CLASSIFYING p-GROUPS BY THEIR SCHUR MULTIPLIERS

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Some recent results devoted to the investigation of the structure of p-groups rely on the study of their Schur multipliers. One of these results states that for any p-group G of order p^n there exists a nonnegative integer $s(G)$ such that the order of the Schur multiplier of G is equal to $p^{\frac{1}{2}(n-1)(n-2)+1-s(G)}$. Characterizations of the structure of all non-abelian p-groups G have been obtained for the case that $s(G) = 0$ or 1. The present paper is devoted to the characterization of all p-groups with $s(G) = 2$.

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Originating in the work of Schur in 1904, the concept of Schur multiplier, $\mathcal{M}(G)$, was studied by several authors, and proved to be an important tool in the classification of p-groups. It is known that the order of Schur multiplier of a given finite p-group of order p^n is equal to $p^{\frac{1}{2}n(n-1)-t(G)}$ for some $t(G) \geq 0$ by a result of Green [5]. It is of interest to know which p-groups have the Schur multiplier of order $p^{\frac{1}{2}n(n-1)-t(G)}$, when $t(G)$ is in hand.

Historically, there are several papers trying to characterize the structure of G by just the order of its Schur multiplier. In [1] and [13], Berkovich and Zhou classified the structure of G when $t(G) = 0, 1$ and 2, respectively.

Later, Ellis in [2] showed that having a new upper bound on the order of Schur multiplier of groups reduces characterization process of structure of G. He reformulated the upper bound due to Gaschütz *et. al.* [4] and classified in a new way to that of [1, 13] the structure of G when $t(G) = 3$.

The result of [9] shows that there exists a nonnegative integer $s(G)$ such that $|\mathcal{M}(G)| = p^{\frac{1}{2}(n-1)(n-2)+1-s(G)}$ which is a reduction of Green's bound for any given non-abelian p-group G of order p^n. One can check that the structure of G can be characterized by using [9, Main Theorem], when $t(G) = 1, 2, 3$.

Moreover, characterizing non-abelian p-groups by $s(G)$ can be significant since for instance the results of [9] and [12] emphasize that the number of groups with a fixed $s(G)$ is larger than the number of groups with fixed $t(G)$. Also, the result of [10] and [11] shows handling the class of p-groups characterized by $s(G) = 0, 1$ may characterize the structure of G by knowing $t(G)$.

In the present paper, we intend to classify the structure of all non-abelian p-groups when $s(G) = 2$.

Throughout this paper, we use the following notations.
Q_8: quaternion group of order 8,
D_8: dihedral group of order 8,
E_1: extra special p-group of order p^3 and exponent p,
E_2: extra special p-group of order p^3 and exponent p^2 $(p \neq 2)$,
$\mathbb{Z}_{p^n}^{(m)}$: direct product of m copies of the cyclic group of order p^n,
G^{ab}: the abelianization of group G,
$H \cdot K$: the central product of H and K,
$E(m)$: $E \cdot Z(E)$, where E is an extra special p-group and $Z(E)$ is a cyclic group of order p^m $(m \geq 2)$,
$\Phi(G)$: the Frattini subgroup of group G.

Also, G has the property $s(G) = 2$ or briefly with $s(G) = 2$ means the order of its Schur multiplier is of order $p^{\frac{1}{2}(n-1)(n-2)-1}$.

The following lemma is a consequence of [9, Main Theorem].

