ON COMMUTATORS IN $P$-GROUPS OF MAXIMAL CLASS
AND SOME CONSEQUENCES

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Let $\Gamma(G)$ denote the set of commutators of a group $G$. In this paper, we first show that if $p$ is a prime number and $G$ is a $p$-group of maximal class then $\Gamma(G) = G'$. As a consequence of this result we present various sufficient conditions implying that the commutators form a subgroup. Next, we prove that if $G$ is a wreath product $G = A \wr P$, with $A$ a nontrivial finite abelian group and $P$ a $p$-group of maximal class, with $|A| > 2$ or $p > 2$, then the commutator length of $G$ is equal to 2.

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1. INTRODUCTION

Recall that the commutator length $c(G)$ of a group $G$ is defined to be the minimal number such that every element of $G'$ can be expressed as a product of at most $c(G)$ commutators. A group $G$ is called a c-group if $c(G)$ is finite. For any positive integer $n$, denote by $c_n$ the class of groups with commutator length equal to $n$. Denote by $\Gamma(G)$ the set of commutators in $G$.

The question as whether every element of the commutator subgroup of a group $G$ is a commutator ($\Gamma(G) = G'$) or not, was studied by many authors. See [12] for a survey article about commutators.

The smallest group with $\Gamma(G) = G'$ has order 96 (see [9]). Indeed there exist many papers providing examples of groups in which $\Gamma(G) = G'$ or $\Gamma(G) \neq G'$. In [10], Guralnick gave interesting examples of wreath product $G := U \wr H$ where $U$ is any nontrivial finite abelian group and $H$ is a finite group with derived subgroup of order at least 3, in which $c(G) \neq 1$. In particular if $H$ is perfect, then $G'$ is perfect and $c(G) \neq 1$.

In [4], we also used wreath product constructions to obtain for any positive integer $n$, solvable groups of derived length $n$ and commutator length equal to 1 or 2. Let $W = G \wr H$ be the wreath product of $G$ by a $n$-generator abelian group $H$. In [3], we proved that every element of $W'$ is a product of
at most \(n + 2\) commutators, and every element of \(W^2\) is a product of at most 
\(3n + 4\) squares in \(W\). This generalizes our previous result.

There are few sufficient conditions implying that the commutators form
a subgroup (see [13]). In this paper, we show that in any \(p\)-group of maximal
class, the commutators form a subgroup. As a consequence of this result we
show in Corollary 3, that if \(P\) is a \(p\)-group of maximal class and \(G = P \wr C_1 \wr \cdots \wr C_n\), where \(C_i\) is a finite cyclic group for \(i = 1, \ldots, n\), then \(\Gamma(G) = G'\).

We also collect in Corollary 4, various sufficient conditions implying that the
commutators form a subgroup. Finally, let \(A\) be a nontrivial finite abelian
group and \(P\) be a \(p\)-group of maximal class. Let \(G = A \wr P\) and assume
that \(|A| > 2\) or \(p > 2\). By using Guralnick’s [10] recent result, we show that
the commutator length of \(G\) is equal to 2. We also give a precise formula for
expressing every element of \(G'\) as a product of two commutators.

The precise statements of results are given in the next section.

2. MAIN RESULTS

Our notation is standard. Let \(G\) be a group and \(x, y\) and \(z \in G\). Then
\[x^y = y^{-1}xy, \ [x, y] = x^{-1}y^{-1}xy.\]

Let \(G\) be a \(p\)-group of maximal class of order \(p^n, n \geq 4\). For each \(i\) with
\(2 \leq i \leq n - 2\) the 2-step centralizer \(K_i\) in \(G\) is defined to be the centralizer
in \(G\) of \(\gamma_i(G)/\gamma_{i+2}(G)\). Clearly \(\gamma_2(G) \leq K_i\) for each \(i\). Define \(P_i = P_i(G)\) by
\(P_0 = G, P_1 = K_2\) and \(P_i = \gamma_i(G)\) for \(2 \leq i \leq n\). Take \(s_1 \in P_1 - P_2, s \in
G - \bigcup_{i=2}^{n-2} K_i\), and define \(s_i = [s_{i-1}, s] = [s_1, s_{i-1} s]\) for \(2 \leq i \leq n - 1\). Note that
\(G = \langle s, s_1 \rangle, P_i = \langle s_1, \cdots, s_{n-1} \rangle\) for \(1 \leq i \leq n - 2\) and \(s_p \in P_{n-1} = Z(G) =
\gamma_{n-1}(G)\) (see [11] and [14]).

In the rest of the paper, we use the above notations.

The main results of this paper are as follows.

**Theorem 1.** If \(G\) is a \(p\)-group of maximal class, then \(\Gamma(G) = G'\).

We have the following consequences of this result.

