To the memory of Majdi Ben Halima

DUAL TOPOLOGY OF GENERALIZED MOTION GROUPS

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Let K be a connected compact Lie group acting on a finite dimensional vector space $(V, \langle , \rangle)$. We consider the semidirect product $G = K \ltimes V$. There are a dense open subset Λ of the dual vector space V^* of V and a subgroup H of K such that for each $\ell \in \Lambda$ the stability group K_ℓ is conjugate to H. Let $\Lambda_H := \{\ell \in \Lambda;\, K_\ell = H\}$ and $\Lambda_{(H,K)} := \Lambda_H/K$ where ℓ and ℓ' in Λ_H are identified if they are on the same K-orbit in V^*. We define the subspace $(\widehat{G})_{H,K}$ of the unitary dual $\widehat{G}$ of G by

$$(\widehat{G})_{H,K} := \{Ind_{H\ltimes V}^{K\ltimes V}(\rho \otimes \chi_\ell);\ \rho \in \widehat{H}, \chi_\ell \in \widehat{V}, \ell \in \Lambda_{(H,K)}\}.$$

In this paper, we give a precise description of the set $(\widehat{G})_{H,K}$ and we determine explicitly the topology of the space $(\mathfrak{g}^\ddagger/G)_{H,K}$ which is formed by all the admissible coadjoint orbits $\mathcal{O}^G_{(\mu,\ell)}$, where μ is the highest weight of $\rho \in \widehat{H}$ and $\ell \in \Lambda_{(H,K)}$. Also, we show that the topological spaces $(\widehat{G})_{H,K}$ and $(\mathfrak{g}^\ddagger/G)_{H,K}$ are homeomorphic.

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1. INTRODUCTION

Let G be a locally compact group. By the unitary dual $\widehat{G}$ of G, we mean the set of all equivalence classes unitary representations of G equipped with the Fell topology (see [6]). One of the most elegant results in the theory of unitary representations is attached to the determination of the topology of $\widehat{G}$. Following the Kirillov mapping, it is well-known that for a simply connected nilpotent Lie group and more generally for an exponential solvable Lie group $G = exp(\mathfrak{g})$, its dual space $\widehat{G}$ is homeomorphic to the coadjoint orbits $\mathfrak{g}^*/G$ (see [14]). In the context of the semi direct product $G = K \ltimes N$ of compact connected Lie groups K acting on simply connected nilpotent Lie groups N, we have again a concrete geometric parametrization of $\widehat{G}$ by the so-called admissible coadjoint orbits (see [15]). A part from this parametrization, we can ask: is the topology of $\widehat{G}$ related to the topology of the admissible coadjoint orbits?

In this direction, an affirmative answer to this question was given for the Euclidean motion groups (see [5]), and it was generalized to a class of Cartan motion groups in [3].

In the present work, we consider the semidirect product $G = K \ltimes V$ of compact connected Lie groups K acting on a finite dimensional vector space V. It is worth mentioning here that there are a dense open subset Λ of the dual vector space V^* of V and a subgroup H of K such that for each $\ell \in \Lambda$ the stability group K_ℓ is conjugate to H (for more details, see [11]). We can assume that Λ is invariant under the action of K and we define the subset Λ_H of Λ by

$$\Lambda_H := \{\ell \in \Lambda;\ K_\ell = H\}.$$

Let $\Lambda_{(H,K)} := \Lambda_H/K$ where ℓ and ℓ' in Λ_H are identified if they are on the same K-orbit in V^*. We define a certain subspace $(\widehat{G})_{H,K} \subset \widehat{G}$ that is generic in some sense as follows:

$$(\widehat{G})_{H,K} := \{Ind_{H\ltimes V}^{K\ltimes V}(\rho \otimes \chi_\ell);\ \rho \in \widehat{H}, \chi_\ell \in \widehat{V}, \ell \in \Lambda_{(H,K)}\}.$$

For every admissible linear form ψ of the Lie algebra $\mathfrak{g}$ of G, we can construct an irreducible unitary representation π_ψ by holomorphic induction and according to Lipsman (see [15]), every irreducible representation of G arises in this manner. Then we get a map from the set $\mathfrak{g}^\ddagger$ of the admissible linear forms onto the dual space $\widehat{G}$ of G. Note that π_ψ is equivalent to $\pi_{\psi'}$ if and only if ψ and ψ' are on the same G-orbit, finally we obtain a bijection between the space $\mathfrak{g}^\ddagger/G$ of admissible coadjoint orbits and the unitary dual $\widehat{G}$.