Lemma 1. *There exists no p-group G with $|G'| \geq p^3$ and $s(G) = 2$.*

Lemma 2. *There exists no p-group G of order p^n $(n \geq 5)$ when G^{ab} is not elementary abelian and $s(G) = 2$.*

Proof. First, suppose that $n = 5$. By virtue of [11, Theorem 3.6], the result follows. In case $n \geq 6$, by invoking [9, Lemma 2.3], we have $|\mathcal{M}(G/G')| \leq p^{\frac{1}{2}(n-2)(n-3)}$, and since $G/Z(G)$ is capable, the rest of proof is obtained by using [3, Proposition 1]. $\square$

Lemma 3. *Let G be a p-group and $|G'| = p$ or p^2 with $s(G) = 2$. Then $Z(G)$ is of exponent at most p^2 and p, respectively.*

Proof. Taking a cyclic central subgroup K of order p^k $(k \geq 3)$ and using [6, Theorem 2.2], we should have

$$\begin{aligned} |\mathcal{M}(G)| \leq p^{-1}|G/K \otimes K| p^{\frac{1}{2}(n-k)(n-k-1)} &\leq p^{n-k-1} p^{\frac{1}{2}(n-k)(n-k-1)} \\ &\leq p^{\frac{1}{2}(n-1)(n-2)-2}, \end{aligned}$$

which is a contradiction. In case $|G'| = p^2$, the result is obtained similarly. $\square$

Lemma 1 indicates that when G has the property $s(G) = 2$, then $|G'| \leq p^2$. First we survey the case $|G'| = p$.

THEOREM 4. *Let G be a p-group with center of order at most p^2 such that G^{ab} is elementary abelian of order p^{n-1} and $s(G) = 2$. Then $G \cong E(2)$, $E_2 \times \mathbb{Z}_p$, Q_8 or H, where H is an extra special p-group of order p^{2m+1} $(m \geq 2)$.*

Proof. First assume that $|Z(G)| = p$. Hence $G \cong Q_8$ or $G \cong H$, where H is an extra special p-group of order p^{2m+1} $(m \geq 2)$ by a result of [8, Theorem 3.3.6]. Now, assume that $|Z(G)| \geq p^2$. Lemma 3 and assumption show that $Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$.

In case $Z(G)$ is of exponent p, we deduce from [9, Lemma 2.1] that $G \cong H \times \mathbb{Z}_p$. It is easily checked that $H \cong E_2$ by using [8, Theorems 2.2.10 and 3.3.6].

In case $Z(G)$ is of exponent p^2, since $\Phi(G) = G'$, [7, Theorem 3.1] shows that

$$p^{\frac{1}{2}(n-1)(n-2)} = |\mathcal{M}(G/\Phi(G))| \leq p\, |\mathcal{M}(G)|,$$

and hence $p^{\frac{1}{2}(n-1)(n-2)-1} \leq |\mathcal{M}(G)|$. On the other hand, the Main Theorems of [9] and [12] imply that $|\mathcal{M}(G)| = p^{\frac{1}{2}(n-1)(n-2)-1}$ since $Z(G)$ is cyclic of order p^2. Moreover, $G \cong E(2)$ by [9, Lemma 2.1], as required. □

THEOREM 5. *Let G be a p-group, G^{ab} be elementary abelian, $|G'| = p$ and $|Z(G)| \geq p^3$ be of exponent p^2. Then $G = E(2) \times Z$, where Z is an elementary abelian p-group.*

Proof. It is known that $G = H \cdot Z(G)$ and $H \cap Z(G) = G'$ by virtue of [9, Lemma 2.1]. Now, for the sake of clarity, we consider two cases.

Case 1. First, assume that G' lies in a central subgroup K of exponent p^2. Therefore, one can check that there exists a central subgroup T such that $G = H \cdot K \times T \cong E(2) \times T$. Thus, when T is an elementary abelian p-group by using [8, Theorem 2.2.10] and Theorem 4, we have

$$\begin{aligned} |\mathcal{M}(G)| &= |\mathcal{M}(E(2))||\mathcal{M}(T)||E(2)^{ab} \otimes T| \\ &= 2m^2+m-1+\tfrac{1}{2}(n-2m-2)(n-2m-3)+2m+1(n-2m-2) \\ &= \tfrac{1}{2}(n-1)(n-2)-1. \end{aligned}$$

In the case T is not elementary abelian, a similar method and [9, Lemma 2.2] asserts that

$$|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n^2-5n+4)} \leq p^{\frac{1}{2}(n-1)(n-2)-2}.$$

