NEW INEQUALITIES AND ERASURES FOR CONTINUOUS G-FRAMES

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This paper addresses continuous g-frames which are extensions of g-frames and continuous frames. Firstly, using operator methods we establish some new inequalities for continuous g-frames and dual continuous g-frames. These results extend and improve ones obtained by Balan, Casazza and Găvruţa. Secondly, we characterize multi-element erasure for continuous g-frames. It generalizes some results previously obtained by M.A. Dehghan and M.A. Hasankhani Fard.

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1. INTRODUCTION

The notion of frame in a general Hilbert space was first introduced by Duffin and Schaeffer in 1952 to study nonharmonic Fourier series [8]. However, the frame theory had not interested many researchers until Daubechies, Crossman and Meyer published their ground breaking work [7] in 1986. In recent years, the study of frame theory has seen great achievements, and discrete frames are widely used in signal processing, quantum measurements, image processing, coding and communication and some other fields [4,6,14,17,18,21]. The study of equalities and inequalities related to Paraeval frames were studied by Balan *et al.* in [3] and many other mathematicians [11,12,16,19]. The notion of frame was generalized to a family indexed by some locally compact space endowed with Radon measure by Ali *et al.* in [2] known as continuous frame. Continuous frames are applied in some fields [10,20]. In particular, Sun in 2006 introduced g-frames in Hilbert space in [22], which includes many generalizations of the discrete frame, for example, frames of subspaces [5], pseudo-frames [15], and bounded quasi-projectors [9], and so on. The notion of g-frame is an extension that includes bounded invertible operators and all mentioned above extensions of discrete frames. The notion of continuous g-frame was firstly introduced

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by M.A. Dehghan and M.A. Hasankhani Fard in [1], which is an extension of g-frames and continuous frames.

In this paper, using the method of operator theory we establish some new inequalities for (dual) continuous g-frames, which extend and improve the results obtained by Balan, Casazza and Găvruţa; we obtain a characterization of multi-element erasure for continuous g-frames, which generalize the results obtained by M.A. Dehghan and M.A. Hasankhani Fard.

This paper is organized as follows. Section 2 is an auxiliary one and in this section, we recall some basic notions, properties and some related results. In Section 3, using the method of operator theory we obtain some important inequalities for (dual) continuous g-frames. In Section 4, we derive an equivalent characterization of multi-element erasure for continuous g-frames.

2. PRELIMINARIES

First we recall some basic notations, notions and properties of frames in Hilbert space. The readers can refer to [1, 6, 20, 22] for details.

Let U, V be separable Hilbert spaces, $(\Omega,\, \mu)$ a positive measure space, and I a countable index set. We denote by I_U the identity operator on U, $\{V_\omega\}_{\omega\in\Omega}$ a sequence of closed subspaces of V, and $L(U,\, V_\omega)$ the set of all bounded linear operators from U into V_ω. Let

$$(\oplus_{\omega\in\Omega} V_\omega,\, \mu)_{L^2} = \left\{ f = \{f_\omega\}_{\omega\in\Omega}, f_\omega : \Omega \to U : \int_\Omega \|f_\omega\|^2 d\mu(\omega) < \infty \right\}.$$

Then $(\oplus_{\omega\in\Omega} V_\omega,\, \mu)_{L^2}$ is a Hilbert space under the following inner product :

$$\langle f,\, g\rangle = \int_\Omega \langle f_\omega,\, g_\omega\rangle d\mu(\omega) \qquad f, g \in (\oplus_{\omega\in\Omega} V_\omega,\, \mu)_{L^2}.$$

Definition 2.1 ([22, Definition 1.1]). A sequence $\{\Lambda_i \in L(U,\, V_i)\}_{i\in I}$ is called a g-frame for U with respect to $\{V_i\}_{i\in I}$ if there exist $0 < A_1 \le B_1 < +\infty$ such that

$$\forall f \in U, \qquad A_1\|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le B_1\|f\|^2.$$

The numbers $A_1,\ B_1$ are called a lower and upper bound for the frame.

Definition 2.2 ([20, Definition 2.1]). Let (X, μ) be a measure space with positive measure μ. Let $f : X \to H$ be weakly measurable (i.e., for all $h \in H$, the mapping $x \to \langle f(x), h\rangle$ is measurable). Then $\{f(x)\}_{x\in X}$ is called a continuous frame for H if there exist constants $0 < A_2 \le B_2 < +\infty$ such that

$$\forall h \in H, \qquad A_2\|h\|^2 \le \int_X |\langle f(x), h\rangle|^2 d\mu(x) \le B_2\|h\|^2. \tag{2.1}$$

We call A_2 and B_2 the lower and upper continuous frame bound, respectively. If only the right-hand inequality of (2.1) is satisfied, we call $\{f(x)\}_{x\in X}$ the continuous Bessel sequence for H with Bessel bound B_2. If $A_2 = B_2 = \lambda$, we call $\{f(x)\}_{x\in X}$ λ-tight continuous frame. Moreover, if $\lambda = 1$, $\{f(x)\}_{x\in X}$ is called Parseval continuous frame.

Definition 2.3 ([1, Definition 2.1]). We say that $\Lambda = \{\Lambda_\omega \in L(U, V_\omega)\}_{\omega\in\Omega}$ is a continuous g-frame for U with respect to (Ω, μ), if

(i) Λ is weakly-measurable, i.e., for $f \in U$, $\omega \to \Lambda_\omega$ is a measurable function on Ω,

(ii) there exist positive constants A, B such that

$$\forall f \in H, \qquad A\|f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2. \tag{2.2}$$

The numbers A, B are called a lower and upper frame bound for the continuous g- frame, respectively. If only the right-hand inequality of (2.2) is satisfied, we call $\{\Lambda_\omega\}_{\omega\in\Omega}$ a Bessel continuous g-mapping for U with respect to (Ω, μ) with bound B. If $A = B = \lambda$, we call $\{\Lambda_\omega\}_{\omega\in\Omega}$ λ-tight continuous g-frame. Moreover, if $\lambda = 1$, $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called Parsevel continuous g-frame.

