SUB-LINEAR OSCILLATIONS *VIA* NONPRINCIPAL SOLUTION

ABDULLAH ÖZBEKLER

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In the paper, we give new oscillation criteria for forced sub-linear differential equations with “oscillatory potentials”under the assumption that corresponding linear homogeneous equation is nonoscillatory.

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1. INTRODUCTION

The concept of the principal solution was introduced in 1936 by W. Leighton and M. Morse [12] in studying positiveness of certain quadratic functional associated with

$$(r(t)x')' + p(t)x = 0, \qquad t \geq t_0. \tag{1.1}$$

Since then the principal and nonprincipal solutions have been used successfully in connection with oscillation and asymptotic theory of (1.1) and related equations, see for instance [1, 4, 6, 8, 12, 17, 19, 23] and the references cited therein. For some extensions to Hamiltonian systems, half-linear differential equations, dynamic equations and impulsive differential equations, we refer in particular to [3, 5, 6, 13, 24].

We recall that a nontrivial solution x_1 of (1.1) is said to be principal (“small”or “recessive”) if for every solution x_2 of (1.1) such that $x_1 \neq cx_2$, $c \in \mathbb{R}$,

$$\lim_{t\to\infty} \frac{x_1(t)}{x_2(t)} = 0.$$

It is well known that a principal solution x_1 of (1.1) exists uniquely up to a multiplication by a nonzero constant if and only if (1.1) is nonoscillatory. A solution x_2 that is linearly independent of x_1 is called a nonprincipal (“large”or “dominant”) solution. For other characterizations of principal and nonprincipal solutions of (1.1), see [8, Theorem 6.4], [11, Theorem 5.59].

In 1999, Wong [23], by employing a nonprincipal solution of (1.1), obtained the following oscillation criterion for

$$(r(t)x')' + p(t)x = f(t). \tag{1.2}$$

For extensions of the theorem to impulsive differential equations and dynamic equations on time scales, see [13, 24].

THEOREM 1.1 (Wong's theorem). *Suppose that* (1.1) *is nonoscillatory. Let* z *be a positive solution of* (1.1) *satisfying*

$$\int_a^\infty \frac{1}{r(s)z^2(s)}\,\mathrm{d}s < \infty \tag{1.3}$$

for some a *sufficiently large, i.e, a nonprincipal solution. If*

$$\overline{\lim_{t\to\infty}}\, H(t) = -\underline{\lim_{t\to\infty}}\, H(t) = \infty, \tag{1.4}$$

where

$$H(t) := \int_a^t \frac{1}{r(s)z^2(s)}\left(\int_a^s z(\tau)f(\tau)\mathrm{d}\tau\right)\mathrm{d}s, \tag{1.5}$$

then Eq. (1.2) is oscillatory.

The aim of our work is to extend the above theorem to Emden-Fowler sub-linear equation

$$(r(t)x')' + q(t)|x|^{\gamma-1}x = f(t), \qquad t \geq t_0, \tag{1.6}$$

where $\gamma \in (0,1)$ and r, q, f are continuous functions on $[t_0,\infty)$ with $r > 0$.

By a solution of (1.6) defined on an interval $[t_1,\infty)$, $t_1 \geq t_0$, we mean a function x, $(rx')' \in \mathrm{C}[t_1,\infty)$, satisfying (1.6). We note that the assumption of r, q, f being continuous is not sufficient to ensure the existence of extendable solutions of (1.6) on $[T,\infty)$, see [20]. However, as usual in the oscillation theory we only consider solutions of (1.6) which are extendable to $[T,\infty)$ and nontrivial in the neighborhood of infinity. Such a solution is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Eq. (1.6) is called oscillatory (nonoscillatory) if all solutions are oscillatory (nonoscillatory).

Usually, nonlinear results require the potential function $a(t)$ in an Emden-Fowler equation

$$x'' + a(t)|x|^{\alpha-1}x = f(t), \quad \alpha > 0$$

to be non-negative, see [21]. Fortunately, we are able to take the potential q to be an oscillatory function in (1.6). On the other hand, letting $\gamma \to 1^-$ in (1.6) results in (1.1) with $p(t) = q(t)$, and thus our result extends Theorem 1.1 by a limiting process $\gamma \to 1$ in (1.6).