Case 2. G' has a complement T in $Z(G)$, and hence $G = H \times T$ where T is not elementary abelian, and so by invoking [9, Lemma 2.2] and [8, Theorems 2.2.10 and 3.3.6], $|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-1)(n-2)-2}$. □

Theorem 6. *Let G be a p-group of order p^n, G^{ab} be elementary abelian of order p^{n-1} and $Z(G)$ be of exponent p. Then G has the property $s(G) = 2$ if and only if it is isomorphic to one of the following groups.*

$$Q_8 \times \mathbb{Z}_2^{(n-3)}, E_2 \times \mathbb{Z}_p^{(n-3)} \text{ or } H \times \mathbb{Z}_p^{(n-2m-1)},$$

where H is extra special of order p^{2m+1} $m \geq 2$.

Proof. It is obtained *via* Theorem 4, [8, Theorems 2.2.10 and 3.3.6] and assumption. □

Lemma 7. *Let G be a p-group of order p^4 and $|G'| = p$. Then G has the property $s(G) = 2$ if and only if G is isomorphic to the one of the following groups.*

(1) $Q_8 \times \mathbb{Z}_2$,
(2) $\langle a, b \mid a^4 = 1, b^4 = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^2 b^2 \rangle$
(3) $\langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$.
(4) $E_4 \cong E_1(2)$,
(5) $E_2 \times \mathbb{Z}_p$,
(6) $\langle a, b \mid a^{p^2} = 1, b^p = 1, [a, b, a] = [a, b, b] = 1 \rangle$,

Proof. It is obtained by using Theorems 4, 6 and a result of [10, Lemma 3.5]. □

The structure of all p-groups of order p^n is characterized with the property $s(G) = 2$ and $|G'| = p$. Now, we suppose that $|G'| = p^2$.

Lemma 8. *There exists no p-group of order p^n $(n \geq 5)$ with $s(G) = 2$, $|G'| = p^2$ and $G' \nsubseteq Z(G)$.*

Proof. First, assume that $|Z(G)| = p^2$, since $Z(G)$ is elementary by Lemma 4. Let K be a central subgroup of order p, such that $|(G/K)'| = p^2$. It is seen that

$$|\mathcal{M}(G)| \leq |\mathcal{M}(G/K)||K \otimes G/(K \times G')| \leq |\mathcal{M}(G/K)|\ p^{n-3}$$

by [7, Theorem 4.1]. On the other hand, [9, 12, Main Theorems] imply that $|\mathcal{M}(G/K)| \leq p^{\frac{1}{2}(n-2)(n-3)-1}$, and hence $|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-1)(n-2)-2}$.

In case $|Z(G)| = p^3$, there exists a central subgroup K of order p^2 such that $G' \cap K = 1$. The rest of the proof is similar to that used in our previous case.

When $|Z(G)| = p$, since G is nilpotent of class 3, the result is deduced by [8, Proposition 3.1.11]. □

Theorem 9. *Let G be a p-group of order p^n $(n \geq 5)$ and $|G'| = p^2$ with $s(G) = 2$. Then*

$$G \cong \mathbb{Z}_p \times \left(\mathbb{Z}_p^{(4)} \rtimes_\theta \mathbb{Z}_p\right)\ (p \neq 2).$$

Proof. By the results of Lemmas 4 and 8, we may assume that $G' \subseteq Z(G)$ and $Z(G)$ is of exponent p. We consider three cases relative to $|Z(G)|$.

Case 1. Assuming that $|Z(G)| = p^4$, there exists a central subgroup K of order p^2 such that $K \cap G' = 1$. [9, Main Theorem] implies that $|\mathcal{M}(G/K)| \leq p^{\frac{1}{2}(n-3)(n-4)}$, and so $|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-1)(n-2)-1}$ due to [7, Theorem 4.1].