Remark 2.1. A continuous g-frame is a generalization of g-frame. Indeed, when Ω is countable, and μ is a counting measure, a continuous g-frame is just a g-frame.

Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous g-frame for U with respect to (Ω, μ). In [1], the authors defined the continuous g-frame operator $S : U \to U$ as follows:

$$S(f) = \int_\Omega \Lambda_\omega^* \Lambda_\omega f d\mu(\omega), \quad \forall f \in U. \tag{2.3}$$

It is easy to check that S is a bounded, positive, self-adjoint and invertible operator. Denote $\tilde{\Lambda}_\omega = \Lambda_\omega S^{-1}$, then $\{\tilde{\Lambda}_\omega\}_{\omega\in\Omega}$ is a continuous g-frame for U with respect to (Ω, μ) with frame bounds $\frac{1}{B}$, $\frac{1}{A}$, the frame operator S^{-1}, which is called the canonical dual continuous g-frame of $\{\Lambda_\omega\}_{\omega\in\Omega}$ (see [1]).

For any $\Omega_1 \subset \Omega$, denote $\Omega_1^c = \Omega\backslash\Omega_1$, and we define the following operators:

$$S_{\Omega_1} f = \int_{\Omega_1} \Lambda_\omega^* \Lambda_\omega f d\mu(\omega), \quad \forall f \in U. \tag{2.4}$$

$$S_{\Omega_1^c} f = \int_{\Omega_1^c} \Lambda_\omega^* \Lambda_\omega f d\mu(\omega), \quad \forall f \in U. \tag{2.5}$$

Then $S = S_{\Omega_1} + S_{\Omega_1^c}$, and S_{Ω_1}, $S_{\Omega_1^c}$ are positive and self-adjoint operators.

Definition 2.4 ([1], Definition 3.1). Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Theta_\omega\}_{\omega\in\Omega}$ be two continuous g-frames for U with respect to $(\Omega,\,\mu)$ such that

$$f=\int_\Omega \Lambda_\omega^*\Theta_\omega f d\mu(\omega)=\int_\Omega \Theta_\omega^*\Lambda_\omega f d\mu(\omega),\qquad f\in U.$$

Then $\{\Theta_\omega\}_{\omega\in\Omega}$ is called an alternate dual continuous g-frame of $\{\Lambda_\omega\}_{\omega\in\Omega}$.

3. NEW INEQUALITIES FOR CONTINUOUS G-FRAMES

Balan *et al.* in [3] obtained the following Theorem 3.1, and P. Găvruţa in [11] obtained the following Theorem 3.2.

THEOREM 3.1. *Let $\{f_j\}_{j\in J}\subset H$ be a Parseval frame. For any $f\in H$, $J_1\subset J$, we have*

$$\sum_{j\in J_1}|\langle f,\,f_j\rangle|^2+\|\sum_{j\in J_1^c}\langle f,\,f_j\rangle f_j\|^2\geq\frac{3}{4}\|f\|^2 \tag{3.6}$$

where $J_1^c=J\setminus J_1$.

THEOREM 3.2. *Let $\{f_j\}_{j\in J}\subset H$ be a frame and $\{g_j\}_{j\in J}\subset H$ be an alternate dual frame of $\{f_j\}_{j\in J}$. Then for any $f\in H$, we have*

$$\mathrm{Re}(\sum_{j\in J_1}\langle f,\,g_j\rangle\overline{\langle f,\,f_j\rangle})+\|\sum_{j\in J_1^c}\langle f,\,g_j\rangle f_j\|^2\geq\frac{3}{4}\|f\|^2 \tag{3.7}$$

This section is devoted to some inequalities for continuous g-frames. Using operator theory method we extend these two theorems to the case of continuous g-frames. We also obtain some other interesting results. For this purpose, we first give a simple property of self-adjoint operators.

LEMMA 3.1. *Let $T\in L(H)$ be a self-adjoint operator and $a,\,b,\,c\in\mathbb{R}$, $U=aT^2+bT+cI_H$, then the following statements hold.*

(i) if $a>0$, then

$$U\geq\frac{4ac-b^2}{4a}I_H.$$

(ii) if $a<0$, then

$$U\leq\frac{4ac-b^2}{4a}I_H.$$

Proof. We only prove (i), and (ii) can be proved similarly. It is easy to check that

$$U=a(T+\frac{b}{2a}I_H)^2+\frac{4ac-b^2}{4a}I_H.$$

Observing that $(T+\frac{b}{2a}I_H)^2$ is a positive operator, we have (i). $\square$

PROPOSITION 3.1. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous g-frame for U with respect to (Ω, μ) with frame operator S. Then $\{\Lambda_\omega S^{-\frac{1}{2}}\}_{\omega\in\Omega}$ is a Parseval continuous g-frame for U with respect to (Ω, μ).*

Proof. Take $T = S^{-\frac{1}{2}}$ in Proposition 3.3 of [1]. $\square$

THEOREM 3.3. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous g-frame for U with respect to (Ω, μ) with frame operator S. Then for $\Omega_1 \subset \Omega$ and $f \in U$, we have*

$$0 \leq \int_{\Omega_1} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega) - \int_{\Omega} \|\tilde{\Lambda}_\omega S_{\Omega_1} f\|^2 \mathrm{d}\mu(\omega) \leq \frac{1}{4} \int_{\Omega} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega), \tag{3.8}$$

$$\begin{aligned}\frac{1}{2} \int_{\Omega} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega) \leq \int_{\Omega} \|\tilde{\Lambda}_\omega S_{\Omega_1} f\|^2 \mathrm{d}\mu(\omega) + \int_{\Omega} \|\tilde{\Lambda}_\omega S_{\Omega_1^c} f\|^2 \mathrm{d}\mu(\omega) \\ \leq \frac{3}{2} \int_{\Omega} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega),\end{aligned} \tag{3.9}$$

$$\begin{aligned}\frac{3}{4} \int_{\Omega} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega) \leq \int_{\Omega_1} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega) + \int_{\Omega} \|\tilde{\Lambda}_\omega S_{\Omega_1} f\|^2 \mathrm{d}\mu(\omega) \\ \leq \int_{\Omega} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega),\end{aligned} \tag{3.10}$$

where $\{\tilde{\Lambda}_\omega\}_{\omega\in\Omega} = \{\Lambda_\omega S^{-1}\}_{\omega\in\Omega}$ is the canonical dual continuous g-frame of $\{\Lambda_\omega\}_{\omega\in\Omega}$.

