold*

(i) *If I and J are graded ideals of R, then $I + J$ and $I \cap J$ are graded ideals.*
(ii) *If K is a graded submodule of M, $r \in h(R)$, $x \in h(M)$ and I is a graded ideal of R, then Rx, IK and rK are graded submodules of M.*
(iii) *If N and K are graded submodules of M, then $N + K$ and $N \cap K$ are also graded submodules of M and $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R.*
(iv) *Let $\{K_\lambda\}$ be a collection of graded submodules of M. Then $\sum_\lambda K_\lambda$ and $\cap_\lambda K_\lambda$ are graded submodules of M.*

Recall that a graded R-module M over a G-graded ring R is said to be graded torsion free (gr-torsion free) R-module whenever $a \in h(R)$ and $m \in M$

with $am = 0$ implies that either $m = 0$ or $a = 0$ (see [5]). A graded R-module M is said to be a graded simple (gr-simple) if (0) and M are its only graded submodules (see [16]).

THEOREM 2.3. *Let R be a G-graded ring and M a gr-multiplication gr-torsion free R-module. If M satisfies the condition $(*)$, then M is gr-simple module.*

Proof. Suppose that M satisfies the condition $(*)$ and let P be the intersection of all non-zero graded submodules K_α of M. Since (0) is a graded prime submodule and M satisfies the condition $(*)$, $P \neq (0)$. Now let $0 \neq t_h \in P \cap h(M)$. Since P is the smallest non-zero graded submodule of M and $Rt_h \subseteq P$, $Rt_h = P$. Let $0 \neq r_g \in h(R)$, then $Rr_gt_h \subseteq Rt_h$. So $Rt_h = P = Rr_gt_h$. Then there exists an $s_\lambda \in h(R)$ such that $t_h = s_\lambda r_g t_h$. Since M is a gr-torsion free, r_g is unit in R. Thus R is a graded field and hence M is a gr-simple module. □

LEMMA 2.4. *Let R be a G-graded ring and M, M' be two graded R-modules and $\varphi : M \to M'$ be a graded epimorphism. Let N' be a graded submodule of M'. Then P' is a graded prime submodule of M' if and only if $\varphi^{-1}(P')$ is a graded prime submodule of M.*

Proof. ($\Rightarrow$) Assume that P' is a graded prime submodule of M' and let $r \in h(R)$ and $m \in h(M)$ such that $rm \in \varphi^{-1}(P')$ and $m \notin \varphi^{-1}(P')$. Then $\varphi(rm) = r\varphi(m) \in P'$. Since P' is a graded prime submodule of M' and $\varphi(m) \notin P'$, we get $r \in (P' :_R M')$, i.e., $rM' \subseteq P'$ and so $r\varphi^{-1}(M') = rM \subseteq \varphi^{-1}(P')$, i.e., $r \in (\varphi^{-1}(P') :_R M)$. Thus $\varphi^{-1}(P')$ is a graded prime submodule of M.
($\Leftarrow$)Assume that $\varphi^{-1}(P')$ is a graded prime submodule of M and let $s \in h(R)$ and $m' \in h(M')$ such that $sm' \in P'$ and $m' \notin P'$. Since φ is an epimorphism, there exists $m \in h(M)$ such that $\varphi(m) = m'$. Thus $s\varphi(m) = \varphi(sm) \in P'$. Hence $sm \in \varphi^{-1}(P')$. Since $\varphi^{-1}(P')$ is a graded prime submodule of M and $m \notin \varphi^{-1}(P')$, we get $s \in (\varphi^{-1}(P') :_R M)$, i.e., $sM \subseteq \varphi^{-1}(P')$ and hence $\varphi(sM) = s\varphi(M) = sM' \subseteq P'$, i.e., $s \in (P' :_R M')$. Therefore P' is a graded prime submodule of M'. □

THEOREM 2.5. *Let R be a G-graded ring, M a gr-multiplication R-module, M' a graded R-module and $\varphi : M \to M'$ be a graded epimorphism. If M satisfies the condition $(*)$, then M' satisfies the condition $(*)$.*