**Corollary 2.** If \(G\) is a \(p\)-group of maximal class of order \(p^n\) \((n \geq 4)\)
and \(s \notin K_i\) for \(2 \leq i \leq n - 2\), then \(G' = \{ [g, s] | g \in G \}\). In particular,
every element of \(G = \langle s, s_1 \rangle\) can be expressed in the form \(s^i s_1^{i_1} [g, s]\) in which
\(i, i_1 \in \mathbb{N}, 0 \leq i \leq p^2\) and \(g \in G\).

By repeated application of Rhemtulla’s [15] result we show that:

**Corollary 3.** Let \(P\) be a \(p\)-group of maximal class and \(G = P \wr C_1 \cdots \wr C_n\),
where \(C_i\) is a finite cyclic group for \(i = 1, \cdots, n\). Then \(\Gamma(G) = G'\).
In the next corollary, we collect various sufficient conditions implying that the commutators form a subgroup.

**Corollary 4.** Let \( G \) be a finite nonabelian \( p \)-group of order \( p^n \). If one of the following assertions holds, then \( \Gamma(G) = G' \):

(i) \( p = 2 \) and \( |G : G'| = 4 \).

(ii) \( G \) has an abelian subgroup of index \( p \) and \( |G : G'| = p^2 \).

(iii) \( G \) has a cyclic subgroup \( U \) of index \( p^2 \) such that \( C_G(U) \) is cyclic.

(iv) \( n > 4 \), and \( G \) has only one abelian subgroup of order \( p^3 \).

Two interesting results are indicated by the following theorems.

**Theorem 5.** Let \( G \) be a \( p \)-group, \( p > 2 \). If \( G \) has no normal elementary abelian subgroup of rank 3, then \( \Gamma(G) = G' \).

**Theorem 6.** Let \( A \) be a nontrivial finite abelian group and \( P = \langle s, s_1 \rangle \) be a \( p \)-group of maximal class. Let \( G = A \wr P \). If \( |A| > 2 \) or \( p > 2 \), then \( c(G) = 2 \).

In particular, every element of \( G' \) is a product of at most two commutators \([b_1, s_1][s, g]^b\), for suitable \( g \) in \( G \) and \( b, b_1 \) in the base group of \( G \).

### 3. PROOFS

**Proof of Theorem 1.** Of course a nonabelian group of order \( p^3 \) is a \( p \)-group of maximal class. It is also well known that if \( G \) is a nonabelian group of order \( p^3 \), where \( p \) is odd, then \( G \) is an extra-special group and \( G \) is isomorphic to

\[
\langle x, y | x^p = y^p = 1, [x, y] = [x, y]^x = [x, y]^y \rangle,
\]
or

\[
\langle x, y | y^p = x^{p^2} = 1, x^y = x^{p+1} \rangle.
\]

Note that these groups have exponent \( p \) and \( p^2 \), respectively.

In both cases one has \( G' = \langle [x, y] \rangle = \{[x, y]^i | 0 \leq i < p \} \); thus \( \Gamma(G) = G' \).

If \( p = 2 \) then \( G \) is isomorphic to \( D_8 \) or \( Q_8 \). If \( \{x, y\} \) is a generating set for \( G \) then \( G' = \{1, [x, y]\} \); thus \( \Gamma(G) = G' \).

For \( n \geq 4 \), we will use induction on \( n \). If \( G = \langle s, s_1 \rangle \) is a \( p \)-group of maximal class of order \( p^4 \), then \( \gamma_2(G) = \langle s_2, s_3 \rangle = \langle [s_1, s], [s_1, s, s] \rangle \), \( \gamma_3(G) = Z(G) = \langle s_3 \rangle = \langle [s_1, s, s] \rangle \), and \( \gamma_4(G) = 1 \). Let \( \gamma \in G' \), then \( \gamma = [s_1, s]^i[s_1, s, s]^j \) where \( i, j \in \mathbb{Z} \). Clearly we have \( \gamma = [s_1, s][s_1, s, s]^{j-i(i-1)/2} \)

\[
= [s_1, s][s_2]^{j-i(i-1)/2}, s] = [s_1 s_2]^{j-i(i-1)/2}, s].
\]

Let \( n > 4 \) and \( G = \langle s, s_1 \rangle \). Let also \( \overline{G} = G/Z(G) \) and observe that \( \overline{G} \) is a \( p \)-group of maximal class. It is easy to prove that the \( i \)th 2–step centralizer in \( G \) is \( \overline{K}_i = K_i/Z(G) \) for \( 2 \leq i \leq n - 3 \). Also, we have:

\[
P_0(\overline{G}) = \overline{P_0(G)}, \quad P_1(\overline{G}) = \overline{P_1(G)} = K_2, \quad P_i(\overline{G}) = \overline{P_i(G)} \quad \text{for} \quad 2 \leq i \leq n - 3,
\]
\( s \in \overline{G} - \bigcup_{i=2}^{n-3} K_i, \ s_1 \in \overline{P_1} - \overline{P_2} \) and \( \overline{G} = \langle s, s_1 \rangle \).

