PRIME IDEALS OF GENERALIZED PRINCIPALLY QUASI-BAER RINGS

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In this paper, we continue to study generalized p.q.-Baer rings. We show that the prime radical is the unique minimal prime ideal of a class of principally right primary rings. Some results are provided which enable us to generate examples of (principally) right primary rings which are not quasi-Baer. We also extend a theorem of Kist for commutative PP-rings to generalized principally quasi-Baer (*e.g.*, completely primary) rings for which every prime ideal contains a unique minimal prime ideal without using topological arguments. Furthermore, various decompositions of generalized right p.q.-Baer rings are determined. Connections to related classes of rings are investigated.

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1. INTRODUCTION

Throughout this paper all rings are associative with identity and all modules are unital. Recall from [20] that R is a *Baer* ring if the right annihilator of every nonempty subset of R is generated by an idempotent. In [20] Kaplansky introduced Baer rings to abstract various properties of AW^*-algebras and von Neumann algebras. The class of Baer rings includes the von Neumann algebras. In [14] Clark defines a ring to be *quasi-Baer* if the left annihilator of every ideal is generated, as a left ideal, by an idempotent. He then uses the quasi-Baer concept to characterize when a finite-dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. Some results on *quasi-Baer* rings can be found in *e.g.* [6–8, 30, 31, 33, 34].

A ring R is called *right (left) PP* if every principal right (left) ideal is projective (equivalently, if the right (left) annihilator of any element of R is generated as a right (left) ideal by an idempotent of R). R is called a *PP ring* (also called a Rickart ring [2, p. 18]), if it is both right and left PP. The concept of PP ring is not left-right symmetric by Chase [12]. A right PP ring R is Baer (so PP) when R is orthogonally finite by Small [36], and a right PP ring R is PP when R is *abelian* (idempotents are central) by Endo [16].

Birkenmeier, Kim and Park in [8] initiated the concept of principally quasi-Baer rings. A ring R is called *right principally quasi-Baer* (or simply *right p.q.-Baer*) if the right annihilator of a principal right ideal is generated by an idempotent. Equivalently, R is right p.q.-Baer if R modulo the right annihilator of any principal right ideal is projective. If R is a semiprime ring, then R is right p.q.-Baer if and only if R is left p.q.-Baer. The class of right p.q.-Baer rings includes properly the class of quasi-Baer rings. Some examples were given in [8] to show that the classes of right p.q.-Baer rings and right PP-rings are distinct.

From [25], a ring R is called *generalized right (principally) quasi-Baer* if for any (principal) right ideal I of R, the right annihilator of I^n is generated by an idempotent for some positive integer n, depending on I. By [25, Proposition 2.2(i)], for a semiprime ring the definition of generalized right p.q.-Baer coincides with that of right p.q.-Baer; and for a ring R with *IFP* (i.e., $ab = 0$ implies $arb = 0$, for each $a, b, r \in R$) the definition of generalized left p.q.-Baer coincides with that of generalized left PP (i.e., rings for any $x \in R$ the left annihilator of x^n is generated by an idempotent for some positive integer n, depending on x). In [25], the authors characterize the generalized right (principally) quasi-Baer property of triangular matrix rings, 2-by-2 generalized triangular matrix rings and full matrix rings. They also prove that, for abelian rings, unlike the Baer or generalized PP condition, the generalized right (principally) quasi-Baer condition is a Morita invariant property.

Recall from [15,17] and [18], that a ring R is *(principally) right primary* if whenever A and B are (principal) ideals of R with $AB = 0$, then $A = 0$ or $B^n = 0$ for some positive integer n. From [11], every prime ring is a right and left primary ring. Moreover (a direct sum of nilpotent rings) a nilpotent ring is (principally) left primary and (principally) right primary. By [11, Proposition 3.6 (i)], a ring R is (principally) left primary if and only if R is semicentral reduced generalized right (principally) quasi-Baer. Since principally right primary rings are indecomposable as rings (in fact, semicentral reduced), we can consider them for components in a decomposition theory for rings. For a prime ideal P of a ring R, the following definition from [4] plays a key role in the sequel:

$$(i)\ O(P) = \{a \in R \mid aRs = 0 \ \text{for some}\ s \in R \setminus P\};$$

$$(ii)\ \overline{O}(P) = \{x \in R \mid x^n \in O(P) \ \text{for some}\ n \in \mathbb{N}\}.$$

Several authors, including [7] and [25] have obtained sheaf representations of rings whose stalks are of the form $R/O(P)$. In this paper, we explore these connections via the sets $O(P)$ and $\overline{O}(P)$. We generalize a result of Kist [22] for commutative PP-rings to generalized principally quasi-Baer rings without using topological arguments, by determining criteria for which every prime ideal contains a unique minimal prime (equivalently, the prime radical of $R/O(P)$ is a prime ideal) in a generalized principally quasi-Baer ring. We deduce that in a local ring with nilpotent Jacobson radical, every prime ideal contains a unique minimal prime. Local rings with nilpotent maximal ideal are often called *completely primary* rings [23]. Moreover, for a generalized principally quasi-Baer ring, we develop conditions for which the various ideals $O(P)$ or P are either essential in R or split-off as direct summands.

