ON COMMUTATORS IN P-GROUPS OF MAXIMAL CLASS AND SOME CONSEQUENCES

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Let $\Gamma(G)$ denote the set of commutators of a group G. In this paper, we first show that if p is a prime number and G is a p-group of maximal class then $\Gamma(G) = G'$. As a consequence of this result we present various sufficient conditions implying that the commutators form a subgroup. Next, we prove that if G is a wreath product $G = A \wr P$, with A a nontrivial finite abelian group and P a p-group of maximal class, with $|A| > 2$ or $p > 2$, then the commutator length of G is equal to 2..

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1. INTRODUCTION

Recall that the commutator length $c(G)$ of a group G is defined to be the minimal number such that every element of G' can be expressed as a product of at most $c(G)$ commutators. A group G is called a c-group if $c(G)$ is finite. For any positive integer n, denote byc_n the class of groups with commutator length equal to n. Denote by $\Gamma(G)$ the set of commutators in G.

The question as whether every element of the commutator subgroup of a group G is a commutator ($\Gamma(G) = G'$) or not, was studied by many authors. See [12] for a survey article about commutators.

The smallest group with $\Gamma(G) = G'$ has order 96 (see [9]). Indeed there exist many papers providing examples of groups in which $\Gamma(G) = G'$ or $\Gamma(G) \neq G'$. In [10], Guralnick gave interesting examples of wreath product $G := U \wr H$ where U is any nontrivial finite abelian group and H is a finite group with derived subgroup of order at least 3, in which $c(G) \neq 1$. In particular if H is perfect, then G' is perfect and $c(G) \neq 1$.

In [4], we also used wreath product constructions to obtain for any positive integer n, solvable groups of derived length n and commutator length equal to 1 or 2. Let $W = G \wr H$ be the wreath product of G by a n-generator abelian group H. In [3], we proved that every element of W' is a product of at most $n+2$ commutators, and every element of W^2 is a product of at most $3n+4$ squares in W. This generalizes our previous result.

There are few sufficient conditions implying that the commutators form a subgroup (see [13]). In this paper, we show that in any p-group of maximal class, the commutators form a subgroup. As a consequence of this result we show in Corollary 3, that if P is a p-group of maximal class and $G = P \wr C_1 \wr \cdots \wr C_n$, where C_i is a finite cyclic group for $i = 1, \cdots, n$, then $\Gamma(G) = G'$. We also collect in Corollary 4, various sufficient conditions implying that the commutators form a subgroup. Finally, let A be a nontrivial finite abelian group and P be a p-group of maximal class. Let $G = A \wr P$ and assume that $|A| > 2$ or $p > 2$. By using Guralnick's [10] recent result, we show that the commutator length of G is equal to 2. We also give a precise formula for expressing every element of G' as a product of two commutators.

The precise statements of results are given in the next section.

2. MAIN RESULTS

Our notation is standard. Let G be a group and x, y and $z \in G$. Then $x^y = y^{-1}xy$, $[x,y] = x^{-1}y^{-1}xy$.

Let G be a p-group of maximal class of order p^n, $n \geq 4$. For each i with $2 \leq i \leq n-2$ the $2-$step centralizer K_i in G is defined to be the centralizer in G of $\gamma_i(G)/\gamma_{i+2}(G)$. Clearly $\gamma_2(G) \leq K_i$ for each i. Define $P_i = P_i(G)$ by $P_0 = G$, $P_1 = K_2$ and $P_i = \gamma_i(G)$ for $2 \leq i \leq n$. Take $s_1 \in P_1 - P_2$, $s \in G - \bigcup_{i=2}^{n-2} \mathrm{K}_i$, and define $s_i = [s_{i-1}, s] = [s_{1,i-1}\, s]$ for $2 \leq i \leq n-1$. Note that $G = \langle s, s_1 \rangle$, $P_i = \langle s_i, \cdots, s_{n-1} \rangle$ for $1 \leq i \leq n-2$ and $s^p \in P_{n-1} = Z(G) = \gamma_{n-1}(G)$ (see [11] and [14]).

In the rest of the paper, we use the above notations.

The main results of this paper are as follows.

Theorem 1. *If G is a p-group of maximal class, then $\Gamma(G) = G'$.*

We have the following consequences of this result.