We note that the oscillation of the solutions of Eq. (1.6) with $\gamma > 0$ has been studied by many authors, see for instance [2, 7, 9, 10, 14–16, 18, 20–23], but to the best of our knowledge there is only one paper in the literature similar to Theorem 1.1 for such nonlinear equations, especially for (1.6) in which Özbekler *et al.* [25] obtained some oscillation results for the nonlinear equations of the form

$$(r(t)x')' + p(t)F(x) - q(t)G(x) = f(t) \tag{1.7}$$

motivated by the less general equation

$$(r(t)x')' + p(t)|x|^{\beta-1}x - q(t)|x|^{\gamma-1}x = f(t), \qquad \beta > 1 > \gamma > 0 \tag{1.8}$$

under the assumption that corresponding linear homogeneous equation

$$(r(t)x')' + [p(t) - q(t)]x = 0$$

is nonoscillatory. In Eq. (1.7) (and Eq. (1.8)), r is assumed to be a positive function, the potential functions p, q are non-negative and no sign restriction is imposed on the forcing term f. Moreover, the nonlinear terms in Eq. (1.7) satisfy

(A) $xF(x) > 0$ and $xG(x) > 0$ for $x \neq 0$;

(B) (a) $\lim_{|x|\to\infty} x^{-1}F(x) > 1, \quad \lim_{|x|\to 0} x^{-1}F(x) < 1,$

(b) $\lim_{|x|\to\infty} x^{-1}G(x) < 1, \quad \lim_{|x|\to 0} x^{-1}G(x) > 1.$

Note that two special cases of Eq. (1.8) are Emden-Fowler super-linear ($q(t) = 0$) and sub-linear ($p(t) = 0$) equations. However, the results presented in [25] do not give any information about the oscillatory behavior of Eq. (1.8) in the case the potentials $p(t)$ and $q(t)$ are oscillatory.

In 2000, Agarwal and Grace [31] studied the super-linear equation

$$x^{(n)} + q(t)|x|^{\alpha-1}x = f(t), \tag{1.9}$$

with $\alpha > 1$ and $q(t) < 0$ by the method of general means, and then Ou and Wong [30] extend their results to more general equation

$$x^{(n)} + q(t)\Phi(x) = f(t), \qquad q(t) \geq 0 \quad (q(t) < 0)$$

imposing some conditions on Φ. As far as the oscillatory potentials are considered, El-Sayed [7] ($n = 2$ and $\alpha = 1$), Wong [23] ($n = 2$ and $\alpha = 1$), Nasr [29] ($n = 2$ and $\alpha > 1$), Sun [28] ($n = 2$ and $\alpha > 1$), Sun and Agarwal [26, 27] ($\alpha > 1$) studied the oscillation of Eq. (1.9).

In 2004, Sun and Wong [32] presented the oscillation criteria for Eq. (1.9) in the sub-linear case, i.e. the case $\alpha \in (0,1)$ by the method similar to that of Agarwal and Grace [31] without assuming any sign condition on the potential function $q(t)$.

In the case where the potential $q(t)$ is non-positive in Eq (1.6), an oscillation criterion was given [25, Corollary 2.4], but it appears to us nothing has been known about the oscillation of Emden-Fowler sub-linear Eq. (1.6) in case of oscillatory potential $q(t)$. In this paper, we will establish an oscillation theorem for Eq. (1.6) under the assumption that the corresponding linear equation

$$(r(t)x')' + q(t)x = 0 \tag{1.10}$$

is nonoscillatory. The results obtained in this paper are analogue to those given in [23] and [25]. Examples are provided to emphasize the main results and we impose some open problems for such critical cases of Eq. (1.6) and related equations.

2. MAIN RESULTS

Associated with equation (1.6) we assume that the linear Eq. (1.10) is nonoscillatory. Since $q(t) \geq -q_-(t)$, equation

$$(r(t)y')' - q_-(t)y = 0 \tag{2.1}$$

is nonoscillatory by Sturm comparison theorem, where $q_-(t) = \max\{-q(t), 0\}$.