Proof. Denote $\Theta_\omega = \Lambda_\omega S^{-\frac{1}{2}}$ for $\omega \in \Omega$, $\{\Theta_\omega\}_{\omega\in\Omega}$ is a continuous g-frame for U with respect to (Ω, μ) by Proposition 3.1. For any $f \in U$, let $\hat{S} f = \int_{\Omega} \Theta_\omega^* \Theta_\omega f \mathrm{d}\mu(\omega) = f$, then

$$\hat{S}_{\Omega_1} f = \int_{\Omega_1} \Theta_\omega^* \Theta_\omega f \mathrm{d}\mu(\omega) = \int_{\Omega_1} S^{-\frac{1}{2}} \Lambda_\omega^* \Lambda_\omega S^{-\frac{1}{2}} f \mathrm{d}\mu(\omega) = S^{-\frac{1}{2}} S_{\Omega_1} S^{-\frac{1}{2}} f. \tag{3.11}$$

Obviously, $\hat{S}_{\Omega_1} + \hat{S}_{\Omega_1^c} = I_U$, furthermore, we have $\hat{S}_{\Omega_1} \hat{S}_{\Omega_1^c} = \hat{S}_{\Omega_1^c} \hat{S}_{\Omega_1}$, so

$$0 \leq \hat{S}_{\Omega_1} \hat{S}_{\Omega_1^c} = \hat{S}_{\Omega_1}(I_U - \hat{S}_{\Omega_1}) = \hat{S}_{\Omega_1} - (\hat{S}_{\Omega_1})^2. \tag{3.12}$$

By Lemma 3.1, we obtain

$$\hat{S}_{\Omega_1} - (\hat{S}_{\Omega_1})^2 \leq \frac{1}{4} I_U. \tag{3.13}$$

Combining (3.11), (3.12) and (3.13), we have

$$0 \leq S^{-\frac{1}{2}} (S_{\Omega_1} - S_{\Omega_1} S^{-1} S_{\Omega_1}) S^{-\frac{1}{2}} \leq \frac{1}{4} I_U. \tag{3.14}$$

(3.14) is equivalent to

$$0 \leq S_{\Omega_1} - S_{\Omega_1} S^{-1} S_{\Omega_1} \leq \frac{1}{4} S. \tag{3.15}$$

For any $f \in U$, we have

$$\begin{aligned}\langle S_{\Omega_1} f,\, f\rangle - \langle S_{\Omega_1} S^{-1} S_{\Omega_1} f,\, f\rangle =& \langle S_{\Omega_1} f,\, f\rangle - \langle S^{-1} S_{\Omega_1} f,\, S_{\Omega_1} f\rangle \\ =& \int_{\Omega_1} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega) - \int_{\Omega} \|\tilde{\Lambda}_\omega S_{\Omega_1} f\|^2 \mathrm{d}\mu(\omega).\end{aligned}$$

Therefore, we obtain (3.8) by (3.15). Next we prove (3.9).

It is easy to check that

$$\begin{aligned}(\hat{S}_{\Omega_1})^2 + (\hat{S}_{\Omega_1^c})^2 =& (\hat{S}_{\Omega_1})^2 + (I_U - \hat{S}_{\Omega_1})^2 \\ =& 2(\hat{S}_{\Omega_1})^2 - 2\hat{S}_{\Omega_1} + I_U\end{aligned} \tag{3.16}$$

We have

$$(\hat{S}_{\Omega_1})^2 + (\hat{S}_{\Omega_1^c})^2 \geq \frac{1}{2} I_U. \tag{3.17}$$

by Lemma 3.1. By simple calculation, we have

$$\begin{aligned}(\hat{S}_{\Omega_1})^2 + (\hat{S}_{\Omega_1^c})^2 =& 2(\hat{S}_{\Omega_1})^2 - 2\hat{S}_{\Omega_1} + I_U \\ =& I_U + 2\hat{S}_{\Omega_1} - 2(\hat{S}_{\Omega_1})^2 + 4((\hat{S}_{\Omega_1})^2 - \hat{S}_{\Omega_1})\end{aligned} \tag{3.18}$$

Thus, we have

$$(\hat{S}_{\Omega_1})^2 + (\hat{S}_{\Omega_1^c})^2 \leq I_H + 2\hat{S}_{\Omega_1} - 2(\hat{S}_{\Omega_1})^2$$

by (3.12). Again by Lemma 3.1, we get

$$(\hat{S}_{\Omega_1})^2 + (\hat{S}_{\Omega_1^c})^2 \leq \frac{3}{2} I_U \tag{3.19}$$

combining (3.17),

$$\frac{1}{2} I_U \leq (\hat{S}_{\Omega_1})^2 + (\hat{S}_{\Omega_1^c})^2 \leq \frac{3}{2} I_U \tag{3.20}$$

(3.20) is equivalent to

$$\frac{1}{2} S \leq S_{\Omega_1} S^{-1} S_{\Omega_1} + S_{\Omega_1^c} S^{-1} S_{\Omega_1^c} \leq \frac{3}{2} S \tag{3.21}$$

For any $f \in U$, we have

$$\begin{aligned}\langle S_{\Omega_1} S^{-1} S_{\Omega_1} f,\, f\rangle + \langle S_{\Omega_1^c} S^{-1} S_{\Omega_1^c} f,\, f\rangle =& \langle S^{-1} S_{\Omega_1} f,\, S_{\Omega_1} f\rangle + \langle S^{-1} S_{\Omega_1^c} f,\, S_{\Omega_1^c} f\rangle \\ =& \int_{\Omega} \|\tilde{\Lambda}_\omega S_{\Omega_1} f\|^2 \mathrm{d}\mu(\omega) + \int_{\Omega} \|\tilde{\Lambda}_\omega S_{\Omega_1^c} f\|^2 \mathrm{d}\mu(\omega)\end{aligned}$$

By using (3.21), we know that (3.9) holds.