Various decompositions of generalized p.q.-Baer rings are determined. In particular, we show that a generalized right quasi-Baer ring R with $\mathcal{S}_\ell(R) = \mathbf{B}(R)$ has a ring decomposition, $R = A \oplus B$, where $Soc(A)$ is left essential in A and $Soc(R) = (Soc(R))^n \oplus Soc(B)$ for some positive integer n.

Furthermore, we show that the prime radical is the unique minimal prime ideal of a class of principally right primary rings, and to illustrate our results, various examples of (principally) right (or left) primary rings which are not quasi-Baer are provided.

2. MINIMAL PRIME IDEALS IN GENERALIZED P.Q.-BAER RINGS

Recall from [5] that an idempotent $e \in R$ is called *left (resp. right) semicentral* if $xe = exe$ (resp. $ex = exe$), for all $x \in R$. The set of left (resp. right) semicentral idempotents of R is denoted by $\mathcal{S}_\ell(R)$ (resp. $\mathcal{S}_r(R)$). A ring R with unity is said to be *semicentral reduced* if $\mathcal{S}_\ell(R) = \{0, 1\}$. Since e is left semicentral if and only if $1 - e$ is right semicentral, being semicentral reduced is left-right symmetric. Thus R is semicentral reduced if and only if $\mathcal{S}_r(R) = \{0, 1\}$. Observe $\mathcal{S}_\ell(R) \cap \mathcal{S}_r(R) = \mathbf{B}(R)$, the set of central idempotents of R, and if R is semiprime or abelian (idempotents are central), then $\mathcal{S}_\ell(R) = \mathcal{S}_r(R) = \mathbf{B}(R)$.

In this section, we show that the prime radical is the unique minimal prime ideal of a class of principally right primary rings. We also give results which enable us to generate examples of (principally) right primary rings which are not (right p.q.-Baer) quasi-Baer.

For notation, let $\mathbf{P}(R)$ and $nil(R)$ denote the prime radical and the set of nilpotent elements of R, respectively. A ring is called 2-*primal* if $\mathbf{P}(R) = nil(R)$. For more details on 2-primal rings see [4] and [3]. Given a ring R, for a nonempty subset X of R, $r_R(X)$ and $\ell_R(X)$ denote the right and left annihilators of X in R, respectively. Also $Z_\ell(R)$ and $Z_r(R)$ denote the left and right singular ideals of R, respectively. The left socle of R will be symbolized by $Soc(R)$. Observe that if R is semiprime, then $Soc(R)$ coincides with the right socle of R. Also $C(R)$ denotes the center of R.

The notions $O(P)$, $\overline{O}(P)$ and $N(P)$ from [4] and [9] are fundamental to the remainder of our discussion.

Definition 2.1. For a prime ideal P of a ring R, denote:

$$N(P) = \{y \in R \mid yRs \subseteq \mathbf{P}(R) \text{ for some } s \in R \setminus P\}.$$

Observe that $O(P) = \bigcup_{s \in R \setminus P} \ell_R(Rs) = \{a \in R \mid r_R(aR) \nsubseteq P\} \subseteq N(P) \subseteq P$. Since $nil(R/O(P)) = \overline{O}(P)/O(P)$ is the set of all nilpotent elements in the ring $R/O(P)$, the condition that $\overline{O}(P) = P$ gives us important information about the ring $R/O(P)$.

LEMMA 2.2. *Let R be a generalized left p.q.-Baer ring and P a prime ideal of R. Then we have:*

$$O(P) = \sum_{e \in P \cap \mathcal{S}_r(R)} Re.$$

Proof. The proof is similar to that of [25, Proposition 5.3 (i)]. □

Definition 2.3 ([5]). A ring R is said to have a set of left (resp. right) *triangulating idempotents* if there exists an ordered set $\{b_1, b_2, \ldots, b_n\}$ of nonzero distinct idempotents such that:

(*i*) $1 = b_1 + b_2 + \cdots + b_n$;

(*ii*) $b_1 \in \mathcal{S}_\ell(R)$ (resp. $b_1 \in \mathcal{S}_r(R)$);

(*iii*) $b_{k+1} \in \mathcal{S}_\ell(c_k R c_k)$ (resp. $b_{k+1} \in \mathcal{S}_r(c_k R c_k)$) where $c_k = 1 - (b_1 + b_2 + \cdots + b_k)$ for $1 \leq k \leq n-1$.

A set of right triangulating idempotents of R is defined similarly, using (i), $b_1 \in \mathcal{S}_r(R)$, and $b_{k+1} \in \mathcal{S}_r(c_k R c_k)$. From part (iii) of the above definition, a set of left (right) triangulating idempotents is a set of pairwise orthogonal idempotents. A set $\{b_1, b_2, \ldots, b_n\}$ of left (right) triangulating idempotents is said to be complete if each $b_i R b_i$ is semicentral reduced. From [5, Proposition 1.3], R is isomorphic to a generalized upper triangular matrix ring with semicentral reduced rings on the main diagonal if and only if R has a *complete* set of left triangulating idempotents.