Corollary 2. *If G is a p-group of maximal class of order p^n $(n \geq 4)$ and $s \notin K_i$ for $2 \leq i \leq n-2$, then $G' = \{[g,s]\} | g \in G\}$. In particular, every element of $G = \langle s, s_1 \rangle$ can be expressed in the form $s^i s_1^{i_1}[g,s]$ in which $i,\ i_1 \in \mathbb{N},\ 0 \leq i \leq p^2$ and $g \in G$.*

By repeated application of Rhemtulla's [15] result we show that:

Corollary 3. *Let P be a p-group of maximal class and $G = P \wr C_1 \wr \cdots \wr C_n$, where C_i is a finite cyclic group for $i = 1, \cdots, n$. Then $\Gamma(G) = G'$.*

In the next corollary, we collect various sufficient conditions implying that the commutators form a subgroup.

Corollary 4. *Let G be a finite nonabelain p-group of order p^n. If one of the following assertions holds, then $\Gamma(G) = G'$:*

(i) $p = 2$ and $|G : G'| = 4$.

(ii) G has an abelian subgroup of index p and $|G : G'| = p^2$.

(iii) G has a cyclic subgroup U of index p^2 such that $C_G(U)$ is cyclic.

(iv) $n > 4$, and G has only one abelian subgroup of order p^3.

Two interesting results are indicated by the following theorems.

Theorem 5. *Let G be a p-group, $p > 2$. If G has no normal elementary abelian subgroup of rank 3, then $\Gamma(G) = G'$.*

Theorem 6. *Let A be a nontrivial finite abelian group and $P = \langle s, s_1\rangle$ be a p-group of maximal class. Let $G = A \wr P$. If $|A| > 2$ or $p > 2$, then $c(G) = 2$. In particular, every element of G' is a product of at most two commutators $[b_1, s_1][s, g]^b$, for suitable g in G and b, b_1 in the base group of G.*

3. PROOFS

Proof of Theorem 1. Of course a nonabelian group of order p^3 is a p-group of maximal class. It is also well known that if G is a nonabelian group of order p^3, where p is odd, then G is an extra-special group and G is isomorphic to

$$\langle x, y | x^p = y^p = 1,\ [x, y] = [x, y]^x = [x, y]^y\rangle,$$

or

$$\langle x, y | y^p = x^{p^2} = 1, x^y = x^{p+1}\rangle.$$

Note that these groups have exponent p and p^2, respectively.

In both cases one has $G' = \langle [x, y]\rangle = \{[x, y^i] | 0 \le i < p\}$; thus $\Gamma(G) = G'$.

If $p = 2$ then G is isomorphic to D_8 or Q_8. If $\{x,\ y\}$ is a generating set for G then $G' = \{1,\ [x, y]\}$; thus $\Gamma(G) = G'$.

For $n \ge 4$, we will use induction on n. If $G = \langle s, s_1\rangle$ is a p-group of maximal class of order p^4, then $\gamma_2(G) = \langle s_2, s_3\rangle = \langle [s_1, s], [s_1, s, s]\rangle$, $\gamma_3(G) = Z(G) = \langle s_3\rangle = \langle [s_1, s, s]\rangle$, $s^p \in \gamma_3(G) = Z(G)$ and $\gamma_4(G) = 1$. Let $\gamma \in G'$, then $\gamma = [s_1, s]^i [s_1, s, s]^j$ where $i,\ j \in \mathbb{Z}$. Clearly we have $\gamma = [s_1^i, s][s_1, s, s]^{j-i(i-1)/2}$ $= [s_1^i, s][s_2^{j-i(i-1)/2}, s] = [s_1^i s_2^{j-i(i-1)/2}, s]$.

Let $n > 4$ and $G = \langle s, s_1\rangle$. Let also $\overline{G} = G/Z(G)$ and observe that $\overline{G}$ is a p-group of maximal class. It is easy to prove that the ith $2-$step centralizer in $\overline{G}$ is $\overline{K_i} = K_i/Z(G)$ for $2 \le i \le n-3$. Also, we have:

$$P_0(\overline{G}) = \overline{P_0(G)},\ P_1(\overline{G}) = \overline{P_1(G)} = \overline{K_2},\ P_i(\overline{G}) = \overline{P_i(G)} \text{ for } 2 \le i \le n-3,$$

$$\overline{s} \in \overline{G} - \bigcup_{i=2}^{n-3} \overline{K}_i,\ \overline{s_1} \in \overline{P_1} - \overline{P_2} \text{ and } \overline{G} = \langle \overline{s}, \overline{s_1} \rangle.$$

By induction on n, $\overline{G}' = \{[\overline{g}, \overline{s}] | g \in G\}$. Hence if $\gamma \in G'$, then $\overline{\gamma} = [\overline{g}, \overline{s}]$ with $g \in G$. Thus $\gamma = [g, s]s_{n-1}^k = [g, s][s_{n-2}^k, s] = [gs_{n-2}^k, s]$ for suitable $k \in \mathbb{Z}$. This shows that $\Gamma(G) = G'$ and completes the proof. $\square$

Corollary 2 is an immediate consequence of this result.