Proof. Assume that M satisfies the condition $(*)$. By [18, Theorem 2], M' is gr-multiplication R-module. Now let P' be a graded prime submodule of M' and let $\{K'_\alpha\}_{\alpha\in\Delta}$ be a family of graded submodules of M' such that $\cap_{\alpha\in\Delta}K'_\alpha \subseteq P'$. Since φ is an epimorphism of graded modules $\varphi^{-1}(\cap_{\alpha\in\Delta}K'_\alpha) \subseteq \varphi^{-1}(P')$. Hence $\cap_{\alpha\in\Delta}\varphi^{-1}(K'_\alpha) \subseteq \varphi^{-1}(P')$. By Lemma 2.4, $\varphi^{-1}(P')$ is a graded prime submodule of M. Since M satisfies the condition $(*)$, there exists $\beta \in \Delta$ such that $\varphi^{-1}(K'_\beta) \subseteq \varphi^{-1}(P')$ and so $K'_\beta \subseteq P'$. Therefore M' satisfies the condition $(*)$. □

LEMMA 2.6. *Let R be a G-graded ring and M a graded R-module. If K_1, $K_2, \ldots, K_n$ are graded submodules of M, then $(\cap_{i=1}^n K_i :_R M) = \cap_{i=1}^n (K_i :_R M)$.*

Proof. The proof is straightforward. □

Recall that a graded R-module M over a G-graded ring R is said to be *gr*-Noetherian (resp. *gr*-Artinian) if M satisfies the ascending (resp. descending) chain condition for graded submodules (see [17]).

THEOREM 2.7. *Let R be a G-graded ring and M a gr-multiplication R-module. If M is a gr-Artinian module, then M satisfies the condition $(*)$.*

Proof. Let P be a graded prime submodule of M and let $\{K_\alpha\}_{\alpha\in\Delta}$ be a family of graded submodules of M such that $\cap_{\alpha\in\Delta} K_\alpha \subseteq P$. Since M is *gr*-Artinian, we get $\cap_{\alpha\in\Delta} K_\alpha = \cap_{i=1}^n K_{\alpha_i}$ for some finite subset $\{\alpha_1, \ldots, \alpha_n\}$ of Δ, where $n \in \mathbb{Z}^+$. Then $(\cap_{\alpha\in\Delta} K_\alpha :_R M) = (\cap_{i=1}^n K_{\alpha_i} :_R M) \subseteq (P :_R M)$. By Lemma 2.6, $(\cap_{i=1}^n K_{\alpha_i} :_R M) = \cap_{i=1}^n (K_{\alpha_i} :_R M) \subseteq (P :_R M)$. Since P is graded prime submodule of M, by [5, Proposition 2.7], $(P :_R M)$ is a graded prime ideal of R. By [19, Proposition 1.4], we conclude that $(K_{\alpha_s} :_R M) \subseteq (P :_R M)$ for some $s \in \mathbb{Z}^+$. Since M is a *gr*-multiplication module, $K_{\alpha_s} = (K_{\alpha_s} :_R M)M \subseteq (P :_R M)M = P$. Therefore M satisfies the condition $(*)$. □

Recall that a graded R-module M over a G-graded ring R is said to be graded finitely generated (finitely *gr*-generated) if there exist $x_{g1}, x_{g2}, ..., x_{gn} \in h(M)$ such that $M = Rx_{g1} + \cdots + Rx_{gn}$ (see [16]).

THEOREM 2.8. *Let R be a G-graded ring and M a finitely gr-generated faithful gr-multiplication R-module. Then M satisfies the condition $(*)$ if and only if R satisfies the condition $(*)$.*

Proof. $(\Rightarrow)$ Assume that M satisfies the condition $(*)$. Let I be a graded prime ideal of R and let $\{J_\alpha\}_{\alpha\in\Delta}$ be a family of graded ideals of R such that $\cap_{\alpha\in\Delta} J_\alpha \subseteq I$. Hence $\cap_{\alpha\in\Delta} J_\alpha M \subseteq IM$. By [18, Corollary 3], IM is a graded prime submodule of M. By using [18, Theorem 8(i)], we have $(\cap_{\alpha\in\Delta} J_\alpha)M = \cap_{\alpha\in\Delta}(J_\alpha M) \subseteq IM$. Since M satisfies the condition $(*)$, we get $J_\beta M \subseteq IM$ for some $\beta \in \Delta$. By [6, Lemma 3.9], $J_\beta \subseteq I$. Therefore R satisfies the condition $(*)$.