In this paper, we show that the subspace $(\widehat{G})_{H,K}$ is homeomorphic to the subspace $(\mathfrak{g}^\ddagger/G)_{H,K}$ of $\mathfrak{g}^\ddagger/G$ which is formed by all the admissible coadjoint orbits $\mathcal{O}^G_{(\mu,\ell)}$, where μ is the highest weight of $\rho \in \widehat{H}$ and $\ell \in \Lambda_{(H,K)}$.

Here we give a short description of the contents of the paper. In Section 2, we recall some results about the construction of the induced symplectic manifold (Marsden-Weinstein reduction). Section 3, serves to fix notations and an important fact worth mentioning here is that every coadjoint orbit of G is always obtained by symplectic induction. Section 4 is devoted to the description via Mackey's little group theory of the unitary dual $\widehat{G}$, mostly in order to give a concrete parametrization for our space $(\widehat{G})_{H,K}$ by Mackey parameters. In Section 5, we study the convergence in $(\mathfrak{g}^\ddagger/G)_{H,K}$ in terms of Mackey parameters. In the last section, the convergence in the space $(\mathfrak{g}^\ddagger/G)_{H,K}$ is studied and our results of this work are derived in Theorem 6.2.

2. SYMPLECTIC INDUCTION

Along this paper, we denote by Ad_L and Ad_L^*, respectively, the adjoint and coadjoint representations for such Lie group L. Let S be a closed Lie subgroup of a connected Lie group L and let (M, ω) be a symplectic manifold. We assume that S acts smoothly on M by symplectomorphisms and M equipped with an equivariant momentum map $J_M : M \longrightarrow \mathfrak{s}^*$, where $\mathfrak{s}$ denotes the Lie algebra of S. Using symplectic induction one can construct in a canonical way a symplectic manifold (M_{ind}, ω_{ind}) on which L acts in a Hamiltonian way with an equivariant momentum map $J_{ind} : M_{ind} \longrightarrow \mathfrak{l}^*$ where $\mathfrak{l}$ is the Lie algebra of L.

For the construction of the Hamiltonian spaces $(M_{ind}, \omega_{ind}, L, J_{ind})$, one proceeds as follows. It is well-known that the subgroup S acts on L by the left multiplication. We denote by φ_{T^*L} the canonical lift of this action to the symplectic manifold T^*L, equipped with its canonical symplectic form $d\nu_L$ and by identifying $\mathfrak{l}^*$ with the left-invariant 1-forms on L, we get the natural isomorphism $T^*L \cong L \times \mathfrak{l}^*$. Then the action of S on T^*L is given by

$$\varphi_{T^*L}(h)(g, f) = (gh^{-1}, Ad_S^*(h)f).$$

Let φ_M be the action of S on M, so S acts on $\widetilde{M} = M \times T^*L$ by the action $\varphi_{\widetilde{M}}$ defined by

$$\varphi_{\widetilde{M}}(h)(m, g, f) \;=\; \Big(\varphi_M(h)(m), gh^{-1}, Ad_S^*(h)f\Big), \tag{2.1}$$

for all $h \in S, (m, g, f) \in M \times T^*L$. This action is symplectic for the symplectic form $\widetilde{\omega} = \omega + d\nu_L$, and it is a proper action because S is closed. Furthermore, this action admits an equivariant momentum map $J_{\widetilde{M}} = J_M + J_{T^*L}$. By observing that the element $0 \in \mathfrak{s}^*$ is a regular value for $J_{\widetilde{M}}$, then the quotient $M_{ind} = J_{\widetilde{M}}^{-1}(0)/S$ is a symplectic manifold (Marsden-Weinstein reduction). We call M_{ind} induced symplectic manifold and some times we denote it by $Ind_S^L M$.