Denote by $z(t)$, $t \in [a, \infty)$, a positive nonprincipal solution of (2.1) i.e., $z(t)$ satisfies

$$\int_a^\infty \frac{\mathrm{d}s}{r(s)z^2(s)} < \infty. \tag{2.2}$$

Define

$$\mathcal{H}_\pm(t) := \int_a^t \frac{1}{r(s)z^2(s)} \left(\int_a^s \left[f(\tau) \pm (1-\gamma) q_-(\tau) \right] z(\tau) \mathrm{d}\tau \right) \mathrm{d}s. \tag{2.3}$$

The main result of this paper is the following theorem.

THEOREM 2.1. *Suppose that* (1.10) *is nonoscillatory and let* $z(t)$ *be a positive solution of* (2.1) *satisfying* (2.2), *i.e. a nonprincipal solution. If*

$$\overline{\lim_{t\to\infty}}\, \mathcal{H}_-(t) = -\underline{\lim_{t\to\infty}}\, \mathcal{H}_+(t) = \infty, \tag{2.4}$$

where $\mathcal{H}_\pm$ *is given by* (2.3), *then Eq.* (1.6) *is oscillatory.*

Proof. Suppose that there is a nonoscillatory solution $x(t)$ of (1.6). We may assume that $x(t) \neq 0$ on $[a, \infty)$ for some $a \geq t_0$ sufficiently large. The change of variables $x = z(t)w$, where $z(t)$ is a positive nonprincipal solution of (2.1), transforms (1.6) into

$$(r(t)z^2\, w')' = \big\{f(t) - q(t)|x|^{\gamma-1}x - q_-(t)x\big\}z, \qquad t \geq a, \tag{2.5}$$

Integration of (2.5) leads to

$$\begin{aligned} w(t) = \int_a^t \frac{1}{r(s)z^2(s)} \int_a^s \big\{f(\tau) - q(\tau)|x|^{\gamma-1}x - q_-(\tau)x\big\}z(\tau)\mathrm{d}\tau\mathrm{d}s \\ +c_1 \int_a^t \frac{\mathrm{d}s}{r(s)z^2(s)} + c_2 \end{aligned} \tag{2.6}$$

where $c_1 = r(a)z^2(a)w'(a)$ and $c_2 = w(a)$ are constants.

If $x(t) > 0$ on $[a, \infty)$, then using (2.6) we obtain

$$\begin{aligned} w(t) \leq \int_a^t \frac{1}{r(s)z^2(s)} \int_a^s \big\{f(\tau) - \gamma q_-(\tau)[x - \frac{1}{\gamma}\, x^\gamma] - (1-\gamma)q_-(\tau)x\big\}z(\tau)\mathrm{d}\tau\mathrm{d}s \\ +c_1 \int_a^t \frac{\mathrm{d}s}{r(s)z^2(s)} + c_2. \end{aligned} \tag{2.7}$$

Define a function $W(u) : (0, \infty) \to \mathbb{R}$ by

$$W(u) := u - \frac{1}{\gamma}\, u^\gamma, \qquad \gamma \in (0, 1).$$

It is not difficult to see that

$$\min_{u \in (0,\infty)} W(u) = 1 - 1/\gamma. \tag{2.8}$$

Using (2.8) in (2.7) yields

$$w(t) \leq \mathcal{H}_+(t) + c_1 \int_a^t \frac{\mathrm{d}s}{r(s)z^2(s)} + c_2, \tag{2.9}$$

and similarly, if $x(t) < 0$ on $[a, \infty)$, then again using (2.8), (2.6) turns out

$$\begin{aligned} w(t) &= \int_a^t \frac{1}{r(s)z^2(s)} \int_a^s \big\{f(\tau) + \gamma q_-(\tau)[|x| - \frac{1}{\gamma}\, |x|^\gamma] \\ &\quad +(1-\gamma)q_-(\tau)|x|\big\}z(\tau)\mathrm{d}\tau\mathrm{d}s + c_1 \int_a^t \frac{\mathrm{d}s}{r(s)z^2(s)} + c_2 \\ &\geq \mathcal{H}_-(t) + c_1 \int_a^t \frac{\mathrm{d}s}{r(s)z^2(s)} + c_2 \end{aligned} \tag{2.10}$$

Note that (2.2), (2.4), (2.9) and (2.10) imply that

$$\overline{\lim_{t\to\infty}}\, w(t) = -\underline{\lim_{t\to\infty}}\, w(t) = +\infty. \tag{2.11}$$

Because $z(t)$ is positive, (2.11) implies that $x(t)$ cannot have a definite sign on $[a, \infty)$, a contradiction. $\square$

Remark 1. Theorem 2.1 is exciting because it reduces to Theorem 1.1 for the linear Eq. (1.2) with $p(t) = q(t)$ by letting $\gamma \to 1^-$ in (1.6).