We continue by giving the proof of Corollary 3.

Proof of Corollary 3. Indeed, Rhemtulla [15] proved that the wreath product of a c_1-group by a finite cyclic group is again a c_1-group. Repeated application of Rhemtulla's result shows that the group G, satisfies the desired property and the proof is complete. $\square$

Next we provide one example for Corollary 3.

Example. Let $q \equiv_8 3$ be a prime number, and $S_2(2^k, q)$ a Sylow 2–subgroup of $GL(2^k, q)$. By [8, p. 142], we have $G = S_2(2^k, q) \simeq S_2(2, q) \wr \mathbb{Z}_2 \wr \cdots \wr \mathbb{Z}_2$, $(k-1$ factors) where $P = S_2(2, q)$ is a semidihedral group of order 16. Since P is a 2–group of maximal class by Corollary 3, we get $\Gamma(G) = G'$.

Next, we prove Corollary 4.

Proof of Corollary 4. Notice that if one of the assertions of Corollary 4 holds, then G is of maximal class (see [6, Lemma J(k)] and [7, Proposition 4.10, Proposition 13.15, Proposition 13.16]). $\square$

Now we turn to the proof of Theorem 5.

Proof of Theorem 5. Let G be a p-group, $p > 2$. If G has no normal elementary abelian subgroup of rank 3, then by [7, Theorem 13.7] one of the following assertions holds:

(a) G is metacyclic.

(b) G is $3-$group of maximal class.

(c) $G = EH$, where $E = \Omega_1(G)$ is nonabelian of order p^3 and exponent p, H is cyclic of index p^2 in G. Furthermore, $Z(G)$ is cyclic and $p^2 \leq |G : Z(G)| \leq p^3$.

First suppose G is metacyclic. We may choose suitable elements s, s_1 in G and a number r, $(1 \leq r \leq |s_1|)$ such that $G = \langle s, s_1 \rangle$ and $s_1^s = s_1^r$. Obviously $G' = \langle [s, s_1] \rangle = s_1^{r-1}$. Now if $\gamma \in G'$, then $\gamma = [s_1, s]^i = (s_1^{-1}s_1^s)^i = s_1^{-i}s_1^{is} = [s_1^{-i}, s]$. Therefore $G' = \{[s_1^{-i}, s] | 1 \leq r \leq |s_1|\} = \Gamma(G)$.

Next, if we suppose that G is 3–group of maximal class, then our result follows from Theorem 1.

Finally if assertion (c) holds, then $G/E \simeq H/H \cap E$ is cyclic. Therefore $1 \neq G' \leq E$. If $G' = E$ then $G = EH = H$ which is a contradiction. Therefore G' is elementary abelian of rank less than or equal 2. By [13, Theorem 2.4] it follows that $\Gamma(G) = G'$, as required. $\square$

Finally, we prove Theorem 6.

Proof of Theorem 6. Let G be the wreath product of a nontrivial finite abelian group A and a p-group of maximal class $P = \langle s, s_1 \rangle$.

Let $B = \underset{1 \leq i \leq |P|}{Dr} A_i$ where $A_i \simeq A$, be the base group of G. Now $G = BP$ is the semidirect product of B by P. Hence $[B, P]$ is a normal subgroup of G and

$$G' = [BP, BP] = [B, P]P'.$$

Since B is a normal abelian subgroup of G, we see that $[P, B] = [G, B]$. Now by Lemma 3 [1], $[B, P] = \{[b_1, s_1][b, s] | b, b_1 \in B\}$. By Corollary 2, every element $\gamma \in G'$ has the form $\gamma = [b_1, s_1][b, s][s, g] = [b_1, s_1][s, gb^{-1}]^b$. Hence $c(\gamma) \leq 2$.

Further if $|A| > 2$ or $p > 2$, then by Theorem 1 [10], some element of $[P, B]$ is not a commutator. Thus $c(G) = 2$, as required.

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