Observe from [5, Corollary 1.7 and Theorem 2.10] that the number of elements in a complete set of left triangulating idempotents is unique for a given ring R (which has such a set) and this is also the number of elements in any complete set of right triangulating idempotents of R. This motivates the following definition: R has triangulating dimension n, written $Tdim(R) = n$, if R has a complete set of left triangulating idempotents with exactly n elements.

Note that R is semicentral reduced if and only if $Tdim(R) = 1$. If R has no complete set of left triangulating idempotents, then we say R has infinite *triangulating dimension*, denoted $Tdim(R) = \infty$. From [5, Proposition 2.14], if R has ACC on ideals, then $Tdim(R) < \infty$.

Lemma 2.4 ([5, Lemma 2.8, Theorems 2.9 and 2.10]). *The following are equivalent:*

(i) R has a complete set of left triangulating idempotents;

(ii) $\{bR \mid b \in \mathcal{S}_\ell(R)\}$ is a finite set;

(iii) R has a complete set of right triangulating idempotents.

Furthermore; if $\{b_1, b_2, \ldots, b_n\}$ and $\{c_1, c_2, \ldots, c_k\}$ are complete sets of left triangulating idempotents; then $k = n$. Also for $b \in \mathcal{S}_\ell(R)$; $bR = \sum\limits_{i\in\Lambda} b_iR$ for a subset $\Lambda \subseteq \{1, 2, \ldots, n\}$

Proposition 2.5. *Let R be a generalized left p.q.-Baer ring with a complete set of left triangulating idempotents. If P is a prime ideal, then $O(P) = Re$ for a right semicentral idempotent e of R.*

Proof. By Lemma 2.2 and Lemma 2.4, $O(P) = Re_1 + Re_2 + \cdots + Re_k$, where $\{e_1, e_2, \ldots, e_k\}$ is a finite subset of $\mathcal{S}_r(R)$. Now, in this case, observe that $e_1 + e_2 - e_1e_2 \in \mathcal{S}_r(R)$ and $Re_1 + Re_2 = R(e_1 + e_2 - e_1e_2)$. By iterating this procedure, there is a right semicentral idempotent e such that $O(P) = Re_1 + \cdots + Re_k = Re$. $\square$

Corollary 2.6. *If P is a prime ideal of a principally right primary ring R, then $O(P) = 0$.*

Proof. The result follows, from [11, Proposition 3.6 (i)], and Proposition 2.5, since a ring R is (principally) right primary if and only if R is a semicentral reduced generalized left (principally) quasi-Baer. $\square$

By [22, Lemma 3.1], in a commutative ring, a minimal prime ideal has been characterized as a prime ideal P such that $\overline{O}(P) = P$. Birkenmeier, Kim and Park, in [4], derived various conditions which ensure that a prime ideal $P = \overline{O}(P)$. Observe that in [22, Lemma 3.1], Kist showed that $\overline{O}(P) = P$, for every minimal prime ideal P of any commutative ring.

By [4] a ring R satisfies *condition* $(CZ2)$ if, whenever $(xy)^n = 0$, for some positive integer n, then $x^mRy^m = 0$, for some positive integer m.

THEOREM 2.7. *Let R be a 2-primal principally right primary ring. If R satisfies condition $(CZ2)$, then the prime radical $\boldsymbol{P}(R)$ is the unique minimal prime ideal of R.*

Proof. Since R is 2-primal and satisfies condition $(CZ2)$, for every minimal prime ideal P of R, we have $\overline{O}(P) = P$, by [4, Theorem 2.3 (ii)]. Hence the definition of $\overline{O}(P)$ and Corollary 2.6 imply that every minimal prime ideal of R is nil. Since R is 2-primal, the prime radical $P(R)$ is the unique minimal prime ideal of R. $\square$

Let R be a ring and α denotes an endomorphism of R with $\alpha(1) = 1$. In [13] the authors defined a skew triangular n-by-n matrix ring as the set of all triangular matrices over R with point-wise addition and a new multiplication subject to the condition $E_{ij}r = \alpha^{j-i}(r)E_{ij}$. So $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = a_{ii}b_{ij} + a_{i,i+1}\alpha(b_{i+1,j}) + \ldots + a_{ij}\alpha^{j-i}(b_{jj})$, for each $i \leq j$ and denoted it by $T_n(R, \alpha)$.

The subring of $T_n(R, \alpha)$ with constant main diagonal is denoted by $S(R, n, \alpha)$ (see [32], for more details). Also, the subring of the skew triangular matrices with constant diagonals is denoted by $T(R, n, \alpha)$ (see [13]). We can denote $A = (a_{ij}) \in T(R, n, \alpha)$ by $(a_0, \ldots, a_{n-1})$. Then $T(R, n, \alpha)$ is a ring with addition pointwise and multiplication given by:

$$(a_0, \ldots, a_{n-1})(b_0, \ldots, b_{n-1}) = (a_0b_0, a_0{*}b_1{+}a_1{*}b_0, \ldots, a_0{*}b_{n-1}{+}\cdots{+}a_{n-1}{*}b_0),$$

with $a_i * b_j = a_i\alpha^i(b_j)$, for each i and j. On the other hand, there is a ring isomorphism $\varphi : R[x; \alpha]/(x^n) \to T(R, n, \alpha)$, given by $\varphi(\sum_{i=0}^{n-1} a_ix^i) = (a_0, a_1, \ldots, a_{n-1})$, with $a_i \in R$, $0 \leq i \leq n-1$. So $T(R, n, \alpha) \cong R[x; \alpha]/(x^n)$, where $R[x; \alpha]$ is the skew polynomial ring with multiplication subject to the condition $xr = \alpha(r)x$ for each $r \in R$, and (x^n) is the ideal generated by x^n.