Remark 2. In case when the potential $q(t)$ is positive, Theorem 2.1 is of particular interest. In this case $q_-(t) = 0$ and Eq. (2.1) turns out to be

$$(r(t)y')' = 0. \tag{2.12}$$

It is clear that Eq. (2.12) is nonoscillatory and has two linearly independent solutions $y_1(t) = 1$ and

$$y_2(t) = \int^t r^{-1}(s)\,\mathrm{d}s.$$

We note that the integral

$$\int^\infty r^{-1}(t)\,\mathrm{d}t$$

determines the (non)principal solutions. Accordingly, we define

$$\mathcal{H}_0(t) := \int_a^t r^{-1}(s) \int_a^s f(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \tag{2.13}$$

and

$$\mathcal{H}_1(t) := \int_a^t r^{-1}(s)\left(\int_a^s r^{-1}(k)\mathrm{d}k\right)^{-2} \int_a^s f(\tau) \int_a^\tau r^{-1}(k)\,\mathrm{d}k\,\mathrm{d}\tau\,\mathrm{d}s. \tag{2.14}$$

PROPOSITION 1. *If*

$$\int^\infty r^{-1}(t)\,\mathrm{d}t < \infty \quad \textit{and} \quad \overline{\lim_{t\to\infty}}\, \mathcal{H}_0(t) = -\underline{\lim_{t\to\infty}}\, \mathcal{H}_0(t) = \infty$$

or

$$\int^\infty r^{-1}(t)\,\mathrm{d}t = \infty \quad \textit{and} \quad \overline{\lim_{t\to\infty}}\, \mathcal{H}_1(t) = -\underline{\lim_{t\to\infty}}\, \mathcal{H}_1(t) = \infty$$

is satisfied, then Eq. (1.6) is oscillatory.

3. CONCLUDING REMARKS

Özbekler *et al.* [25, Corollary 2.4] presented an oscillation criterion analogous to Theorem 2.1 for Eq. (1.6) with non-positive potential, *i.e.*, $q_-(t) = -q(t)$.

Theorem 3.1 ([25]). *Let $z(t)$ be a positive solution of Eq.* (1.10) *satisfying* (2.2), *i.e., a nonprincipal solution. If*

$$\varlimsup_{t\to\infty} \mathcal{N}_-(t) = -\varliminf_{t\to\infty} \mathcal{N}_+(t) = \infty, \tag{3.1}$$

where

$$\mathcal{N}_\pm(t) := \int_a^t \frac{1}{r(s)z^2(s)} \left(\int_a^s \left[f(\tau) \pm (1-\gamma)\gamma^{\gamma/(1-\gamma)} q(\tau) \right] z(\tau) \mathrm{d}\tau \right) \mathrm{d}s,$$

then Eq. (1.6) is oscillatory.

However, this oscillation result is little weaker than the one we obtained in Theorem 2.1. We note that $\mathcal{H}_+(t) > \mathcal{N}_+(t)$ and $\mathcal{H}_-(t) < \mathcal{N}_-(t)$, since $\gamma^{\gamma/(1-\gamma)} < 1$ for any $\gamma \in (0,1)$, and hence we conclude that condition (2.4) of Theorem 2.1 implies condition (3.1) of Theorem 3.1.

Consider a slightly more general equation than (1.6)

$$(r(t)x')' + q(t)\mathcal{F}(x) = f(t), \qquad t \geq t_0, \tag{3.2}$$

where r, q, f are defined as previously and the function $\mathcal{F} \in C(\mathbb{R}, \mathbb{R})$ satisfies the following conditions:

(i) $x\mathcal{F}(x) > 0$ for $x \neq 0$;

(ii) $\lim_{|x|\to\infty} x^{-1}\mathcal{F}(x) < 1, \qquad \lim_{|x|\to 0} x^{-1}\mathcal{F}(x) > 1.$

Using (i) and (ii), it is easy to find positive constants γ_0, δ_0 such that

$$\max_{x\leq 0}[x - \mathcal{F}(x)] = \delta_0, \qquad \min_{x\geq 0}[x - \mathcal{F}(x)] = -\gamma_0. \tag{3.3}$$

