ON CENTRAL TENDENCIES OF THE PARTS IN CERTAIN RESTRICTED PARTITIONS

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In this paper, we calculate the approximate values of the central tendencies of the parts with some restrictions across all the ordinary partitions, n-color partitions and compositions of ν. Further, these approximate values are compared graphically with the actual values of central tendencies of the parts with some restrictions in compositions. These results provide us a new way to explore statistical behavior of the parts in partitions and compositions.

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1. INTRODUCTION

Enumerative combinatorics is the most interesting area of combinatorics and it concentrates on counting the number of certain combinatorial objects, although counting the number of elements in a set is a rather broad mathematical problem. Analytic combinatorics concerns the enumeration of combinatorial structures using tools from complex analysis and probability theory and aims at obtaining asymptotic formulae, in contrast with enumerative combinatorics, which uses explicit combinatorial formulae and generating functions to describe the results. In literature, we find several applications of analytical number theory and combinatorial number theory, see for instance [7,12]. Partition theory [5] studies various enumeration and asymptotic problems related to integer partitions and is closely related to q-series, special functions and orthogonal polynomials. Originally a part of number theory and analysis, it is now considered a part of combinatorics or an independent field. It incorporates the bijective approach and various tools in analysis, analytic number theory and has connections with statistical mechanics [6]. While studying combinatorial objects, partitions are often studied with some restrictions on parts [4,8,9,13]. Recently, Hirschhorn [10,11] has studied statistical distribution of parts of ordinary partitions. In the present paper, we study the central tendencies of the restricted parts across all the ordinary partitions, n-color partitions and compositions of ν. We shall do it by considering a general part λ in the ordinary partitions, a general part m_i in the n-color partitions. Before proceeding further we recall some basic definitions.

A partition [5] of positive integer ν is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, \dots \lambda_r$ such that

$$\sum_{i=1}^{r} \lambda_i = \nu$$

where $\lambda_i's$ are called summands of the partition. The number of partitions of ν is denoted by $p(\nu)$.

For example, $p(3) = 3$, where the relevant ordinary partitions of 3 are 3, 21, 111.

The generating function for ordinary partitions [5] is given by

$$\sum_{\nu=0}^{\infty} p(\nu) q^{\nu} = \prod_{\nu=1}^{\infty} \frac{1}{(1-q^{\nu})}.$$

An "n-color partition" [1] is a partition in which a summand of size n, $(n \geq 0)$ can come in n different colors denoted by the subscripts: $n_1, n_2, n_3, \cdots, n_n$. The number of "$n$-color partitions" of ν is denoted by $P(\nu)$.

For example, $P(3) = 6$, where the relevant colored partitions of 3 are: 3_1, 3_2, 3_3, $2_1 1_1$, $2_2 1_1$, $1_1 1_1 1_1$.

The generating function for n-color partitions [2] is given by

$$\sum_{\nu=0}^{\infty} P(\nu) q^{\nu} = \prod_{\nu=1}^{\infty} \frac{1}{(1-q^{\nu})^{\nu}}.$$

A composition [5] of positive integer ν is an ordered partition. The number of compositions of ν is denoted by $c(\nu)$. Thus $c(3) = 4$, where the relevant compositions of 3 are: 3, 21, 12, 111.

The generating function for compositions [5] is given by $\sum\limits_{\nu=0}^{\infty} c(m,\nu) q^{\nu} = \frac{q^m}{(1-q)^m}$, where $c(m,\nu)$ denotes the number of compositions of ν with exactly m parts.

2. CENTRAL TENDENCIES USING ORDINARY PARTITIONS

In this section, we will discuss the mean, variance and standard deviation of a general part λ in the ordinary partitions of ν.

2.1. EXACT CALCULATIONS

Let $p(\nu)$ be the number of ordinary partitions of ν. Out of the $p(\nu)$ ordinary partitions of ν, it is easy to see that there are $p(\nu-\lambda)$ partitions with at least one λ as a part: simply delete λ as a part from those partitions and we obtain the partitions of $\nu-\lambda$ (and the process is reversible). In the same way, we see that the number of ordinary partitions of ν with at least two λ's as a part is $p(\nu-2\lambda)$ (delete one λ as a part from the $p(\nu-\lambda)$ partitions of ν with at least one λ). Continuing this way, we see that the number of partitions of ν with at least k λ's is $p(\nu-k\lambda)$. It follows that the number of partitions of ν with exactly k λ's is $p(\nu-k\lambda)-p(\nu-k\lambda-\lambda)$. If we let X be the number of λ's in a partition of ν and let f_k be the relative frequency with which $X=k,\ k=0,1,2,\cdots\left[\frac{\nu}{\lambda}\right]$, then