According to Krempa [21], an endomorphism α of a ring R is said to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$, for $a \in R$. A ring R is said to be α-*rigid* if there exists a rigid endomorphism α of R. In [19], the authors introduced α-compatible rings and studied their properties. A ring R is α-*compatible* if for each $a, b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$. Basic properties of rigid and compatible endomorphisms are proved by Hashemi and the second author in [19, Lemma 2.2 and 2.1].

According to [26], a ring R with an automorphism α is called α-*weakly rigid* if for each $a, b \in R$, $aRb = 0$ if and only if $aR\alpha(b) = 0$. Any compatible (and hence rigid) automorphism is weakly rigid. For any positive integer n, a ring R with an automorphism α is α-weakly rigid if and only if, the n-by-n upper triangular matrix ring $T_n(R)$ is $\overline{\alpha}$-weakly rigid if and only if, the matrix ring $M_n(R)$ is $\overline{\alpha}$-weakly rigid, where $\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))$ for each $(a_{ij}) \in M_n(R)$. Also if R is a semiprime α-weakly rigid ring, then the ring of polynomials $R[X]$, for X a nonempty set of commuting indeterminates, is a semiprime $\overline{\alpha}$-weakly rigid ring, where $\overline{\alpha}(\sum\limits_{i=0}^{n} r_i x^i) = \sum\limits_{i=0}^{n} \alpha(r_i)x^i$. For every prime ring R and any automorphism α, the rings $M_n(R)$, $T_n(R)$, $R[X]$ and the power series ring $R[[X]]$ are $\overline{\alpha}$-weakly rigid rings.

Lemma 2.8. *Let R be an α-weakly rigid ring. Then*

(i) $\mathcal{S}_r(R) = \boldsymbol{B}(R)$ if and only if $\mathcal{S}_r(S(R, n, \alpha)) = \boldsymbol{B}(S(R, n, \alpha))$;

(ii) $\mathcal{S}_r(R) = \boldsymbol{B}(R)$ if and only if $\mathcal{S}_r(T(R, n, \alpha)) = \boldsymbol{B}(T(R, n, \alpha))$.

In particular, every central idempotent in $S(R, n, \alpha)$ (resp., $T(R, n, \alpha)$) is of the form eI_n, where $e^2 = e \in \boldsymbol{B}(R)$ and I_n is the identity matrix, for some positive integer $n \geq 2$.

Proof. We only prove (i), because the proof of the other case is similar. First, assume that $\mathcal{S}_r(R) = \mathbf{B}(R)$, we show that $\mathcal{S}_r(S(R, n, \alpha)) = \mathbf{B}(S(R, n, \alpha))$. It is sufficient to show that $\mathcal{S}_r(S(R, n, \alpha)) \subseteq \mathbf{B}(S(R, n, \alpha))$. We proceed by induction on n. Assume $E = \begin{pmatrix} e & a \\ 0 & e \end{pmatrix} \in \mathcal{S}_r(S(R, 2, \alpha))$. Then $ES(R, 2, \alpha)(1 - E) = 0$. Hence we obtain the following:

$$eR(1 - e) = 0 \tag{2.1}$$

$$eRa = a\alpha(R(1 - e)). \tag{2.2}$$

From Eq. (2.1), it implies that $e \in \mathcal{S}_r(R)$. Hence e is a central element. Also, since R is an α-weakly rigid ring, from Eq. (2.1), we obtain

$$e\alpha(R(1 - e)) = 0. \tag{2.3}$$

By multiplying Eq. (2.2) by e, from the left, we have $eRa = ea\alpha(R(1 - e))$. Now, from Eq. (2.3) and the fact that e is central, we obtain $eRa = 0$. Hence $a\alpha(R(1 - e)) = 0$. Now the weakly rigidness of R and the fact that e is central implies that $a = 0$. Therefore $\mathcal{S}_r(S(R, 2, \alpha)) \subseteq \mathbf{B}(S(R, 2, \alpha))$.

Now assume that $E = \begin{pmatrix} e & a_{12} & \cdots & a_{1n} \\ 0 & e & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e \end{pmatrix} \in \mathcal{S}_r(S(R, n, \alpha))$ and that I_1 is a matrix obtained by deleting the nth row and the nth column of E and I_2 is a matrix obtained by deleting the first row and the first column of E in the ring $S(R, n-1, \alpha)$. Then $I_1, I_2 \in \mathcal{S}_r(S(R, n-1, \alpha))$. So the a_{ij}'s entries are zero and e is central, by the induction hypothesis. Then we have

$$eRa_{1n} = a_{1n}\alpha^{n-1}(R(1 - e)). \tag{2.4}$$

Multiplying Eq.(2.4) by e from the left and using again the weakly rigidness of R, we obtain $eRa_{1n} = 0$. Now Eq.(2.4) implies that $a_{1n} = 0$. Therefore, $\mathcal{S}_r(S(R,n,\alpha)) = \mathbf{B}(S(R,n,\alpha))$. Conversely, let $e \in \mathcal{S}_r(R)$. Then $eR(1-e) = 0$. So $(eI_n)S(R,n,\alpha)((1-e)I_n) = 0$. Thus $eI_n \in \mathcal{S}_r(S(R,n,\alpha)) = \mathbf{B}(S(R,n,\alpha))$, and hence $e \in \mathbf{B}(R)$, and the result follows. $\square$