Finally, we prove (3.10). Observe that

$$\begin{aligned} \hat{S}_{\Omega_1} + (\hat{S}_{\Omega_1})^2 =& \hat{S}_{\Omega_1} + (I_U - \hat{S}_{\Omega_1})^2 \\ =& (\hat{S}_{\Omega_1})^2 - \hat{S}_{\Omega_1} + I_U. \end{aligned} \tag{3.22}$$

and that $\hat{S}_{\Omega_1} - (\hat{S}_{\Omega_1})^2 \geq 0$ by (3.12). We have

$$\frac{3}{4} I_U \leq \hat{S}_{\Omega_1} + (\hat{S}_{\Omega_1})^2 \leq I_U, \tag{3.23}$$

by Lemma 3.1. Therefore, we have

$$\frac{3}{4} S \leq S_{\Omega_1} + S_{\Omega_1} S^{-1} S_{\Omega_1} \leq S, \tag{3.24}$$

by (3.11). For $f \in U$, we have

$$\begin{aligned} \langle (S_{\Omega_1} + S_{\Omega_1} S^{-1} S_{\Omega_1}) f,\, f\rangle =& \langle S_{\Omega_1} f,\, f\rangle + \langle S_{\Omega_1} S^{-1} S_{\Omega_1} f,\, f\rangle \\ =& \langle S_{\Omega_1} f,\, f\rangle + \langle S^{-1} S_{\Omega_1} f,\, S_{\Omega_1} f\rangle \\ =& \int_{\Omega_1} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega) + \int_{\Omega} \|\tilde{\Lambda}_\omega S_{\Omega_1} f\|^2 \mathrm{d}\mu(\omega) \end{aligned}$$

Combining this and (3.24), (3.10) holds. The proof is completed. □

Corollary 3.1. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous Parseval g-frame for U with respect to $(\Omega,\, \mu)$. Then for $\Omega_1 \subset \Omega$ and $f \in U$, we have*

$$0 \leq \int_{\Omega_1} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega) - \left\| \int_{\Omega_1} \Lambda_\omega^* \Lambda_\omega f \mathrm{d}\mu(\omega) \right\| \leq \frac{1}{4} \|f\|^2. \tag{3.25}$$

$$\frac{1}{2}\|f\|^2 \leq \left\| \int_{\Omega_1} \Lambda_\omega^* \Lambda_\omega f \mathrm{d}\mu(\omega) \right\| + \left\| \int_{\Omega_1^c} \Lambda_\omega^* \Lambda_\omega f \mathrm{d}\mu(\omega) \right\| \leq \frac{3}{2}\|f\|^2. \tag{3.26}$$

$$\frac{3}{4}\|f\|^2 \leq \int_{\Omega_1} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega) + \left\| \int_{\Omega_1} \Lambda_\omega^* \Lambda_\omega f \mathrm{d}\mu(\omega) \right\| \leq \|f\|^2. \tag{3.27}$$

Proof. $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous Parseval g-frame for U with respect to $(\Omega,\, \mu)$, for $f \in U$, we have

$$\int_{\Omega} \|\Lambda_\omega\|^2 \mathrm{d}\mu(\omega) = \|f\|^2.$$

Observe that the frame operator of $\{\Lambda_\omega\}_{\omega\in\Omega}$ is I_U, therefore, for $f \in U$, we also have

$$\int_{\Omega} \|\tilde{\Lambda}_\omega S_{\Omega_1} f\|^2 \mathrm{d}\mu(\omega) = \int_{\Omega} \|\Lambda_\omega S_{\Omega_1} f\|^2 \mathrm{d}\mu(\omega) = \|S_{\Omega_1} f\|^2 = \left\| \int_{\Omega_1} \Lambda_\omega^* \Lambda_\omega f \mathrm{d}\mu(\omega) \right\|,$$

and

$$\int_{\Omega} \|\tilde{\Lambda}_{\omega} S_{\Omega_1^c} f\|^2 \mathrm{d}\mu(\omega) = \left\| \int_{\Omega_1^c} \Lambda_{\omega}^* \Lambda_{\omega} f \mathrm{d}\mu(\omega) \right\|.$$

Hence (3.25), (3.26), (3.27) hold. The proof is completed. $\square$

Observe that $\{\frac{1}{\sqrt{\lambda}}\Lambda_{\omega}\}_{\omega\in\Omega}$ is a continuous Parseval g-frame if $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ is a continuous λ-tight g-frame for U with respect to (Ω, μ). As an immediate consequence of Corollary 3.1, we have

COROLLARY 3.2. *Let $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ be a continuous λ-tight g-frame for U with respect to (Ω, μ). Then for $\Omega_1 \subset \Omega$ and $f \in U$, we have*

$$0 \le \int_{\Omega_1} \|\Lambda_{\omega} f\|^2 \mathrm{d}\mu(\omega) - \left\| \int_{\Omega_1} \Lambda_{\omega}^* \Lambda_{\omega} f \mathrm{d}\mu(\omega) \right\| \le \frac{\lambda}{4}\|f\|^2. \tag{3.28}$$

$$\frac{\lambda}{2}\|f\|^2 \le \left\| \int_{\Omega_1} \Lambda_{\omega}^* \Lambda_{\omega} f \mathrm{d}\mu(\omega) \right\| + \left\| \int_{\Omega_1^c} \Lambda_{\omega}^* \Lambda_{\omega} f \mathrm{d}\mu(\omega) \right\| \le \frac{3\lambda}{2}\|f\|^2. \tag{3.29}$$

$$\frac{3\lambda}{4}\|f\|^2 \le \int_{\Omega_1} \|\Lambda_{\omega} f\|^2 \mathrm{d}\mu(\omega) + \left\| \int_{\Omega_1} \Lambda_{\omega}^* \Lambda_{\omega} f \mathrm{d}\mu(\omega) \right\| \le \lambda\|f\|^2. \tag{3.30}$$

Next we will give an inequality for dual continuous g-frames. To do so, we first give the following lemma.