In order to obtain a Hamiltonian action of L on M_{ind} we remark that the group L acts naturally on itself on the left, the canonical lift of this action to T^*L is Hamiltonian and we let also L act trivially on M, then we obtain a Hamiltonian action of L on $\widetilde{M}$ with equivariant momentum map $\widetilde{J} : \widetilde{M} \longrightarrow \mathfrak{l}^*$ defined by

$$\widetilde{J}(m, g, f) = Ad_L^*(g)f.$$

The S-action on $\widetilde{M}$ commutes with the action of L on $\widetilde{M}$, hence we obtain a symplectic action of L on M_{ind}. The fact that $\widetilde{J}$ is invariant under the S-action, it descends as an equivariant momentum map for the L-action on M_{ind} denoted by J_{ind}. This finishes the production of the induced symplectic manifold, and it was pointed out in [2], that M_{ind} is a fibre bundle over $T^*(L/S)$ with typical fibre M.

3. GENERALIZED MOTION GROUPS AND SYMPLECTIC INDUCTION

Let K be a connected compact Lie group acting unitary on a finite dimensional vector space $(V, \langle , \rangle)$. We write $k.v$ and $A.v$ (resp. $k.\ell$ and $A.\ell$) for the result of applying elements $k \in K$ and $A \in \mathfrak{k} := Lie(K)$ to $v \in V$ (resp. to $\ell \in V^*$).

Now, one can form the semidirect product $G := K \ltimes V$ which is the so-called generalized motion group. As a set $G = K \times V$ and the multiplication in this group is given by

$$(k, v)(h, u) = (kh, v + k.u), \forall (k, v), (h, u) \in G.$$

The Lie algebra of G is $\mathfrak{g} = \mathfrak{k} \oplus V$ (as a vector space) and the Lie algebra structure is given by the bracket

$$[(A, a), (B, b)] = ([A, B], A.b - B.a), \forall (A, a), (B, b) \in \mathfrak{g}.$$

Under the identification of the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ with $\mathfrak{k}^* \oplus V^*$, we can express the duality between $\mathfrak{g}$ and $\mathfrak{g}^*$ as $F(A, a) = f(A) + \ell(a)$, for all $F = (f, \ell), (A, a) \in \mathfrak{g}$. The adjoint and coadjoint representations of G are given respectively, by the following relations

$$\begin{aligned} Ad_G(k, v)(A, a) &= (Ad_K(k)A, k.a - Ad_K(k)A.v), \forall (k, v) \in G, (A, a) \in \mathfrak{g}, \\ Ad^*_G(k, v)(f, \ell) &= (Ad^*_K(k)f + k.\ell \odot v, k.\ell), \forall (k, v) \in G, (f, \ell) \in \mathfrak{g}^*, \end{aligned}$$

where $\ell \odot v$ is the element of $\mathfrak{k}^*$ defined by

$$\ell \odot v(A) = \ell(A.v) = -(A.\ell)(v), \forall A \in \mathfrak{k}, \ell \in V^*, v \in V.$$

Therefore, the coadjoint orbit of G passing through $(f, \ell) \in \mathfrak{g}^*$ is given by

$$\mathcal{O}^G_{(f,\ell)} = \left\{ \left(Ad^*_K(k)f + k.\ell \odot v, k.\ell \right), k \in K, v \in V \right\}. \tag{3.1}$$

For $\ell \in V^*$, we define $K_\ell := \{k \in K;\ k.\ell = \ell\}$ the isotropy subgroup of ℓ in K and the Lie algebra of K_ℓ is given by the vector space $\mathfrak{k}_\ell = \{A \in \mathfrak{k};\ A.\ell = 0\}$. Hence, if we define the linear map $\psi_\ell : \mathfrak{k} \ni A \longmapsto -A.\ell \in V^*$, we obtain the equality $\mathfrak{k}_\ell = Ker(\psi_\ell)$.