$(\Leftarrow)$ Assume that R satisfies the condition $(*)$. Let P be a graded prime submodule of M and let $\{K_\alpha\}_{\alpha\in\Delta}$ be a family of graded submodules of M such that $\cap_{\alpha\in\Delta} K_\alpha \subseteq P$. Since M is *gr*-multiplication R-module by [18, Corollary

3], there exist graded ideals J_α and graded prime ideal I such that $K_\alpha = J_\alpha M$ and $P = IM$. By [18, Theorem 8(i)], $\cap_{\alpha\in\Delta} K_\alpha = \cap_{\alpha\in\Delta}(J_\alpha M) = (\cap_{\alpha\in\Delta} J_\alpha)M \subseteq P = IM$. By [6, Lemma 3.9], $\cap_{\alpha\in\Delta} J_\alpha \subseteq I$. Since R satisfies the condition $(*)$, $J_\beta \subseteq I$ for some $\beta \in \Delta$. Hence $K_\beta = J_\beta M \subseteq IM = P$. Therefore M satisfies the condition $(*)$. □

Theorem 2.9. *Let R be a G-graded ring, M a gr-multiplication R-module and $S \subseteq h(R)$ a multiplicative closed subset of R. If M satisfies the condition $(*)$, then $S^{-1}M$ satisfies the condition $(*)$ as an $S^{-1}R$-module.*

Proof. Since M is gr-multiplication module, by [11, Proposition 5.8], $S^{-1}M$ is gr-multiplication as an $S^{-1}R$. Now let P be a graded prime submodule of $S^{-1}M$ and let $\{K_\alpha\}_{\alpha\in\Delta}$ be a family of graded submodules of $S^{-1}M$ such that $\cap_{\alpha\in\Delta} K_\alpha \subseteq P$. Hence $(\cap_{\alpha\in\Delta} K_\alpha) \cap M \subseteq P \cap M$ and so $\cap_{\alpha\in\Delta}(K_\alpha \cap M) \subseteq P \cap M$. Since $P \cap M$ is graded prime submodule of M and M satisfies the condition $(*)$, we conclude that there exists $\beta \in \Delta$ such that $K_\beta \cap M \subseteq P \cap M$. Hence $S^{-1}(K_\beta \cap M) \subseteq S^{-1}(P \cap M)$ and so $K_\beta \subseteq P$. Therefore $S^{-1}M$ satisfies the condition $(*)$. □

3. *gr*-MULTIPLICATION MODULES

In this section, we obtain some results on gr-multiplication modules.

Theorem 3.1. *Let R be a G-graded ring. Then a graded R-module M is a gr-multiplication R-module if and only if for all graded submodules N and K of M with $Ann_R(M/N) = Ann_R(M/K)$, we have $N = K$.*

Proof. Assume that M is a gr-multiplication R-module and N and K are two graded submodules of M such that $Ann_R(M/N) = Ann_R(M/K)$. Since M is gr-multiplication, we get $N = Ann_R(M/N)M = Ann_R(M/K)M = K$. Conversely, let L be a graded submodule of M. Since $Ann_R(M/L) = Ann_R(M/(L :_R M)M)$, we get $L = (L :_R M)M$. Therefore M is gr-multiplication modules. □

Theorem 3.2. *Let R be a G-graded ring. Then a graded R-module M is a gr-multiplication R-module if and only if for any $m_g \in h(M)$ and graded submodule N of M, $(Rm_g :_R M) \subseteq (N :_R M)$ implies that $m_g \in N$.*

Proof. ($\Rightarrow$) Assume that $(Rm_g :_R M) \subseteq (N :_R M)$ for some $m_g \in h(M)$ and graded submodule N of M. Since M is a gr-multiplication R-module, $Rm_g = (Rm_g :_R M)M \subseteq (N :_R M)M = N$. Thus $m_g \in N$.

($\Leftarrow$) Let $x_g \in h(M)$. Then $(Rx_g :_R M)M$ is a graded submodule of the gr-cyclic module Rx_g. Since Rx_g is a gr-multiplication module, $(Rx_g :_R$

$M)M = ((Rx_g :_R M)M : Rx_g)Rx_g$. Hence $(Rx_g :_R M) \subseteq (((Rx_g :_R M)M :_R Rx_g)Rx_g :_R M)$ and so, by hypothesis, $x_g \in ((Rx_g :_R M)M :_R Rx_g)Rx_g \subseteq (Rx_g :_R M)M$. Hence $Rx_g = (Rx_g :_R M)M$. It follows that M is gr-multiplication module by [14, Proposition 2.3]. □