LEMMA 3.2. *Let $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ be a continuous g-frame for U with respect to (Ω, μ), $\{\Gamma_{\omega}\}_{\omega\in\Omega}$ be an alternate dual continuous g-frame of $\{\Lambda_{\omega}\}_{\omega\in\Omega}$, and $a = \{a_{\omega}\}_{\omega\in\Omega} \in l^{\infty}(\Omega)$. Define the operator T_a as follows:*

$$T_a : U \to U, T_a f = \int_{\Omega} a_{\omega} \Gamma_{\omega}^* \Lambda_{\omega} f \mathrm{d}\mu(\omega), \quad \forall f \in U,$$

then T_a is a bounded linear operator, and

$$T_a^* f = \int_{\Omega} \bar{a}_{\omega} \Lambda_{\omega}^* \Gamma_{\omega} f \mathrm{d}\mu(\omega).$$

Where $l^{\infty}(\Omega) = \{\{a_{\omega}\}_{\omega\in\Omega} : \sup\limits_{\omega\in\Omega} |a_{\omega}| < \infty\}$.

Proof. For $\Omega_1 \subset \Omega$ and $f \in U$, we have

$$\begin{aligned} \left\| \int_{\Omega_1} a_{\omega} \Gamma_{\omega}^* \Lambda_{\omega} f \mathrm{d}\mu(\omega) \right\| &= \sup_{g\in U, \|g\|=1} \left| \left\langle \int_{\Omega_1} a_{\omega} \Gamma_{\omega}^* \Lambda_{\omega} f \mathrm{d}\mu(\omega),\, g \right\rangle \right| \\ &= \sup_{g\in U, \|g\|=1} \left| \int_{\Omega_1} \left\langle \Gamma_{\omega}^* \Lambda_{\omega} f,\, \bar{a}_{\omega} g \right\rangle \mathrm{d}\mu(\omega) \right| \\ &= \sup_{g\in U, \|g\|=1} \left| \int_{\Omega_1} \left\langle \Lambda_{\omega} f,\, \bar{a}_{\omega} \Gamma_{\omega} g \right\rangle \mathrm{d}\mu(\omega) \right| \end{aligned}$$

$$\le \sup_{g\in U, \|g\|=1} \left(\int_{\Omega_1} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega)\right)^{\frac{1}{2}}$$
$$\cdot \left(\int_{\Omega_1} \|\bar{a}_\omega \Gamma_\omega g\|^2 \mathrm{d}\mu(\omega)\right)^{\frac{1}{2}}$$
$$\le \sqrt{BB'} M\|f\|,$$

where $M = \sup_{\omega\in\Omega} |a_\omega|$, $\overline{a}_\omega$ is the conjugate of a_ω and B' is the upper bound of $\{\Gamma_\omega\}_{\omega\in\Omega}$. Hence T_a is well-defined and $\|T_a f\| \le \sqrt{BB'}M\|f\|$. Therefore, T_a is a bounded linear operator. Now let us compute $(T_a)^*$.

$$\begin{aligned}\langle f,\, (T_a)^* g\rangle = \langle T_a f,\, g\rangle &= \left\langle \int_\Omega a_\omega \Gamma_\omega^* \Lambda_\omega f \mathrm{d}\mu(\omega),\, g \right\rangle \\ &= \int_\Omega \langle \Lambda_\omega f,\, \bar{a}_\omega \Gamma_\omega g\rangle \mathrm{d}\mu(\omega) \\ &= \int_\Omega \langle f,\, \bar{a}_\omega \Lambda_\omega^* \Gamma_\omega g\rangle \mathrm{d}\mu(\omega) \\ &= \left\langle f,\, \int_\Omega \bar{a}_\omega \Lambda_\omega^* \Gamma_\omega g \mathrm{d}\mu(\omega)\right\rangle.\end{aligned}$$

The proof is completed. □

THEOREM 3.4. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous g-frame for U with respect to (Ω, μ), $\{\Gamma_\omega\}_{\omega\in\Omega}$ be an alternate dual continuous g-frame of $\{\Lambda_\omega\}_{\omega\in\Omega}$, and $\{a_\omega\}_{\omega\in\Omega} \in l^\infty(\Omega)$. Then for $f \in U$, we have*

$$\begin{aligned}\frac{3}{4}\|f\|^2 &\le \left\|\int_\Omega a_\omega \Gamma_\omega^* \Lambda_\omega f \mathrm{d}\mu(\omega)\right\|^2 + \mathrm{Re}\Big(\int_\Omega (1-a_\omega)\langle \Lambda_\omega f,\, \Gamma_\omega f\rangle \mathrm{d}\mu(\omega)\Big) \\ &= \left\|\int_\Omega (1-a_\omega) \Gamma_\omega^* \Lambda_\omega f \mathrm{d}\mu(\omega)\right\|^2 + \mathrm{Re}\Big(\int_\Omega a_\omega\langle \Lambda_\omega f,\, \Gamma_\omega f\rangle \mathrm{d}\mu(\omega)\Big) \\ &\le \frac{3 + \|T_a - T_{1-a}\|^2}{4}\|f\|^2 \end{aligned} \tag{3.31}$$