COROLLARY 2.9. *Let R be an α-weakly rigid ring. Then R is semicentral reduced if and only if the ring $S(R,n,\alpha)$ (resp., $T(R,n,\alpha)$) is semicentral reduced, for some positive integer $n \geq 2$.*

THEOREM 2.10. *Let R be an α-weakly rigid ring with $\mathcal{S}_r(R) = \boldsymbol{B}(R)$. Then R is generalized left (resp., principally) quasi-Baer if and only if the ring $S(R,n,\alpha)$ (resp., $T(R,n,\alpha)$) is generalized left (resp., principally) quasi-Baer, for some positive integer $n \geq 2$.*

Proof. We proceed by induction on n. First, we claim that $S(R,2,\alpha)$ is generalized left p.q.-Baer. Assume $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ is an element in $S(R,2,\alpha)$. Since R is generalized left p.q.-Baer, there is a positive integer m and a central idempotent e in $\mathcal{S}_r(R)$ such that $\ell_R(Ra)^m = Re$. So, by [25, Lemma 2.6], $\ell_R(Ra)^m = \ell_R(Ra)^{2m}$. We show that $\ell_{S(R,2,\alpha)}(S(R,2,\alpha)A)^{2m} = S(R,2,\alpha)eI_2$. Let

$$B = \begin{pmatrix} x & x_{12} \\ 0 & x \end{pmatrix} \in \ell_{S(R,2,\alpha)}(S(R,2,\alpha)A)^{2m}$$

and

assume that $X = \begin{pmatrix} r_1a\cdots r_{2m}a & z \\ 0 & r_1a\cdots r_{2m}a \end{pmatrix}$ is an arbitrary element in $(S(R,2,\alpha)A)^{2m}$ such that $\begin{pmatrix} r_i & z_i \\ 0 & r_i \end{pmatrix} \in S(R,2,\alpha)$, and z has $2m$ terms, where one of them is $r_ib+z_i\alpha(a)$ and the others in the form $\alpha^j(r_ia)$. We have $BX = 0$, so

$$xr_1ar_2a\cdots r_{2m}a = 0 \tag{2.5}$$

and

$$xz + x_{12}\alpha(r_1ar_2a\cdots r_{2m}a) = 0. \tag{2.6}$$

Thus $x \in \ell_R(Ra)^{2m} = \ell_R(Ra)^m = Re$. Hence $x = xe$. Now by replacing x with xe in Eq. (2.6), and using the weakly rigidness of R and the fact that e is central and that $e(Ra)^m = 0$, we get

$$x_{12}\alpha(r_1ar_2a\cdots r_{2m}a) = 0.$$

Thus we have $x_{12}\alpha(Ra)^{2m} = 0$. Now using again the weakly rigidness of R it implies that $x_{12}(Ra)^{2m} = 0$. So $x, x_{12} \in \ell_R(Ra)^m = Re$, and hence $x = xe$ and $x_{12} = x_{12}e$. Then $B = BeI_2$. Therefore $\ell_{S(R,2,\alpha)}(S(R,2,\alpha)A)^{2m} \subseteq S(R,2,\alpha)eI_2$. Since e is a central element and $e \in \ell_R(Ra)^m$, we deduce that $\ell_{S(R,2,\alpha)}(S(R,2,\alpha)A)^{2m} = S(R,2,\alpha)eI_2$, and if $B \in \ell_{S(R,2,\alpha)}(S(R,2,\alpha)A)^{2m}$, then all entries of B are in $\ell_R(Ra)^m$.

Now suppose that $A = \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in S(R,n,\alpha)$, and that I_1 is a matrix obtained by deleting the nth row and the nth column of A and I_2 is a matrix obtained by deleting the first row and the first column of A in the ring $S(R,n-1,\alpha)$. Since R is also generalized left p.q.-Baer, there is a positive integer m and $e \in \mathcal{S}_r(R)$ such that $\ell_R(Ra)^m = Re$. Thus, by [25, Lemma 2.6], $\ell_R(Ra)^m = \ell_R(Ra)^{mn}$. Let

$$B = \begin{pmatrix} x & x_{12} & \cdots & x_{1n} \\ 0 & x & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix}$$

be an element in $\ell_{S(R,n,\alpha)}(S(R,n,\alpha)A)^{mn}$, B_1 is a matrix obtained by deleting the nth row and the nth column of B, B_2 is a matrix obtained by deleting the first row and the first column of B, and

$$X = \begin{pmatrix} r_1a\cdots r_{nm}a & y_{12} & \cdots & y_{1n} \\ 0 & r_1a\cdots r_{nm}a & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_1a\cdots r_{nm}a \end{pmatrix} \in (S(R,n,\alpha)A)^{mn}.$$