THEOREM 3.3. *Let R be a G-graded ring and M a graded R-module. If for every non-zero graded submodule N of M, we have that M/N is a gr-multiplication module and $(N :_R M) \neq Ann_R(M)$, then M is gr-multiplication module.*

Proof. Let N be a nonzero graded submodule of M. Set $J = (N :_R M)$. If $JM = 0$, then $J = Ann_R(M)$, which is a contradiction. So $JM \neq 0$. Hence by the assumption, $N/JM = (N/JM :_R M/JM)(M/JM) = 0$. Thus M is gr-multiplication module. □

THEOREM 3.4. *Let R be a G- graded ring and M be a gr-semisimple R-module. If M is gr-multiplication module, then for each graded endomorphism φ of M, we have $M = ker(\varphi) \bigoplus Im(\varphi)$.*

Proof. Let M be a gr-multiplication R-module and φ be a graded endomorphism of M. Since M is gr-semisimple and $Im(\varphi)$ is a graded submodule of M, there exists a graded submodule N of M, such that $M = Im(\varphi) \bigoplus N$. Hence $M/ker(\varphi) \cong Im(\varphi) \cong M/N$. By Theorem 3.1, $ker(\varphi) = N$. □

THEOREM 3.5. *Let R be a G-graded ring and M a graded R-module. If M is a gr-multiplication module, then for each graded endomorphism φ of M, we have $ker(\varphi) = (0 :_M Ann_R(M/Im(\varphi)))$.*

Proof. Since M is gr-multiplication module and $Im(\varphi)$ is a graded submodule of M, we have $Im(\varphi) = (Im(\varphi) :_R M)M$. Then $M/(ker(\varphi)) \cong (Im(\varphi) :_R M)M$. Since $Ann_R((Im(\varphi) :_R M)M) = Ann_R(M/(0 :_M (Im(\varphi) :_R M)))$, we conclude that $Ann_R(M/ker(\varphi)) = Ann_R(M/(0 :_M (Im(\varphi) :_R M)))$. Since M is gr-multiplication module, $ker(\varphi) = (0 :_M Ann_R(M/Im(\varphi)))$. □

Let R be a G-graded ring and M be a graded R-module. A graded zero divisor on M is an element $r \in h(R)$ for which there exists $m \in h(M)$ such that $m \neq 0$ but $r\ m = 0$. The set of all graded zero-divisors on M is denoted by $G\text{-}zdv_R(M)$. A graded module M over a G-graded ring R is said to be gr-domain if $G\text{-}zdv_R(M) = 0$ (see [4]).

THEOREM 3.6. *Let R be a G-graded ring and M a graded R-module. If M is gr-multiplication gr-domain such that M has a gr-minimal submodule, then M is gr-simple module.*

Proof. Let N be a minimal graded submodule of M. Then $(N :_R M)N = 0$ or $(N :_R M)N = N$. If $(N :_R M)N = 0$, then $(N :_R M) = 0$ because M is

gr-domain. Since M is gr-multiplication module, $N = (N :_R M)M = 0$, which is a contradiction. Hence $(N :_R M)N = N$. There exists $0 \neq m_g \in h(M)$, such that $N = Rm_g$. Hence $(Rm_g :_R M)Rm_g = Rm_g$. Thus $m_g = t_h m_g$ for some $t_h \in (Rm_g :_R M) \cap h(R)$. Since M is gr-domain, $t_h = 1$. Hence $(N :_R M) = R$ and so $N = M$. □

THEOREM 3.7. *Let R be a G-graded ring and M a graded R-module. If M is gr-semisimple module such that $(N :_R M) \neq Ann_R(M)$ for every gr-minimal submodule N of M, then M is gr-multiplication module.*

Proof. Let N be a graded submodule of M. Since N is a gr-semisimple module, there exists a collection $\{K_\alpha\}_{\alpha\in I}$ of gr-minimal submodules of M such that $N = \sum_{\alpha\in I} K_\alpha$. Since K_α is gr-minimal and $(K_\alpha :_R M) \neq Ann_R(M)$ for each $\alpha \in I$, we have $K_\alpha = (K_\alpha :_R M)M$. Hence $N = \sum_{\alpha\in I} K_\alpha = (\sum_{\alpha\in I}(K_\alpha :_R M))M$. Thus R is gr-multiplication module. □