Proof. First we prove the "equality" part. Let $T_{1-a} f = \int_\Omega (1-a_\omega)\Gamma_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega)$, $\forall f \in U$, then

$$T_{1-a}^* f = \int_\Omega (1-\bar{a}_\omega)\Lambda_\omega^* \Gamma_\omega f \mathrm{d}\mu(\omega), \quad \forall f\in U,$$

and $T_a + T_{1-a} = I_U$. So for $f \in U$, we have

$$\begin{aligned}&\left\|\int_\Omega a_\omega \Gamma_\omega^* \Lambda_\omega f \mathrm{d}\mu(\omega)\right\|^2 + \int_\Omega (1-a_\omega)\langle \Lambda_\omega f,\, \Gamma_\omega f\rangle \mathrm{d}\mu(\omega) \\ &= \|T_a f\|^2 + \langle T_{1-a} f,\, f\rangle = \langle T_a f,\, T_a f\rangle + \langle (I_U - T_a) f,\, f\rangle\end{aligned}$$

$$= \langle T_a f, T_a f\rangle + \langle f, f\rangle - \langle T_a f, f\rangle. \tag{3.32}$$

On the other hand,

$$\begin{aligned}
&\left\|\int_\Omega (1-a_\omega)\Gamma_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega)\right\|^2 + \overline{\int_\Omega a_\omega \langle \Lambda_\omega f, \Gamma_\omega f\rangle \mathrm{d}\mu(\omega)}\\
&= \|T_{1-a} f\|^2 + \overline{\langle T_a f, f\rangle} = \langle T_{1-a} f, T_{1-a} f\rangle + \langle f, T_a f\rangle\\
&= \langle (I_U - T_a) f, (I_U - T_a) f\rangle + \langle f, T_a f\rangle\\
&= \langle f, f\rangle - \langle T_a f, f\rangle + \langle T_a f, T_a f\rangle.
\end{aligned} \tag{3.33}$$

Therefore, by (3.32) and (3.33), we have

$$\begin{aligned}
&\left\|\int_\Omega a_\omega \Gamma_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega)\right\|^2 + \int_\Omega (1-a_\omega)\langle \Lambda_\omega f, \Gamma_\omega f\rangle \mathrm{d}\mu(\omega)\\
&= \left\|\int_\Omega (1-a_\omega)\Gamma_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega)\right\|^2 + \overline{\int_\Omega a_\omega \langle \Lambda_\omega f, \Gamma_\omega f\rangle \mathrm{d}\mu(\omega)}.
\end{aligned} \tag{3.34}$$

Thus

$$\begin{aligned}
&\left\|\int_\Omega a_\omega \Gamma_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega)\right\|^2 + \mathrm{Re}\Big(\int_\Omega (1-a_\omega)\langle \Lambda_\omega f, \Gamma_\omega f\rangle \mathrm{d}\mu(\omega)\Big)\\
&= \left\|\int_\Omega (1-a_\omega)\Gamma_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega)\right\|^2 + \mathrm{Re}\Big(\int_\Omega a_\omega \langle \Lambda_\omega f, \Gamma_\omega f\rangle \mathrm{d}\mu(\omega)\Big).
\end{aligned} \tag{3.35}$$

Next we prove the "left-hand inequality" part. By Lemma 3.2, we have

$$\mathrm{Re}\Big(\int_\Omega a_\omega \langle \Lambda_\omega f, \Gamma_\omega f\rangle \mathrm{d}\mu(\omega)\Big) = \left\langle \frac{T_a + T_a^*}{2} f, f\right\rangle.$$

Thus for $h \in H$, we have

$$\begin{aligned}
&\left\|\int_\Omega (1-a_\omega)\Gamma_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega)\right\|^2 + \mathrm{Re}\Big(\int_\Omega a_\omega \langle \Lambda_\omega f, \Gamma_\omega f\rangle \mathrm{d}\mu(\omega)\Big)\\
&= \left\langle \left(T_{1-a}^* T_{1-a} + \frac{T_a + T_a^*}{2}\right) f, f\right\rangle\\
&= \left\langle \left((I_U - T_a^*)(I_U - T_a) + \frac{T_a + T_a^*}{2}\right) f, f\right\rangle\\
&= \left\langle \left(I_U + T_a^* T_a - \frac{T_a + T_a^*}{2}\right) f, f\right\rangle\\
&= \left\langle \left[\left(T_a - \frac{1}{2} I_U\right)^* \left(T_a - \frac{1}{2} I_U\right) + \frac{3}{4} I_U\right] f, f\right\rangle\\
&= \left\|\left(T_a - \frac{1}{2} I_U\right) f\right\|^2 + \frac{3}{4}\|f\|^2 \geq \frac{3}{4}\|f\|^2.
\end{aligned}$$

At last we prove the "right-hand inequality" part. Observe that $T_a+T_{1-a}=I_U$. For $f\in U$, we have

$$\begin{aligned}
&\left\|\int_\Omega (1-a_\omega)\Gamma_\omega^*\Lambda_\omega f\mathrm{d}\mu(\omega)\right\|^2+\mathrm{Re}(\int_\Omega a_\omega\langle\Lambda_\omega f,\,\Gamma_\omega f\rangle\mathrm{d}\mu(\omega))\\
&=\langle T_{1-a}f,\,T_{1-a}f\rangle+\mathrm{Re}\,\langle T_a f,\,f\rangle\\
&=\langle T_{1-a}f,\,T_{1-a}f\rangle+\langle f,\,f\rangle-\mathrm{Re}\,\langle T_{1-a}f,\,f\rangle\\
&=\frac{3}{4}\langle f,\,f\rangle+\frac{1}{4}\langle f,\,f\rangle-\mathrm{Re}\,\langle T_{1-a}f,\,f\rangle+\langle T_{1-a}f,\,T_{1-a}f\rangle\\
&=\frac{3}{4}\langle f,\,f\rangle+\frac{1}{4}\left(\langle f,\,f\rangle-4\mathrm{Re}\,\langle T_{1-a}f,\,f\rangle+4\,\langle T_{1-a}f,\,T_{1-a}f\rangle\right)\\
&=\frac{3}{4}\langle f,\,f\rangle+\frac{1}{4}\left(\langle f,\,f\rangle-2\,\langle T_{1-a}f,\,f\rangle-2\,\langle f,\,T_{1-a}f\rangle+4\,\langle T_{1-a}f,\,T_{1-a}f\rangle\right)\\
&=\frac{3}{4}\langle f,\,f\rangle+\frac{1}{4}\,\langle (I_U-2T_{1-a})f,\,(I_U-2T_{1-a})f\rangle\\
&=\frac{3}{4}\langle f,\,f\rangle+\frac{1}{4}\,\langle (T_a-T_{1-a})f,\,(T_a-T_{1-a})f\rangle\\
&\le\frac{3}{4}\|f\|^2+\frac{1}{4}\|T_a-T_{1-a}\|^2\|f\|^2\\
&=\frac{3+\|T_a-T_{1-a}\|^2}{4}\|f\|^2.
\end{aligned}$$