$$f_k=\frac{p(\nu-k\lambda)-p(\nu-k\lambda-\lambda)}{p(\nu)}.$$

It can be easily verified that

$$\sum_{k=0}^{\left[\frac{\nu}{\lambda}\right]} f_k=1.$$

Thus, the mean of number of λ's is

$$\begin{aligned}\mu=E(X)&=\sum_{k=0}^{\left[\frac{\nu}{\lambda}\right]} f_k k\\ &=0\left(\frac{p(\nu)-p(\nu-\lambda)}{p(\nu)}\right)+1\left(\frac{p(\nu-\lambda)-p(\nu-2\lambda)}{p(\nu)}\right)\\ &\qquad+2\left(\frac{p(\nu-2\lambda)-p(\nu-3\lambda)}{p(\nu)}\right)+\cdots\\ &\qquad\qquad=\frac{p(\nu-\lambda)+p(\nu-2\lambda)+p(\nu-3\lambda)+\cdots}{p(\nu)}\end{aligned}$$

and

$$\begin{aligned}E(X^2)&=\sum_{k=0}^{\left[\frac{\nu}{\lambda}\right]} f_k k^2\\ &=0\left(\frac{p(\nu)-p(\nu-\lambda)}{p(\nu)}\right)+1\left(\frac{p(\nu-\lambda)-p(\nu-2\lambda)}{p(\nu)}\right)\\ &\qquad+4\left(\frac{p(\nu-2\lambda)-p(\nu-3\lambda)}{p(\nu)}\right)+\cdots\end{aligned}$$

$$= \frac{p(\nu-\lambda)+3p(\nu-2\lambda)+5p(\nu-3\lambda)+\cdots}{p(\nu)}.$$

Then,

$$\sigma^2 = E(X^2) - (E(X))^2.$$

2.2. APPROXIMATE CALCULATIONS

In order to approximate μ, σ and f_k, we will make use of the rough approximation [5]

$$p(\nu) \approx \frac{1}{4\nu\sqrt{3}} exp\{K\sqrt{\nu}\}, \qquad \text{where } K = \pi\sqrt{\frac{2}{3}}.$$

Thus

$$\begin{aligned}
\mu &= \frac{1}{p(\nu)} \sum_{k=0}^{[\frac{\nu}{\lambda}]} p(\nu - k\lambda) - 1 \\
&\approx \frac{\nu}{\exp\{K\sqrt{\nu}\}} \int_1^{[\frac{\nu}{\lambda}]} \frac{\exp\{K\sqrt{\nu - x\lambda}\}}{\nu - x\lambda} dx - 1 \\
&\approx \frac{\nu}{\exp\{K\sqrt{\nu}\}} \int_1^{[\frac{\nu}{\lambda}]} \frac{1}{\sqrt{\nu - x\lambda}} \frac{\exp\{K\sqrt{\nu - x\lambda}\}}{\sqrt{\nu - x\lambda}} dx - 1 \\
&\approx \frac{\nu}{\exp\{K\sqrt{\nu}\}} \left\{ \left[\frac{1}{\sqrt{\nu - x\lambda}} \frac{\exp\{K\sqrt{\nu - x\lambda}\}}{K} \left(\frac{-2}{\lambda}\right) \right]_1^{[\frac{\nu}{\lambda}]} \right. \\
&\qquad \left. + \frac{1}{K} \int_1^{[\frac{\nu}{\lambda}]} \frac{\exp\{K\sqrt{\nu - x\lambda}\}}{\sqrt[3/2]{\nu - x\lambda}} dx \right\} \\
&\approx \frac{\nu}{K\exp\{K\sqrt{\nu}\}} \left\{ \left[\frac{-2\exp\{K\sqrt{\nu - x\lambda}\}}{\lambda\sqrt{\nu - x\lambda}} \right]_1^{[\frac{\nu}{\lambda}]} \right. \\
&\qquad \left. + \left[\frac{-2\exp\{K\sqrt{\nu - x\lambda}\}}{K\lambda(\nu - x\lambda)} \right]_1^{[\frac{\nu}{\lambda}]} \right\} \\
&\approx \frac{2\nu}{K\lambda\exp\{K\sqrt{\nu}\}} \frac{\exp\{K\sqrt{\nu - \lambda}\}}{\sqrt{\nu - \lambda}} \left(1 + \frac{1}{K\sqrt{\nu - \lambda}}\right).
\end{aligned}$$