Then by the induction hypothesis and Lemma 2.8, for $i = 1, 2$, there exists $e_i^2 = e_i$ in $S(R, n-1, \alpha)$ such that $e_i = f_iI_{n-1}$, $f_i^2 = f_i \in R$,

$$\ell_{S(R,n-1,\alpha)}(S(R,n-1,\alpha)I_i)^{(n-1)m} = e_iS(R,n-1,\alpha)$$

and $B_i \in \ell_{S(R,n-1,\alpha)}(S(R,n-1,\alpha)I_i)^{(n-1)m}$, for each i. So by the hypothesis, all the entries of B_i are in $\ell_R(Ra)^m$. Thus all x_{ij}'s except x_{1n} are in $\ell_R(Ra)^{(n-1)m}$. Now we have:

$$xy_{1n} + x_{12}\alpha(y_{2n}) + \cdots + x_{1n}\alpha^{n-1}(r_1ar_2a\cdots r_{nm}a) = 0. \tag{2.7}$$

Since x_{ij}'s except x_{1n} are in $\ell_R(Ra)^m = Re$, so $x_{1i} = x_{1i}e$ for each $1 \leq i \leq n-1$. Now by replacing x_{1i} with $x_{1i}e$ in Eq. (2.7), and using the fact that

e is central and using weakly rigidness of R and that $e(Ra)^m = 0$, we get $xy_{1n} + x_{1n}\alpha^{n-1}(r_1ar_2a\cdots r_{nm}a) = 0$. But $x \in \ell_R(Ra)^m$, so $xy_{1n} = 0$. Thus using again weakly rigidness of R it implies that $x_{1n} \in \ell_R(Ra)^m = \ell_R(Ra)^{mn}$. Hence all the entries of B are in $\ell_R(Ra)^m$. So $x_{ij} = x_{ij}e$, for each i, j. Then $B = BeI_n$. Therefore $\ell_{S(R,n,\alpha)}(S(R,n,\alpha)A)^{nm} \subseteq S(R,n,\alpha)eI_n$. Since e is a central element and $e \in \ell_R(Ra)^m$, we conclude that $\ell_{S(R,n,\alpha)}(S(R,n,\alpha)A)^{nm} = S(R,n,\alpha)eI_n$. Therefore $S(R,n,\alpha)$ is a generalized left p.q.-Baer ring.

Conversely, assume that $S(R,n,\alpha)$ is generalized left p.q.-Baer, we show that R is also generalized left p.q.-Baer. Let $a \in R$ and $A = aI_n$. Since $S(R,n,\alpha)$ is generalized left p.q.-Baer, $\ell_{S(R,n,\alpha)}(S(R,n,\alpha)A)^k = S(R,n,\alpha)E$ for some positive integer k and $E^2 = E = eI_n$, where $e^2 = e \in R$, by Lemma 2.8. Hence for any $r_i \in R$, where $1 \le i \le k$, we have $e(r_1a)(r_2a)\cdots(r_ka)I_n = 0$, since $E(S(R,n,\alpha)A)^k = 0$. It follows that $Re \subseteq \ell_R(Ra)^k$. Conversely if $b \in \ell_R(Ra)^k$, then we have $b(r_1a)(r_2a)\cdots(r_ka) = 0$, for any $r_i \in R$. Thus $bI_n \in \ell_{S(R,n,\alpha)}(S(R,n,\alpha)A)^k = S(R,n,\alpha)E$. It follows that $b \in eR$. Therefore $\ell_R(Ra)^k = Re$, and R is generalized left p.q.-Baer, and the proof is complete. $\square$

Now we give results which enable us to generate examples of (principally) right primary rings which are not (right p.q.-Baer) quasi-Baer.

THEOREM 2.11. *Let R be an α-weakly rigid ring. Then R is (resp., principally) right primary if and only if $S(R,n,\alpha)$ is a (resp., principally) right primary if and only if $R[x;\alpha]/\langle x^n\rangle$ is a (resp., principally) right primary, for some positive integer $n \ge 2$.*

Proof. The result follows from Corollary 2.9 and Theorem 2.10. $\square$

COROLLARY 2.12. *A ring R is (resp., principally) right primary if and only if $S(R,n)$ is a (resp., principally) right primary if and only if $R[x]/\langle x^n\rangle$ is a (resp., principally) right primary, for some positive integer $n \ge 2$.*

The next example allows us to construct numerous examples of generalized left p.q.-Baer rings which are not principally right primary.

Example 2.13. Let R be a nonprime left p.q.-Baer with $\mathcal{S}_r(R) = \mathbf{B}(R)$ (*e.g.*, the ring in [8, Example 1.5 (i)]). For each $n \ge 2$, the rings $S(R,n)$ and $R[x]/\langle x^n\rangle$ are generalized left p.q.-Baer, by Theorem 2.10, but they are not principally right primary, since R is not semicentral reduced.

Using the above results we can give the following examples to illustrate Theorem 2.7.