By induction on \( n \), \( \overline{G}' = \{ [\overline{g}, s] | g \in G \} \). Hence if \( \gamma \in G' \), then \( \overline{\gamma} = [\overline{g}, s] \) with \( g \in G \). Thus \( \gamma = [g, s]s_{n-1}^k = [g, s][s_{n-2}^k, s] = [gs_{n-2}^k, s] \) for suitable \( k \in \mathbb{Z} \). This shows that \( \Gamma(G) = G' \) and completes the proof. □

Corollary 2 is an immediate consequence of this result.

We continue by giving the proof of Corollary 3.

**Proof of Corollary 3.** Indeed, Rhemtulla [15] proved that the wreath product of a \( c_1 \)--group by a finite cyclic group is again a \( c_1 \)--group. Repeated application of Rhemtulla’s result shows that the group \( G \), satisfies the desired property and the proof is complete. □

Next we provide one example for Corollary 3.

**Example.** Let \( q \equiv_8 3 \) be a prime number, and \( S_2(2^k, q) \) a Sylow \( 2 \)--subgroup of \( GL(2^k, q) \). By [8, p. 142], we have \( G = S_2(2^k, q) \simeq S_2(2, q) \ltimes \mathbb{Z}_2 \cdots \mathbb{Z}_2 \), \((k - 1 \text{ factors})\) where \( P = S_2(2, q) \) is a semidihedral group of order 16. Since \( P \) is a \( 2 \)--group of maximal class by Corollary 3, we get \( \Gamma(G) = G' \).

Next, we prove Corollary 4.

**Proof of Corollary 4.** Notice that if one of the assertions of Corollary 4 holds, then \( G \) is of maximal class (see [6, Lemma J(k)] and [7, Proposition 4.10, Proposition 13.15, Proposition 13.16]). □

Now we turn to the proof of Theorem 5.

**Proof of Theorem 5.** Let \( G \) be a \( p \)--group, \( p > 2 \). If \( G \) has no normal elementary abelian subgroup of rank 3, then by [7, Theorem 13.7] one of the following assertions holds:

(a) \( G \) is metacyclic.

(b) \( G \) is 3--group of maximal class.

(c) \( G = EH \), where \( E = \Omega_1(G) \) is nonabelian of order \( p^3 \) and exponent \( p \), \( H \) is cyclic of index \( p^2 \) in \( G \). Furthermore, \( Z(G) \) is cyclic and \( p^2 \leq |G : Z(G)| \leq p^3 \).

First suppose \( G \) is metacyclic. We may choose suitable elements \( s, s_1 \) in \( G \) and a number \( r \), \((1 \leq r \leq |s_1|)\) such that \( G = \langle s, s_1 \rangle \) and \( s_1^s = s_1^r \). Obviously \( G' = \langle [s, s_1] \rangle = s_1^{r-1} \). Now if \( \gamma \in G' \), then \( \gamma = [s_1, s]^i = (s_1^{-1}s_1^i)^i = s_1^{-i}s_1^is_1^{-i} = [s_1^{-i}, s] \). Therefore \( G' = \{[s_1^{-i}, s]|1 \leq r \leq |s_1|\} = \Gamma(G) \).

Next, if we suppose that \( G \) is 3--group of maximal class, then our result follows from Theorem 1.
Finally if assertion (c) holds, then $G/E \cong H/H \cap E$ is cyclic. Therefore $1 \neq G' \leq E$. If $G' = E$ then $G = EH = H$ which is a contradiction. Therefore $G'$ is elementary abelian of rank less than or equal to 2. By [13, Theorem 2.4] it follows that $\Gamma(G) = G'$, as required. □

Finally, we prove Theorem 6.

Proof of Theorem 6. Let $G$ be the wreath product of a nontrivial finite abelian group $A$ and a $p$-group of maximal class $P = \langle s, s_1 \rangle$.

Let $B = \prod_{1 \leq i \leq |P|} A_i$ where $A_i \cong A$, be the base group of $G$. Now $G = BP$ is the semidirect product of $B$ by $P$. Hence $[B, P]$ is a normal subgroup of $G$ and

$$G' = [BP, BP] = [B, P]P'.$$

Since $B$ is a normal abelian subgroup of $G$, we see that $[P, B] = [G, B]$. Now by Lemma 3 [1], $[B, P] = \{[b_1, s_1][b, s]b, b_1 \in B\}$. By Corollary 2, every element $\gamma \in G'$ has the form $\gamma = [b_1, s_1][b, s][s, g] = [b_1, s_1][s, gb^{-1}]b$. Hence $c(\gamma) \leq 2$.

Further if $|A| > 2$ or $p > 2$, then by Theorem 1 [10], some element of $[P, B]$ is not a commutator. Thus $c(G) = 2$, as required.

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