Also,

$$E(X^2) = \frac{1}{p(\nu)} \sum_{k=0}^{[\frac{\nu}{\lambda}]} (2k+1) p(\nu - \lambda k - \lambda)$$

$$\approx \frac{\nu}{\exp\{K\sqrt{\nu}\}} \int_1^{[\frac{\nu}{\lambda}]} \frac{(2x+1)\exp\{K\sqrt{\nu - x\lambda - \lambda}\}}{\nu - x\lambda - \lambda} dx$$

$$\approx \frac{\nu}{\exp\{K\sqrt{\nu}\}} \int_1^{[\frac{\nu}{\lambda}]} \frac{2x+1}{\sqrt{\nu - x\lambda - \lambda}} \frac{\exp\{K\sqrt{\nu - x\lambda - \lambda}\}}{\sqrt{\nu - x\lambda - \lambda}} dx$$

$$\approx \frac{2\nu}{\lambda K \exp\{K\sqrt{\nu}\}} \left\{ \left[\frac{-(2x+1)\exp\{K\sqrt{\nu - x\lambda - \lambda}\}}{\sqrt{\nu - x\lambda - \lambda}} \right]_1^{[\frac{\nu}{\lambda}]} \right\}$$

$$+ \frac{2\nu}{\lambda K \exp\{K\sqrt{\nu}\}} \left\{ \frac{1}{2} \int_1^{[\frac{\nu}{\lambda}]} \frac{4\nu - 2x\lambda - 3\lambda}{\nu - x\lambda - \lambda} \frac{\exp\{K\sqrt{\nu - x\lambda - \lambda}\}}{\sqrt{\nu - x\lambda - \lambda}} dx \right\}$$

$$\approx \frac{2\nu}{\lambda K \exp\{K\sqrt{\nu}\}} \left\{ \left[\frac{-(2x+1)\exp\{K\sqrt{\nu - x\lambda - \lambda}\}}{\sqrt{\nu - x\lambda - \lambda}} \right]_1^{[\frac{\nu}{\lambda}]} \right\}$$

$$- \frac{2\nu}{\lambda K \exp\{K\sqrt{\nu}\}} \left\{ \left[\frac{(4\nu - 2x\lambda - 3\lambda)\exp\{K\sqrt{\nu - x\lambda - \lambda}\}}{\lambda k(\nu - x\lambda - \lambda)} \right]_1^{[\frac{\nu}{\lambda}]} \right\}$$

$$\approx \frac{2\nu}{\lambda K \exp\{K\sqrt{\nu}\}} \left\{ \frac{3\exp\{K\sqrt{\nu - 2\lambda}\}}{\sqrt{\nu - 2\lambda}} \frac{(4\nu - 5\lambda)\exp\{K\sqrt{\nu - 2\lambda}\}}{\lambda K(\nu - 2\lambda)} \right\}$$

$$\approx \frac{2\nu}{\lambda K \exp\{K\sqrt{\nu}\}} \frac{\exp\{K\sqrt{\nu - 2\lambda}\}}{\sqrt{\nu - 2\lambda}} \left(3 + \frac{4\nu - 5\lambda}{\lambda K\sqrt{\nu - 2\lambda}} \right).$$

And,

$$\sigma^2 = E(X^2) - (E(X))^2.$$

3. CENTRAL TENDENCIES USING N-COLOR PARTITIONS

Here, central tendencies are studied by observing a general part m_i of the n-color partitions of ν.

3.1. EXACT CALCULATIONS

Let $P(\nu)$ denote the number of n-color partitions of ν. Out of the $P(\nu)$ n-color partitions of ν, it is easy to see that there are $P(\nu - m)$ n-color partitions with at least one m_i as a part: simply delete m_i as a part from these n-color partitions and we obtain the partitions of $\nu - m$ (and the process is reversible). In the same way, we see that the number of n-color partitions of ν with at least two m_i's is $P(\nu - 2m)$ (delete a i_j as a part from the $P(\nu - m)$ partitions of ν with at least one m_i). Continuing this way, we see that the number of n-color partitions of ν with at least k m_i's is $P(\nu - km)$.