THEOREM 3.8. *Let R be a G-graded ring and M a graded R-module. If M is a finitely gr-generated gr-multiplication R-module such that for each graded ideal I of R and for each collection $\{K_\alpha\}_{\alpha\in\Delta}$ of graded submodules of M, we have $(\sum_{\alpha\in\Delta} K_\alpha :_M I) = \sum_{\alpha\in\Delta}(K_\alpha :_M I)$, then M is a gr-Noetherian R-module.*

Proof. Let $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ be an ascending chain of graded submodules of M. Set $I = (\sum_{i\in 1}^{\infty} K_i :_R M)$. By assumption, $(\sum_{i\in 1}^{\infty} K_i :_M I) = \sum_{i\in 1}^{\infty}(K_i :_M I)$. Hence $\sum_{i\in 1}^{\infty}(K_i :_M I) = M$. Since M is a finitely gr-generated R-module, there exists a positive integer j such that $(K_j :_M I) = M$. Hence $IM \subseteq K_j$. Since M is a gr-multiplication module, we get $\sum_{i\in 1}^{\infty} K_i \subseteq K_j$. Hence $\sum_{i\in 1}^{\infty} K_i = K_j$. Therefore M is a gr-Noetherian R-module. □

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REFERENCES

[1] K. Al-Zoubi and R. Abu-Dawwas, *On graded quasi-prime submodules.* Kyungpook Math. J. **55** (2015), *2*, 259–266.

[2] K. Al-Zoubi, M. Jaradat and R. Abu-Dawwa, *On graded classical prime and graded prime submodules.* Bull. Iranian Math. Soc. **41** (2015), *1*, 217–225.

[3] K. Al-Zoubi and F. Qarqaz, *An intersection condition for graded prime ideals.* Boll. Unione Mat. Ital. (In press)

[4] H. Ansari-Toroghy and F. Farshadifar, *Graded comultiplication modules.* Chiang Mai J. Sci. **38** (2011), *3*, 311–320.

[5] S.E. Atani, *On graded prime submodules.* Chiang Mai. J. Sci. **33** (2006), *1*, 3–7.

[6] S.E. Atani and R.E. Atani, *Graded multiplication modules and the graded ideal* $\theta_g(M)$. Turkish J. Math. **33** (2009), 1–9.

[7] S.E Atani and F. Farzalipour, *Notes on the graded prime submodules.* Int. Math. Forum 1 **38** (2006), 1871–1880.

[8] S.E. Atani and F. Farzalipour, *On graded secondary modules.* Turkish J. Math. **31** (2007), 371–378.

[9] S.E. Atani and F.E.K. Saraei, *Graded modules which satisfy the Gr-Radical formula.* Thai J. Math. **8** (2010), *1*, 161–170.

[10] J. Escoriza and B. Torrecillas, *Multiplication rings and graded rings.* Comm. Algebra **27** (1999), *12*, 6213–6232.

[11] J. Escoriza and B. Torrecillas, *Multiplication objects in commutative Grothendieck categories.* Comm. Algebra **26** (1998), *6*, 1867–1883.

[12] R. Hazrat, *Graded Rings and Graded Grothendieck Groups.* Cambridge Univ. Press, Cambridge, 2016.

[13] P. Ghiasvand and F. Farzalipour, *On graded weak multiplication modules.* Tamkang J. Math. **43** (2012), 171–177.

[14] P. Ghiasvand and F. Farzalipour, *Some properties of graded multiplication modules.* Far East J. Math. Sci. **34** (2009), 341–352.

[15] S.C. Lee and R. Varmazyar, *Semiprime submodules of graded multiplication modules.* J. Korean Math. Soc. **49** (2012), 435–447.

[16] C. Nastasescu and V.F. Oystaeyen, *Graded Ring Theory.* North-Holland Mathematical Library **28**, North-Holand, Amsterdam, 1982.

[17] C. Nastasescu and V.F. Oystaeyen, *Methods of Graded Rings.* Lecture Notes in Math. **1836**, Berlin-Heidelberg: Springer-Verlag, 2004.

[18] K.H Oral, U. Tekir and A.G. Agargun, *On graded prime and primary submodules.* Turkish J. Math. **35** (2011), 159–167.

[19] M. Refai and K. Al-Zoubi, *On graded primary ideals.* Turkish J. Math. **28** (2004), 217–229.

[20] R.N. Uregen, U. Tekir and K.H. Oral, *On the union of graded prime ideals.* Open Phys. **14** (2016), 114–118.

[21] N. Zamani, *Finitely generated graded multiplication modules.* Glasg. Math. J. **53** (2011), *3*, 693–705.

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