The proof is completed. $\square$

4. ERASURES FOR CONTINUOUS G-FRAMES

In [1, Theorem 3.7] the authors gave a proposition for a continuous g-frame to be a continuous g-frame for one element erasure, only one element being deleted. So it is natural to ask whether there is a general result for erasure of [1, Theorem 3.7]? We can erase some elements of a continuous g-frame, and the remainder after erasure is also a continuous g-frame. In this section, we will give some results for erasure. To do that we first give the following lemma:

LEMMA 4.1 ([13, Theorem 2.29]). *Suppose that X is a Banach space and $Q\in L(X)$. If $\|Q\|<1$, then I_X-Q is invertible on X. Moreover, we have $\|(I_X-Q)^{-1}\|\le\frac{1}{1-\|Q\|}$.*

THEOREM 4.1. *Let $\Omega_1\subset\Omega$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous g-frame for U with respect to $(\Omega,\,\mu)$ with frame bounds A, B, frame operator S. S_{Ω_1} is defined as in (2.4). Then the following are equivalent:*

(i) $I_U - S^{-1}S_{\Omega_1}$ *is invertible on* U.

(ii) $I_U - S_{\Omega_1}S^{-1}$ *is invertible on* U.

(iii) $\{\Lambda_\omega\}_{\omega\in\Omega\backslash\Omega_1}$ *be a continuous g-frame for* U *with respect to* $(\Omega\backslash\Omega_1,\, \mu)$.

In addition, if (i) or (ii) is satisfied, the continuous g-frame $\{\Lambda_\omega\}_{\omega\in\Omega\backslash\Omega_1}$ *has the lower frame bound* $A(1-\|S^{-1}S_{\Omega_1}\|)$. *Otherwise if there exists* $0 \neq f \in U$ *such that* $f = S^{-1}S_{\Omega_1}f$, *then* $\{\Lambda_\omega\}_{\omega\in\Omega\backslash\Omega_1}$ *is not a continuous g-frame for* U *with respect to* $(\Omega \setminus \Omega_1,\, \mu)$.

Proof. (i)⇔(ii) Observe that S and S_{Ω_1} are self-adjoint. Therefore, we have

$$(I_U - S^{-1}S_{\Omega_1})^* = I_U - (S^{-1}S_{\Omega_1})^* = I_U - S_{\Omega_1}S^{-1},$$

So $I_U - S^{-1}S_{\Omega_1}$ is invertible on U if and only if $I_U - S_{\Omega_1}S^{-1}$ is invertible on U.

(i)⇔(iii) Denote the frame operator of $\{\Lambda_\omega\}_{\omega\in\Omega\backslash\Omega_1}$ by $S_{\Omega_1^c}$. It is easy to check that

$$S_{\Omega_1^c} = S - S_{\Omega_1} = S(I_U - S^{-1}S_{\Omega_1}). \tag{4.36}$$

Therefore, $\{\Lambda_\omega\}_{\omega\in\Omega\backslash\Omega_1}$ is a continuous g-frame for U with respect to $(\Omega\backslash\Omega_1,\, \mu)$ if and only if $S(I_U - S^{-1}S_{\Omega_1})$ is a bounded and invertible operator on U by (4.36). Observe that $S(I_U - S^{-1}S_{\Omega_1})$ is a bounded and invertible operator on U, equivalent to $I_U - S^{-1}S_{\Omega_1}$ and a bounded and invertible operator on U. Thus we prove the equivalence between (ii) and (iii).

Next we prove that the continuous g-frame $\{\Lambda_\omega\}_{\omega\in\Omega\backslash\Omega_1}$ has the lower frame bound $\frac{A}{\|(I_U - S^{-1}S_{\Omega_1})^{-1}\|^2}$ if (i) or (ii) is satisfied. Suppose (i) holds. Note that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous g-frame for U with respect to $(\Omega,\, \mu)$ with frame bounds A, B, frame operator S and $\Omega_1 \subset \Omega$. For $f \in U$, we have

$$\begin{aligned}
f = S^{-1}Sf &= S^{-1}\Big(\int_\Omega \Lambda_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega)\Big)\\
&= \int_{\Omega\backslash\Omega_1} S^{-1}\Lambda_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega) + S^{-1}\Big(\int_{\Omega_1} \Lambda_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega)\Big)\\
&= S^{-1}S_{\Omega_1}f + \int_{\Omega\backslash\Omega_1} S^{-1}\Lambda_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega),
\end{aligned}$$

that is

$$(I_U - S^{-1}S_{\Omega_1})f = \int_{\Omega\backslash\Omega_1} S^{-1}\Lambda_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega). \tag{4.37}$$

Therefore, we obtain

$$\|(I_U - S^{-1}S_{\Omega_1})f\| = \left\| \int_{\Omega\setminus\Omega_1} S^{-1}\Lambda_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega) \right\|. \tag{4.38}$$

$$\begin{aligned}
\left\| \int_{\Omega\setminus\Omega_1} S^{-1}\Lambda_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega) \right\| &= \sup_{g\in U,\, \|g\|=1} \left| \left\langle \int_{\Omega\setminus\Omega_1} S^{-1}\Lambda_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega), g \right\rangle \right| \\
&= \sup_{g\in U,\, \|g\|=1} \left| \int_{\Omega\setminus\Omega_1} \langle \Lambda_\omega f, \tilde{\Lambda}_\omega g\rangle \mathrm{d}\mu(\omega) \right| \\
&\le \sup_{g\in U,\, \|g\|=1} \left(\int_{\Omega\setminus\Omega_1} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega) \right)^{\frac{1}{2}} \left(\int_{\Omega\setminus\Omega_1} \|\tilde{\Lambda}_\omega g\|^2 \mathrm{d}\mu(\omega) \right)^{\frac{1}{2}} \\
&\le \sup_{g\in U,\, \|g\|=1} \frac{1}{\sqrt{A}} \|g\| \left(\int_{\Omega\setminus\Omega_1} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega) \right)^{\frac{1}{2}} \\
&\le \frac{1}{\sqrt{A}} \left(\int_{\Omega\setminus\Omega_1} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega) \right)^{\frac{1}{2}}. \qquad (4.39)
\end{aligned}$$