Let $\imath_\ell : \mathfrak{k}_\ell \hookrightarrow \mathfrak{k}$ be the injection map, then $\imath^*_\ell : \mathfrak{k}^* \longrightarrow \mathfrak{k}^*_\ell$ is the projection map and we have

$$\mathfrak{k}^\circ_\ell = Ker(\imath^*_\ell) \tag{3.2}$$

where $\mathfrak{k}^\circ_\ell$ is the annihilator of $\mathfrak{k}_\ell$ (for more details see [2]).

It is well-known that the coadjoint orbit $\mathcal{O}^G_{(f,\ell)}$ of G is symplectomorphic to a subbundle of a modified cotangent bundle, obtained by symplectic induction from a point. In this context, the following result give a more general property of the coadjoint orbit $\mathcal{O}^G_{(f,\ell)}$ and was discussed by Baguis in [2].

LEMMA 3.1. *The coadjoint orbit $\mathcal{O}^G_{(f,\ell)}$ is always obtained by symplectic induction from the coadjoint orbit $\mathcal{O}^{G_\ell}_\nu$ of G_ℓ passing through $\nu = (\imath^*_\ell(f), \ell) \in \mathfrak{g}^*_\ell$, where $G_\ell = K_\ell \ltimes V$ and $\mathfrak{g}_\ell$ is the Lie algebra of G_ℓ.*

Using the convention of Section 1 and notations of the previous Lemma, we can write

$$\mathcal{O}^G_{(f,\ell)} = Ind^G_{G_\ell}(\mathcal{O}^{G_\ell}_\nu).$$

Note that the coadjoint orbit $\mathcal{O}^{G_\ell}_\nu$ is canonically isomorphic to the coadjoint orbit $\mathcal{O}^{K_\ell}_{\imath^*_\ell(f)}$ of K_ℓ passing through $\imath^*_\ell(f)$. Let us choose the symplectic manifold M as $M = \mathcal{O}^{G_\ell}_\nu$. By applying the symplectic induction from M with the groups L and H as $L = G$ and $H = G_\ell$, we obtain that

$$\mathcal{O}^G_{(f,\ell)} = M_{ind} = J^{-1}_{\widetilde{M}}(0)/G_\ell,$$

where $J_{\widetilde{M}} : \widetilde{M} = M \times T^*G \longrightarrow \mathfrak{g}^*_\ell$ is the momentum map of $\widetilde{M}$ and the zero level set $J^{-1}_{\widetilde{M}}(0)$ is given by $J^{-1}_{\widetilde{M}}(0) = \Big\{\Big(\big(Ad^*_K(k)(\imath^*_\ell(f)), \ell\big), g, \big(Ad^*_K(k)f + \ell \odot v, \ell\big)\Big),\ k \in K_\ell, g \in G, v \in V\Big\}$ (see [2]).

4. DUAL SPACES OF GENERALIZED MOTION GROUPS

We start with recalling briefly the description of the unitary dual of G via Mackey's little group theory. For every non-zero linear form ℓ on V, we denote by χ_ℓ the unitary character of the vector Lie group V given by $\chi_\ell = e^{i\ell}$. Let ρ be an irreducible unitary representation of K_ℓ on some Hilbert space $\mathcal{H}_\rho$. The map

$$\rho \otimes \chi_\ell : (k, v) \longmapsto e^{i\ell(v)}\rho(k)$$

is a representation of the Lie group $K_\ell \ltimes V$ such that one induces in order to get a unitary representation of G. We denote by $\mathcal{H}_{(\rho,\ell)} := L^2(K, \mathcal{H}_\rho)^\rho$ the subspace of $L^2(K, \mathcal{H}_\rho)$ consisting of all the maps ξ which satisfy the covariance condition

$$\xi(kh) = \rho(h^{-1})\xi(k), \forall k \in K, h \in K_\ell.$$

The induced representation

$$\pi_{(\rho,\ell)} := Ind^{K\ltimes V}_{K_\ell \ltimes V}(\rho \otimes \chi_\ell)$$