Case 2. In the case $|Z(G)| = p^2$, we have $G' = Z(G)$. Moreover [12, Main Theorem] implies that $n \geq 6$ and so there exists a central subgroup K such that $G/K \cong H \times Z(G/K)$ where H is an extra special p-group of order p^{2m+1} $m \geq 2$, and so

$$|\mathcal{M}(G)| \leq p^{n-3}|\mathcal{M}(G/K)| \leq p^{n-3}p^{\frac{1}{2}(n-1)(n-4)} \leq p^{\frac{1}{2}(n-1)(n-2)-2}.$$

Case 3. Now, we may assume that $|Z(G)| = p^3$. Let K be a complement of G' in $Z(G)$, so [11, Main Theorem] implies that $|\mathcal{M}(G/K)| \leq p^{\frac{1}{2}(n-2)(n-3)}$. On the other hand, [7, Theorem 4.1] and our assumption imply that

$$\begin{aligned} p^{\frac{1}{2}(n-1)(n-2)-1} = |\mathcal{M}(G)| &\leq |\mathcal{M}(G/K)||K \otimes G/Z(G)| \\ &\leq \big|\mathcal{M}(G/K)\big|p^{n-3}, \end{aligned}$$

so we should have $|\mathcal{M}(G/K)| = p^{\frac{1}{2}(n-2)(n-3)}$ and $G/Z(G)$ is elementary abelian. Now, since $|\mathcal{M}(G/K)| = p^{\frac{1}{2}(n-2)(n-3)}$ and $|(G/K)'| = p^2$, by using [12, Main Theorem], $G/K \cong \mathbb{Z}_p^{(4)} \rtimes_\theta \mathbb{Z}_p$ $(p \neq 2)$. Moreover, [3, Proposition 1] and our assumption show that G^{ab} is elementary abelian. Hence, it is readily shown that

$$G \cong \mathbb{Z}_p \times \big(\mathbb{Z}_p^{(4)} \rtimes_\theta \mathbb{Z}_p\big)\ (p \neq 2). \quad \square$$

THEOREM 10. *Let G be a group of order p^4 with $s(G) = 2$ and $|G'| = p^2$. Then G is isomorphic to the one of the following groups.*

(1) $\langle a, b \mid a^9 = b^3 = 1, [a,b,a] = 1, [a,b,b] = a^6, [a,b,b,b] = 1\rangle$,

(2) $\langle a, b \mid a^p = 1, b^p = 1, [a,b,a] = [a,b,b,a] = [a,b,b,b] = 1\rangle (p \neq 3)$.

Proof. The structure of these groups has been characterized in [10, Lemma 3.6]. $\square$

We summarize all results as follows,

THEOREM 11. *Let G be a group of order p^n. Then $s(G) = 2$ if and only if G is isomorphic to one of the following groups.*

(1) $E(2) \times \mathbb{Z}_p^{(n-2m-2)}$,

(2) $E_2 \times \mathbb{Z}_p^{(n-3)}$,

(3) $Q_8 \times \mathbb{Z}_2^{(n-3)}$,

(4) $H \times \mathbb{Z}_p^{(n-2m-1)}$, where H is an extra special p-group of order p^{2m+1} $(m \geq 2)$,
(5) $\langle a, b \mid a^4 = 1, b^4 = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^2b^2 \rangle$,
(6) $\langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$,
(7) $\langle a, b \mid a^{p^2} = 1, b^p = 1, [a, b, a] = [a, b, b] = 1 \rangle$,
(8) $\mathbb{Z}_p \times \left(\mathbb{Z}_p^{(4)} \rtimes_\theta \mathbb{Z}_p\right)$ $(p \neq 2)$,
(9) $\langle a, b \mid a^9 = b^3 = 1, [a, b, a] = 1, [a, b, b] = a^6, [a, b, b, b] = 1 \rangle$,
(10) $\langle a, b \mid a^p = 1, b^p = 1, [a, b, a] = [a, b, b, a] = [a, b, b, b] = 1 \rangle (p \neq 3)$.

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