In what follows, we define

$$\mathcal{N}_1(t) := \int_a^t \frac{1}{r(s)z^2(s)} \left(\int_a^s \left[f(\tau) - \delta_0 q_-(\tau) \right] z(\tau) \mathrm{d}\tau \right) \mathrm{d}s. \tag{3.4}$$

and

$$\mathcal{N}_2(t) := \int_a^t \frac{1}{r(s)z^2(s)} \left(\int_a^s \left[f(\tau) + \gamma_0 q_-(\tau) \right] z(\tau) \mathrm{d}\tau \right) \mathrm{d}s, \tag{3.5}$$

Theorem 3.2. *Suppose that* (1.10) *is nonoscillatory and let $z(t)$ be a positive solution of* (2.1) *satisfying* (2.2), *i.e. a nonprincipal solution. If*

$$\varlimsup_{t\to\infty} \mathcal{N}_1(t) = -\varliminf_{t\to\infty} \mathcal{N}_2(t) = \infty, \tag{3.6}$$

where $\mathcal{H}$ is given by (2.3), *then Eq.* (1.6) *is oscillatory.*

Proof. Suppose that there is a nonoscillatory solution $x(t)$ of (3.2). We may assume that $x(t) \neq 0$ on $[a, \infty)$ for some $a \geq t_0$ sufficiently large. The change of variables $x = z(t)w$, where $z(t)$ is a positive nonprincipal solution of (2.1), transforms (3.2) into

$$(r(t)z^2 w')' = \{f(t) - q(t)\mathcal{F}(x) - q_-(t)x\}z, \qquad t \geq a, \tag{3.7}$$

Integration of (3.7) leads to

$$\begin{aligned} w(t) = & \int_a^t \frac{1}{r(s)z^2(s)} \int_a^s \{f(\tau) - q(\tau)\mathcal{F}(x) - q_-(\tau)x\}z(\tau)\mathrm{d}\tau\mathrm{d}s \\ & + k_1 \int_a^t \frac{\mathrm{d}s}{r(s)z^2(s)} + k_2 \end{aligned} \tag{3.8}$$

where $k_1 = r(a)z^2(a)w'(a)$ and $k_2 = w(a)$ are constants.

If $x(t) > 0$ on $[a, \infty)$, then using (3.3) and (3.8) we obtain

$$w(t) \leq \mathcal{N}_2(t) + k_1 \int_a^t \frac{\mathrm{d}s}{r(s)z^2(s)} + k_2. \tag{3.9}$$

Similarly, if $x(t) < 0$ on $[a, \infty)$, then again using (3.3) and (3.8) we get

$$w(t) \geq \mathcal{N}_1(t) + k_1 \int_a^t \frac{\mathrm{d}s}{r(s)z^2(s)} + k_2. \tag{3.10}$$

Note that (2.2), (3.6), (3.9) and (3.10) imply that

$$\overline{\lim_{t\to\infty}}\, w(t) = -\underline{\lim_{t\to\infty}}\, w(t) = +\infty. \tag{3.11}$$

Because $z(t)$ is positive, (3.11) implies that $x(t)$ cannot have a definite sign on $[a, \infty)$, so we obtain a similar contradiction as in the proof of Theorem 2.1. □

If we take $\mathcal{F}(x) = |x|^{\gamma-1}x$ where $\gamma \in (0, 1)$, then it can be easily calculated that

$$\delta_0 = \gamma_0 = (1 - \gamma)\gamma^{\gamma/(1-\gamma)} > 0.$$

Finally, we pose an open problem concerning a possible extension of Theorems 2.1 or 3.2. Consider the Emden-Fowler super-linear equation

$$(r(t)x')' + p(t)|x|^{\beta-1}x = f(t), \qquad \beta > 1 \tag{3.12}$$

where the potential function $p(t)$ has no definite sign. Suppose the unforced equation

$$(r(t)z')' + p(t)z = 0$$

is nonoscillatory. Find the conditions on the forcing term $f(t)$ which will determine the oscillatory character of Eq. (3.12).

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Atilim University 06836,
Department of Mathematics
Incek, Ankara, Turkey
aozbekler@gmail.com
abdullah.ozbekler@atilim.edu.tr