is defined on $\mathcal{H}_{(\rho,\ell)}$ by

$$\pi_{(\rho,\ell)}(k,v)\xi(h) = e^{i\ell(h^{-1}.v)}\xi(k^{-1}h)$$

where $(k,v) \in G, h \in K$ and $\xi \in \mathcal{H}_{(\rho,\ell)}$. By Mackey's theory we can say that the induced representation $\pi_{(\rho,\ell)}$ is irreducible and every infinite dimensional irreducible unitary representation of G is equivalent to one of $\pi_{(\rho,\ell)}$. Moreover, tow representations $\pi_{(\rho,\ell)}$ and $\pi_{(\rho\prime,\ell')}$ are equivalent if and only if ℓ and ℓ' are contained in the same K-orbit and the representation ρ and ρ' are equivalent under the identification of the conjugate subgroups K_ℓ and $K_{\ell'}$. All irreducible representations of G which are not trivial on the normal subgroup V, are obtained by this manner. On the other hand, we denote also by τ the extension of every unitary irreducible representation τ of K on G, simply defined by $\tau(k,v) = \tau(k)$ for $k \in K$ and $v \in V$. There exists a so-called principal stability subgroup for the action of K on V^*, *i.e.*, there are a dense open subset Λ of V^* and a subgroup H of K such that for each $\ell \in \Lambda$ the stability group K_ℓ is conjugate to H (see [11]). We can assume that Λ is invariant under the action of K and we define the subset Λ_H of Λ by

$$\Lambda_H := \{\ell \in \Lambda;\ K_\ell = H\}.$$

Let $\Lambda_{(H,K)} := \Lambda_H/K$ where ℓ and ℓ' in Λ_H are identified if they are on the same K-orbit in V^*. Below we give a certain subspace $(\widehat{G})_{H,K} \subset \widehat{G}$ which is so-called generic in some sense:

$$(\widehat{G})_{H,K} := \{Ind_{H\ltimes V}^{K\ltimes V}(\rho \otimes \chi_\ell);\ \rho \in \widehat{H}, \chi_\ell \in \widehat{V}, \ell \in \Lambda_{(H,K)}\}.$$

Applying Mackey's analysis, we obtain the bijection

$$(\widehat{G})_{H,K} \simeq \widehat{H} \times \Lambda_{(H,K)}.$$

In the remainder of this paper, we shall assume that G is exponential, *i.e.*, K_ℓ is connected for all $\ell \in V^*$. Let ρ_μ be an irreducible representation of K_ℓ with highest weight μ. For simplicity, we shall write $\pi_{(\mu,\ell)}$ instead of $\pi_{(\rho_\mu,\ell)}$ and $\mathcal{H}_{(\mu,\ell)}$ instead of $\mathcal{H}_{(\rho_\mu,\ell)}$.

5. CONVERGENCE OF IRREDUCIBLE REPRESENTATIONS OF G

Let N be an abelian group, and assume that a compact Lie group K acts on the left on N by automorphisms. As sets, the semidirect product $K \ltimes N$ is the Cartesian product $K \times N$ and the group multiplication is given by

$$(k_1, x_1) \cdot (k_2, x_2) = (k_1k_2, x_1 + k_1x_2).$$

Let χ be a unitary character of N, and let K_χ be the stabilizer of χ under the action of K on $\widehat{N}$ defined by

$$(k \cdot \chi)(x) = \chi(k^{-1}x).$$

If ρ is an element of $\widehat{K_\chi}$, then the triple $(\chi, (K_\chi, \rho))$ is called a cataloguing triple. From the notations of [1], we denote by $\pi(\chi, K_\chi, \rho)$ the induced representation $Ind_{K_\chi \ltimes N}^{K \ltimes N}(\rho \otimes \chi)$. Referring to [1, p. 187], we have

Proposition 1. *The mapping* $(\chi, (K_\chi, \rho)) \longrightarrow \pi(\chi, K_\chi, \rho)$ *is onto* $\widehat{K \ltimes N}$.