Example 2.14. Let D be a domain with an automorphism α. Then the rings $S(D,n,\alpha)$ and $D[x;\alpha]/\langle x^n\rangle$ are principally right primary, by Theorem 2.11. On the other hand, [4, Theorem 2.9] and [28, Theorem 2.12], yields that $S(D,n,\alpha)$ and $D[x;\alpha]/\langle x^n\rangle$ are 2-primal and it is straightforward to show that $S(D,n,\alpha)$ and $D[x;\alpha]/\langle x^n\rangle$ satisfy condition $(CZ2)$. Hence by Theorem 2.7, the prime radical is the unique minimal prime ideal of $S(D,n,\alpha)$ and $D[x;\alpha]/\langle x^n\rangle$.

For a ring R and an (R,R)-bimodule M, let $T(R,M)=\{(a,x)|a\in R, x\in M\}$ with the multiplication defined by $(a_1,x_1)(a_2,x_2)=(a_1a_2,a_1x_2+x_1a_2)$. Then $T(R,M)$ is a ring which is called the *trivial extension (also called the split-null extension)* of R by M. Notice that $T(R,M)$ is isomorphism to the ring of matrices $\begin{pmatrix} a & x \\ 0 & a \end{pmatrix}$, where $a\in R, x\in M$ and the usual matrix operations are used.

Example 2.15. Let D be a commutative domain and $M=\bigoplus_{i\in I} D_i$ for a nonempty index set I, where $D_i=D$ for each i. It is easy to show that $R:=T(D,M)$ is a principally primary ring which is not p.q.-Baer. Since R is commutative, by Theorem 2.7, $\mathbf{P}(R)$ is the unique minimal prime ideal of $T(D,M)$. In particular, by [29, Theorem 2.2, Theorem 3.15 and Theorem 4.13(3)] the polynomial ring $R[x]$, the power series ring $R[[x]]$, the Laurent polynomial ring $R[x;x^{-1}]$, the Laurent series ring $R[[x;x^{-1}]]$ and the monoid ring $R[S]$, for every commutative, torsion-free, and cancellative monoid S, are commutative principally primary rings. Therefore, the prime radical for each of these rings is the unique minimal ideal prime.

Example 2.16. Assume that R is a 2-primal principally right primary ring such that it satisfies condition $(CZ2)$. From [3, Proposition 2.2], subrings of 2-primal rings are 2-primal. Since the condition $(CZ2)$ is inherited by subrings, we see that if $e\in R$ is an idempotent, so the subring eRe of R is 2-primal and satisfies condition $(CZ2)$. On the other hand by [11, Proposition 3.4], the ring eRe is principally right primary. Hence, by Theorem 2.7, $\mathbf{P}(eRe)$ is the unique minimal prime ideal of eRe.

Proposition 2.17. *Let R be a principally right primary ring. Let P be a nonzero prime ideal.*

(1) *If X is a nonzero right ideal of R, then $r_R(X)\subseteq P$.*
(2) *P is left essential in R.*
(3) *Let K be a nonzero left ideal of R such that $\ell_R(K)\neq 0$. Then:*
  (i) *$K\subseteq P$;*
  (ii) *if $\ell_R(\ell_R(K))\neq 0$, then $\ell_R(K)\subseteq P$ and P is right essential in R;*
  (iii) *if $\ell_R(\ell_R(K))=0$, then $K[\ell_R(K)]$ is a nonzero nilpotent ideal of R.*

(4) *If P is a minimal prime ideal and Q is a prime ideal of R such that $Q \neq P$, then $l_R(Q) = 0$. Hence Q is right essential in R.*

Proof. The result follows from Corollary 2.6 and [7, Lemma 3.3]. □

PROPOSITION 2.18. *Let R be a principally right primary ring and P be a nonzero prime ideal. If K is a nonzero principal left ideal of R, then exactly one of the following conditions is satisfied:*

(1) *$\ell_R(K) = 0$; or*

(2) *$K \subseteq P$ and K is a nilpotent ideal of R.*

Proof. Assume that $\ell_R(K) \neq 0$. By Proposition 2.17(3), we have $K \subseteq P$. On the other hand, by [11, Proposition 3.6 (i)], R is a semicentral reduced generalized left p.q.-Baer. Hence $\ell_R(K^n) = 0$ or $\ell_R(K^n) = R$, for some positive integer n. Thus K is a nilpotent ideal of R, since $\ell_R(K) \neq 0$. □

Remark 2.19. For any domain D, the rings $T(D, n)$ and $S(D, n)$ are not quasi-Baer but they are 2-primal (see *e.g.*, [27]) and principally right primary, by Corollary 2.12.

By the following result, nonzero prime ideals of a principally right primary ring R, are left essential in R.

PROPOSITION 2.20. *Let R be a principally right primary ring. Then:*

(1) *If P is a nonzero prime ideal of R, then P is left essential in R.*

(2) *If R is 2-primal and P is a minimal prime ideal of R, then the prime radical $\boldsymbol{P}(R)$ is right essential in P.*

Proof. (1) Since R is right primary, by Corollary 2.6, $O(P) = 0$. Now, assume that P is not left essential in R. Then there exists a nonzero left ideal I such that $P \cap I = 0$. Hence $PI = 0$. So $P \subseteq O(P)$, a contradiction.