It follows that the number of m_i's in an n-color partition of ν with exactly k m_i's is $P(\nu-km)-P(\nu-km-m)$. If we let X be the number of m_i's in an n-color partition of ν, and let f_k be the relative frequency with which $X=k,\ k=0,1,2,\cdots,\left[\frac{\nu}{m}\right]$, then

$$f_k = \frac{P(\nu-km)-P(\nu-km-m)}{P(\nu)},$$

$$\text{and} \qquad \sum_{k=0}^{\left[\frac{\nu}{m}\right]} f_k = 1.$$

Thus, the mean of number of m_i's is

$$\begin{aligned}
\mu = E(X) &= \sum_{k=0}^{\left[\frac{\nu}{m}\right]} f_k k \\
&= 0\left(\frac{P(\nu)-P(\nu-m)}{P(\nu)}\right) + 1\left(\frac{P(\nu-m)-P(\nu-2m)}{P(\nu)}\right) \\
&\qquad + 2\left(\frac{P(\nu-2m)-P(\nu-3m)}{P(\nu)}\right) + \cdots \\
&= \frac{P(\nu-m)+P(\nu-2m)+P(\nu-3m)+\cdots}{P(\nu)}
\end{aligned}$$

and

$$\begin{aligned}
E(X^2) &= \sum_{k=0}^{\left[\frac{\nu}{m}\right]} f_k k^2 \\
&= 0\left(\frac{P(\nu)-P(\nu-m)}{P(\nu)}\right) + 1\left(\frac{P(\nu-m)-P(\nu-2m)}{P(\nu)}\right) \\
&\qquad + 4\left(\frac{P(\nu-2m)-P(\nu-3m)}{P(\nu)}\right) + \cdots \\
&= \frac{P(\nu-m)+3P(\nu-2m)+5P(\nu-3m)+\cdots}{P(\nu)}.
\end{aligned}$$

$$\text{So,} \qquad \sigma^2 = E(X^2) - (E(X))^2.$$

4. CENTRAL TENDENCIES USING COMPOSITIONS

Let $c(\nu)$ denote the number of compositions of ν. From [5], we have, $c(\nu) = 2^{\nu-1}$. We first prove a proposition.

PROPOSITION 4.1. *Let $c(\nu)$ denote the number of compositions of ν. Then*

$c(\nu) = c(\nu-1)+c(\nu-2)+c(\nu-3)+\cdots+c(0)$, *where,* $c(0)=1$ *and* $c(\nu<0)=0$.

Analytical Proof:

$$\begin{aligned}
&c(\nu-1)+c(\nu-2)+c(\nu-3)+\cdots+c(1)+c(0)\\
&=2^{\nu-2}+2^{\nu-3}+2^{\nu-4}+\cdots+1+1\\
&=\frac{2^{\nu-2}\left(1-\left(\frac{1}{2}\right)^{\nu-1}\right)}{1-\frac{1}{2}}+1\\
&=2^{\nu-1}=c(\nu).
\end{aligned}$$

Combinatorial Proof:

It can be seen easily, there are $c(\nu-1)$ compositions with 1 as a last part: simply delete the last part 1 from these compositions and we obtain the compositions of $\nu-1$ (and the process is reversible). In the same way, we see that the number of compositions of ν with 2 as a last part is $c(\nu-2)$ (delete the last part 2 from $c(\nu)$ compositions of ν with the last part 2). Continuing this way, we see that

$$c(\nu)=c(\nu-1)+c(\nu-2)+\cdots+c(0)$$

If h be the last part in a composition of ν and let f_k be the relative frequency with which $h=k,\ k=1,2,\cdots,\nu$, then

$$f_k=\frac{c(\nu-k)}{c(\nu)},\qquad \text{clearly,}\qquad \sum_{k=1}^{\nu}f_k=1.$$

Thus,

$$\mu=E(h)=\sum_{k=1}^{\nu}f_kk=\frac{c(\nu-1)+2c(\nu-2)+\cdots\nu c(0)}{c(\nu)},$$

$$E(h^2)=\sum_{k=1}^{\nu}f_kk^2=\frac{c(\nu-1)+4c(\nu-2)+\cdots\nu^2c(0)}{c(\nu)}.$$