Let $\mathcal{A}(K)$ be the set of all pairs (K', ρ'), where K' is a closed subgroup of K and ρ' is an irreducible representation of K'. We equip $\mathcal{A}(K)$ with the Fell topology (see [6]). Therefore, every element in $\widehat{K \ltimes N}$ can be catalogued by elements in the topological space $\widehat{N} \times \mathcal{A}(K)$. The following result of Baggett (see [1, Theorem 6.2-A]) provides a precise and neat description of the topology of $\widehat{K \ltimes N}$.

Theorem 5.1. *Let Y be a subset of $\widehat{K \ltimes N}$ and π an element of $\widehat{K \ltimes N}$. Then π is weakly contained in Y if and only if there exist: a cataloguing triple $(\chi, (K_\chi, \rho))$ for π, an element (K', ρ') of $\mathcal{A}(K)$, and a net $\{(\chi_n, (K_{\chi_n}, \rho_n))\}$ of cataloguing triples such that:*

- *(i) for each n, the irreducible unitary representation $\pi(\chi_n, K_{\chi_n}, \rho_n)$ of $K \ltimes N$ is an element of Y;*
- *(ii) the net $\{(\chi_n, (K_{\chi_n}, \rho_n))\}$ converges to $(\chi, (K', \rho'))$;*
- *(iii) K_χ contains K', and the induced representation $Ind_{K'}^{K_\chi}(\rho')$ contains ρ.*

Let us now return to the context and notations of Section 4. To an irreducible representation ρ_μ of H with highest weight μ and a linear form $\ell \in \Lambda_{(H,K)}$, we associate the representation $\pi_{(\mu,\ell)}$ of G and its corresponding cataloguing triple $(\chi_\ell, (H, \rho_\mu))$. By a direct application of Theorem 5.1 it gives us the following result.

Proposition 2. *Let $(\pi_{(\mu^n,\ell_n)})_n$ be a sequence of elements in $(\widehat{G})_{H,K}$. Then $(\pi_{(\mu^n,\ell_n)})_n$ converges to $\pi_{(\mu,\ell)}$ in $(\widehat{G})_{H,K}$ if and only if $(\ell_n)_n$ tends to ℓ as $n \longrightarrow +\infty$ and $\mu^n = \mu$ for n large enough.*

Remark 5.2. By Proposition 2, we easily see that $(\widehat{G})_{H,K}$ has a Hausdorff topology.

6. CONVERGENCE OF ADMISSIBLE COADJOINT ORBITS OF G

We shall freely use the notations of the previous sections. Let $\mathfrak{t}_\mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$, and let $\mathfrak{t}_\mathfrak{h} \subset \mathfrak{t}_\mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{h} := Lie(H)$.

Now, we fix a linear form ℓ in $\Lambda_{(H,K)}$ and we consider an irreducible representation ρ_μ of H with highest weight μ. Then the stabilizer G_ψ of $\psi = (\mu, \ell)$ in G is given by

$$\begin{aligned} G_\psi &= \Big\{(k,v) \in G;\ (Ad_K^*(k)\mu + k.\ell \odot v, k.\ell) = (\mu, \ell)\Big\} \\ &= \Big\{(k,v) \in G;\ k \in H, Ad_K^*(k)\mu + \ell \odot v = \mu\Big\} \\ &= \Big\{(k,v) \in G;\ k \in H,\ Ad_K^*(k)\mu = \mu\Big\} \end{aligned}$$

since $\imath^*(\ell \odot v) = 0$ (see [2]) where $\imath^* : \mathfrak{k}^* \longrightarrow \mathfrak{h}^*$ is the projection map. Thus, we have $G_\psi = K_\psi \ltimes V_\psi$, then ψ is aligned (see [15]). A linear form $\psi \in \mathfrak{g}^*$ is called admissible if there exists a unitary character χ of the identity component of G_ψ such that $d\chi = i\psi_{|_{\mathfrak{g}_\psi}}$. According to Lipsman (see [15]), the representation of G obtained by holomorphic induction from (μ, ℓ) is equivalent to the representation $\pi_{(\mu,\ell)}$. We denote by $\mathfrak{g}^\ddagger \subset \mathfrak{g}^*$ the set of all admissible linear forms on $\mathfrak{g}$. The quotient space $\mathfrak{g}^\ddagger/G$ is called the space of admissible coadjoint orbits of G. Let $(\mathfrak{g}^\ddagger/G)_{H,K}$ the subspace of $\mathfrak{g}^\ddagger/G$, that is the subspace formed by all the coadjoint orbits $\mathcal{O}^G_{(\mu,\ell)}$.