By (4.38) and (4.39), we know that $I_U - S^{-1}S_{\Omega_1}$ is bounded on U. Therefore, we have

$$\begin{aligned}
S_{\Omega_1^c} = S - S_{\Omega_1} &= S^{\frac{1}{2}}(I_U - S^{-\frac{1}{2}}S_{\Omega_1}S^{-\frac{1}{2}})S^{\frac{1}{2}} \\
&\ge \sqrt{A}(1 - \|S^{-\frac{1}{2}}S_{\Omega_1}S^{-\frac{1}{2}}\|)\sqrt{A}I_U \\
&\ge A(1 - \|S^{-1}S_{\Omega_1}\|)I_U.
\end{aligned}$$

It follows that

$$A(1 - \|S^{-1}S_{\Omega_1}\|)\|f\|^2 \le \int_{\Omega\setminus\Omega_1} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega).$$

We finally prove the last part. If there exists $0 \ne f \in U$ such that $f = S^{-1}S_{\Omega_1}f$. By (4.37), we have

$$\int_{\Omega\setminus\Omega_1} S^{-1}\Lambda_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega) = 0.$$

Observe that S^{-1} is invertible, then

$$\int_{\Omega\setminus\Omega_1} \Lambda_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega) = 0.$$

Therefore, we have

$$\left\langle \int_{\Omega\setminus\Omega_1} \Lambda_\omega^*\Lambda_\omega f \mathrm{d}\mu(\omega),\, f \right\rangle = \int_{\Omega\setminus\Omega_1} \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega),$$

since $f \neq 0$, thus $\{\Lambda_\omega\}_{\omega\in\Omega\setminus\Omega_1}$ is not a continuous g-frame for U with respect to $(\Omega \setminus \Omega_1, \mu)$. The proof is completed. □

By the arguments in Theorem 4.1 and Lemma 4.1, we have

COROLLARY 4.1. *Let $\Omega_1 \subset \Omega$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous g-frame for U with respect to (Ω, μ) with frame bounds A, B, frame operator S. S_{Ω_1} is defined as in (2.4). If $\|S^{-1}S_{\Omega_1}\| < 1$, then $\{\Lambda_\omega\}_{\omega\in\Omega\setminus\Omega_1}$ is a continuous g-frame for U with respect to $(\Omega \setminus \Omega_1, \mu)$ with the lower frame bound $A(1 - \|S^{-1}S_{\Omega_1}\|)$.*

Observe that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous tight g-frame for U with respect to (Ω, μ) with frame bounds A, then $S = AI_U$. As an immediate consequence of Theorem 4.1, we have

COROLLARY 4.2. *Let $\Omega_1 \subset \Omega$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous tight g-frame for U with respect to (Ω, μ) with frame bounds A, frame operator S. S_{Ω_1} is defined as in (2.4). If $\|S_{\Omega_1}\| < A$, then $\{\Lambda_\omega\}_{\omega\in\Omega\setminus\Omega_1}$ is a continuous g-frame for U with respect to $(\Omega \setminus \Omega_1, \mu)$ with the lower frame bound $(A - \|S_{\Omega_1}\|)$.*

COROLLARY 4.3. *Let $\Omega_1 \subset \Omega$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous Parseval g-frame for U with respect to (Ω, μ). S_{Ω_1} is defined as in (2.4). If $\|S_{\Omega_1}\| < 1$, then $\{\Lambda_\omega\}_{\omega\in\Omega\setminus\Omega_1}$ is a continuous g-frame for U with respect to $(\Omega \setminus \Omega_1, \mu)$ with the lower frame bound $(1 - \|S_{\Omega_1}\|)$.*

If $\Omega_1 = \{\omega_0\}$, by Theorem 4.1 we have the following corollary:

COROLLARY 4.4. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous g-frame for U with respect to (Ω, μ) with frame operator S. Then the following are equivalent.*

(i) $I_U - \mu(\omega_0)\tilde{\Lambda}_{\omega_0}^*\Lambda_{\omega_0}$ is invertible on U.

(ii) $I_U - \mu(\omega_0)\Lambda_{\omega_0}^*\tilde{\Lambda}_{\omega_0}$ is invertible on U.

(iii) $\{\Lambda_\omega\}_{\omega\in\Omega\setminus\{\omega_0\}}$ be a continuous g-frame for U with respect to $(\Omega \setminus \{\omega_0\}, \mu)$.

In addition, if (i) or (ii) is satisfied, the continuous g-frame $\{\Lambda_\omega\}_{\omega\in\Omega\setminus\{\omega_0\}}$ has the lower frame bound $A(1 - \|\mu(\omega_0)\tilde{\Lambda}_{\omega_0}^*\Lambda_{\omega_0}\|)$. Otherwise if there exists $0 \neq f \in U$ such that $f = \mu(\omega_0)\tilde{\Lambda}_{\omega_0}^*\Lambda_{\omega_0} f$, then $\{\Lambda_\omega\}_{\omega\in\Omega\setminus\{\omega_0\}}$ is not a continuous g-frame for U with respect to $(\Omega \setminus \{\omega_0\}, \mu)$.

Remark 4.1. The part (i)⇒(iii) in Corollary 4.4 was first stated in Theorem 3.7 in [1].

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