and

$$\sigma^2=E(h^2)-(E(h))^2.$$

4.1. ALMOST EXACT CALCULATIONS

In order to approximate μ, σ and f_k, we will make use of the exact formula

$$c(\nu)=2^{\nu-1}.$$

Thus

$$\begin{aligned}
\mu = E(h) &= \frac{1}{c(\nu)} \sum_{k=1}^{\nu} kc(\nu - k) \\
&= \frac{1}{2^{\nu-1}} \left(\sum_{k=1}^{\nu-1} k2^{\nu-k-1} + \nu \right) \\
&\approx \frac{1}{2^{\nu-1}} \left(\int_1^{\nu-1} x2^{\nu-x-1} dx + \nu \right) \\
&= \int_1^{\nu-1} x2^{-x} dx + \frac{\nu}{2^{\nu-1}} \\
&= \int_1^{\nu-1} x2^{-x} dx + \frac{\nu}{2^{\nu-1}} \\
&= \left[\frac{-x2^{-x}}{\ln 2} \right]_1^{\nu-1} + \int_1^{\nu-1} \frac{2^{-x}}{\ln 2} dx + \frac{\nu}{2^{\nu-1}} \\
&= \frac{1}{2^{\nu-1}} \left[\frac{2^{\nu-2} - \nu + 1}{\ln 2} + \frac{2^{\nu-2} - 1}{(\ln 2)^2} + \nu \right].
\end{aligned}$$

And

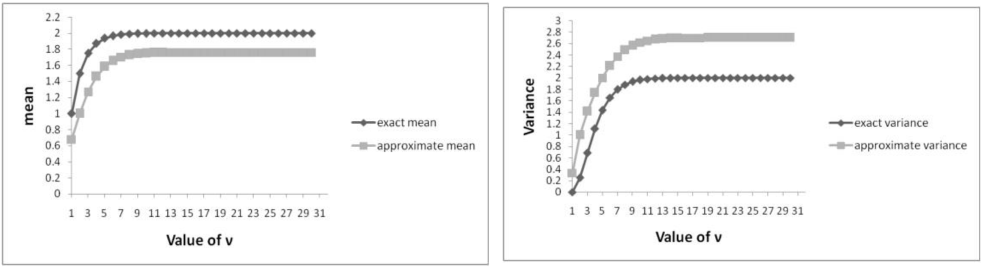
$$\begin{aligned}
E(h^2) &= \frac{1}{c(\nu)} \sum_{k=1}^{\nu} k^2 c(\nu - k) \\
&= \frac{1}{2^{\nu-1}} \left(\sum_{k=1}^{\nu-1} k^2 2^{\nu-k-1} + \nu^2 \right) \\
&\approx \frac{1}{2^{\nu-1}} \left(\int_1^{\nu-1} x^2 2^{\nu-x-1} dx + \nu^2 \right) \\
&= \int_1^{\nu-1} x^2 2^{-x} dx + \frac{\nu^2}{2^{\nu-1}} \\
&= \left[\frac{-x^2 2^{-x}}{\ln 2} \right]_1^{\nu-1} + \int_1^{\nu-1} \frac{2x2^{-x}}{\ln 2} dx + \frac{\nu^2}{2^{\nu-1}} \\
&= \left[\frac{-x^2 2^{-x}}{\ln 2} \right]_1^{\nu-1} + \frac{2}{\ln 2} \left[\frac{-x2^{-x}}{\ln 2} \right]_1^{\nu-1} + \frac{2}{(\ln 2)^2} \int_1^{\nu-1} 2^{-x} dx + \frac{\nu^2}{2^{\nu-1}} \\
&= \frac{1}{2^{\nu-1}} \left[\frac{2^{\nu-2} - (\nu-1)^2}{\ln 2} + \frac{2^{\nu-1} - 2(\nu-1)}{(\ln 2)^2} + \frac{2^{\nu-1} - 2}{(\ln 2)^3} + \nu^2 \right].
\end{aligned}$$

So,

$$\sigma^2 = E(h^2) - (E(h))^2.$$

5. GRAPHICAL ILLUSTRATION

For the illustration, Figure 1a shows the variation in exact mean and approximate mean for compositions and Figure 1b shows the variation in exact variance and approximate variance for compositions calculated in Section 4.1.



(a) (b)

Fig. 1

From Figures 1a and 1b, we see that the mean $\mu = E(h)$ and variance σ^2 (both exact and approximated values) increase steeply for $1 \leq \nu \leq 10$, then these become almost constant for any large value of ν.

6. CONCLUSION

The paper gives a new direction to study partition theory and their statistical behavior (such as statistical distributions). A very obvious problem arising here is: As there is no exact formula to calculate the colored partitions, it would be interesting to use generating functions of colored partitions to study the statistical distributions and central tendencies of their parts.

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