Let T_K and T_H be maximal tori respectively in K and H such that $T_H \subset T_K$. Their corresponding Lie algebras are denoted by $\mathfrak{t}_\mathfrak{k}$ and $\mathfrak{t}_\mathfrak{h}$. We denote by W_K and W_H the Weyl groups of K and H associated respectively to the tori T_K and T_H. Notice that every element $\lambda \in P_K$ takes pure imaginary values on $\mathfrak{t}_\mathfrak{k}$, where P_K is the integral weight lattice of T_K. Hence such an element $\lambda \in P_K$ can be considered as an element of $(i\mathfrak{t}_\mathfrak{k})^*$. Let C_K^+ be a positive Weyl chamber in $(i\mathfrak{t}_\mathfrak{k})^*$, and we define the set P_K^+ of dominant integral weights of T_K by $P_K^+ := P_K \cap C_K^+$. For $\lambda \in P_K^+$, denote by $\mathcal{O}_\lambda^K$ the K-coadjoint orbit passing through the vector $-i\lambda$. It was proved by Kostant in [13], that the projection of $\mathcal{O}_\lambda^K$ on $\mathfrak{t}_\mathfrak{k}^*$ is a convex polytope with vertices $-i(w.\lambda)$ for $w \in W_K$, and that is the convex hull of $-i(W_K.\lambda)$. For the same manner, we fix a positive Weyl chamber C_H^+ in $\mathfrak{t}_\mathfrak{h}^*$ and we define the set P_H^+ of dominant integral weights of T_H.

It is well-known that $\widehat{K}$ (resp. $\widehat{H}$) is in bijective correspondence with P_K^+ (resp. P_H^+), and hence

$$(\mathfrak{g}^\ddagger/G)_{H,K} \simeq P_H^+ \times \Lambda_{(H,K)}.$$

Before the study of convergence in the quotient space $\mathfrak{g}^*/G$, we need the following lemma (see [14, p. 135]).

Lemma 6.1. *Let G be a unimodular Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^*$ be the vector dual space of $\mathfrak{g}$. We denote $\mathfrak{g}^*/G$ the space of coadjoint orbits and by $p_G : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*/G$ the canonical projection. We equip this space with* *the quotient topology, i.e., a subset V in $\mathfrak{g}^*/G$ is open if and only if $p_G^{-1}(V)$ is open in $\mathfrak{g}^*$. Therefore, a sequence $(\mathcal{O}_k^G)_k$ of elements in $\mathfrak{g}^*/G$ converges to the orbit $\mathcal{O}^G$ in $\mathfrak{g}^*/G$ if and only if for any $l \in \mathcal{O}^G$, there exist $l_k \in \mathcal{O}_k^G$, $k \in \mathbb{N}$, such that $l = \lim_{k \longrightarrow +\infty} l_k$.*

Now we are ready to prove the following result.

PROPOSITION 3. *Let $\left(\mathcal{O}^G_{(\mu^n,\ell_n)}\right)_n$ be a sequence of $(\mathfrak{g}^\ddagger/G)_{H,K}$. Then $\left(\mathcal{O}^G_{(\mu^n,\ell_n)}\right)_n$ converges to $\mathcal{O}^G_{(\mu,\ell)}$ in $(\mathfrak{g}^\ddagger/G)_{H,K}$ if and only if $(\ell_n)_n$ tends to ℓ as $n \longrightarrow +\infty$ and $\mu^n = \mu$ for n large enough.*

Proof. Let at first recall from Lemma 3.1 that the coadjoint orbit $\mathcal{O}^G_{(\mu,\ell)}$ is always obtained by symplectic induction from the coadjoint orbit $M = \mathcal{O}^{H \ltimes V}_{(\mu,\ell)}$ of $H \ltimes V$ passing through $(\mu, \ell) \in \mathfrak{h}^* \oplus V^*$, *i.e.*,

$$\mathcal{O}^G_{(\mu,\ell)} = M_{ind} = J^{-1}_{\widetilde{M}}(0)/(H \ltimes V),$$

where $J_{\widetilde{M}} : \widetilde{M} = M \times T^*G \longrightarrow \mathfrak{h} \ltimes V := Lie(H \ltimes V)$ is the momentum map of $\widetilde{M}$ and the zero level set $J^{-1}_{\widetilde{M}}(0)$ is given by

$$J^{-1}_{\widetilde{M}}(0) = \Big\{\Big((Ad^*_K(k)\mu, \ell), g, (Ad^*_K(k)\mu + \ell \odot v, \ell)\Big),\ k \in H, g \in G, v \in V\Big\}.$$

So if the sequence of admissible coadjoint orbit $\left(\mathcal{O}^G_{(\mu^n,\ell_n)}\right)_n$ converges to $\mathcal{O}^G_{(\mu,\ell)}$ in $(\mathfrak{g}^\ddagger/G)_{H,K}$, then by Lemma 6.1 and the relation (2.1) there exist sequences $(k_n)_n, (h_n)_n \subset H$, $(v_n)_n, (w_n)_n \subset V$, and $(g_n)_n \subset G$ such that the sequence $(F_n)_n$ defined by

$$\begin{aligned} F_n &= \varphi_{\widetilde{M}}(k_n, v_n)\big((Ad^*_K(h_n)\mu^n, \ell_n), g_n, (Ad^*_K(h_n)\mu^n + \ell_n \odot w_n, \ell_n)\big) \\ &= \Big(Ad^*_K(k_n h_n)\mu^n + \imath^*(\ell_n \odot v_n), \ell_n\Big), g_n(k_n, v_n)^{-1}, (Ad^*_K(k_n h_n)\mu^n \\ &+ Ad^*_K(k_n)\ell_n \odot w_n + \ell_n \odot v_n, \ell_n)\Big) \end{aligned}$$

converges to $\big((\mu,\ell), e_G, (\mu,\ell)\big)$. In particular, we have

$$\big(Ad^*_K(k_n h_n)\mu^n + \imath^*(\ell_n \odot v_n), \ell_n\big) \longrightarrow (\mu, \ell) \tag{6.1}$$

as $n \longrightarrow +\infty$. It is clear that $(\ell_n)_n$ tends to ℓ as $n \longrightarrow +\infty$. By observing that $\imath^*(\ell_n \odot v_n) = 0$ and by compactness of H we may assume that the sequence $(k_n h_n)_n$ converges to $h_0 \in H$. From the relation (6.1) we get

$$\mu^n = Ad^*_K(h_0^{-1})\mu$$

for n large enough. Furthermore, we know that there exists s in the Weyl group W_H such that

$$Ad^*_K(h_0^{-1})\mu = s.\mu$$

(see [10]). Hence $\mu^n = s.\mu$ for n large enough. Since the weights μ^n and μ are contained in the set iC_H^+ and since every W_H-orbit in $\mathfrak{h}^*$ intersects the closure $\overline{iC_H^+}$ in exactly one point (see [4, p. 203]), it follows that $\mu^n = \mu$ for n large enough.

Conversely, let us assume that $(\ell_n)_n$ converges to ℓ and $\mu^n = \mu$ for n large enough, then we have

$$\lim_{n \longrightarrow +\infty} (\mu^n, \ell_n) = (\mu, \ell).$$

By Lemma 6.1, we deduce that $(\mathcal{O}^G_{(\mu^n,\ell_n)})_n$ converges to $\mathcal{O}^G_{(\mu,\ell)}$. $\square$

According to Proposition 2 and Proposition 3, we get the following result.

THEOREM 6.2. *The topological space $(\widehat{G})_{H,K}$ is homeomorphic to the subspace $(\mathfrak{g}^\ddagger/G)_{H,K}$ of $\mathfrak{g}^\ddagger/G$.*

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