(2) This is a consequence of Corollary 2.6 and [4, Proposition 3.1(ii)]. □

PROPOSITION 2.21. *Let R be a generalized left p.q.-Baer ring with a complete set of triangulating idempotents and P be a prime ideal. Then $R/O(P)$ is a generalized left p.q.-Baer ring.*

Proof. By Proposition 2.5, $O(P) = Re$ for some right semicentral idempotent e. Since $eR(1-e) = 0$, $R/O(P) \simeq (1-e)R(1-e)$. Hence $R/O(P)$ is generalized left p.q.-Baer by [25, Theorem 4.7]. □

Remark 2.22. For a prime ideal P of a ring R, let

$$S_P = \{e \in R \mid e \text{ is a left semicentral idempotent such that } e \notin P\}.$$

By [7, Proposition 2.4], S_P is a right denominator set. By [37, Proposition 1.4], $R[S_P^{-1}]$ is a right ring of fractions of R. In [25, Theorem 5.6], the authors proved that, when R is a generalized right quasi-Baer ring, $O(P) \neq 0$ for every minimal prime ideal P. Then R has a nontrivial representation as a subdirect product of the right ring of fractions $R[S_P^{-1}]$, where P ranges through all minimal prime ideals.

3. PRIME IDEALS CONTAINING A UNIQUE MINIMAL PRIME IDEAL

In [9], Birkenmeier, Kim and Park showed that if $\overline{O}(P) = P$ for every minimal prime ideal P of a right p.q.-Baer ring R, then every prime ideal of R contains a unique minimal prime ideal. In this section, we investigate the condition, every prime ideal contains a unique minimal prime ideal, in a generalized left p.q.-Baer ring. We derive some conditions which ensure that a prime ideal $P = \overline{O}(P)$ in the class of generalized left p.q.-Baer rings. There exists a large class of generalized left p.q.-Baer rings which are neither p.q.-Baer nor PP, but for every minimal prime ideal P of these rings, $\overline{O}(P) = P$.

Lemma 3.1. *Let P and Q be prime ideals of a generalized left p.q.-Baer ring R, such that $Q \subseteq P$. Then $O(P) = O(Q)$.*

Proof. By definition it is clear that $O(P) \subseteq O(Q)$ and by Lemma 2.2, $O(Q) \subseteq O(P)$. □

By an adaptation of [9, Theorem 1.4], in the following Theorem 3.2, part (2) generalizes a result of Kist [22] for commutative PP rings and avoids the topological argument of his proof. Observe that in [22] Kist showed that $\overline{O}(P) = P$ for each minimal prime ideal P in the case of commutative rings.

Theorem 3.2. *Let R be a generalized left p.q.-Baer ring such that $\overline{O}(P) = P$ for every minimal prime ideal P of R. Then:*

(1) *R is 2-primal.*

(2) *Every prime ideal of R contains a unique minimal prime ideal.*

(3) *For every prime ideal P of R, the ring $R/O(P)$ is 2-primal and $\boldsymbol{P}(R/O(P))$ is a completely prime ideal.*

(4) *For a prime ideal P of R, every prime ideal of $R/O(P)$ which properly contains $\boldsymbol{P}(R/O(P))$ is right essential in $R/O(P)$.*

Proof. (1) It follows by [4, Lemma 2.1(i), (ii)].

(2) Let Q be a prime ideal of R which contains minimal prime ideals P_1 and P_2. By Lemma 3.1, $O(P_1) = O(Q)$ and $O(P_2) = O(Q)$. So $\overline{O}(P_1) = \overline{O}(P_2)$. By the assumption $P_1 = \overline{O}(P_1) = \overline{O}(P_2) = P_2$.

(3) The proof is similar to that of [9, Theorem 1.4(iii)] with use of Lemma 2.2 and Lemma 3.1.

(4) This part follows from [4, Lemma 3.4]. □

The following result gives an explicit description of the unique minimal prime ideal of generalized left p.q.-Baer rings.

Corollary 3.3. *Let R be a 2-primal generalized left p.q.-Baer ring which satisfies in condition $(CZ2)$ and P is a prime ideal of R. Then $\overline{O}(P)$ is the unique minimal prime ideal of R such that $\overline{O}(P) \subseteq P$ and $\overline{O}(P)$ is completely prime.*

Proof. Since R is 2-primal and satisfies $(CZ2)$ condition, by [4, Theorem 2.3 (ii)] for every minimal prime ideal P in R, $\overline{O}(P) = P$. Therefore by Theorem 3.2, for every prime ideal P of R there exists a unique minimal prime ideal $Q \subseteq P$. So by Lemma 3.1, $O(P) = O(Q)$ and hence $\overline{O}(P) = \overline{O}(Q)$. Thus $\overline{O}(Q) = Q$ implies that $\overline{O}(P) = Q$. Therefore $\overline{O}(P)$ is the unique minimal prime ideal of R such that $\overline{O}(P) \subseteq P$. Moreover, since $\overline{O}(P) = \overline{O}(Q) = Q$ is an ideal of R, by [4, Proposition 1.1 (iii)] $\overline{O}(P)$ is completely prime and the proof is complete. $\square$

Recall from [1] that a ring R satisfies *the IFP* (*insertion of factors property*) if and only if $r_R(x)$ is an ideal of R for all $x \in R$. Note that by [8, Proposition 1.14], every right p.q.-Baer ring R with IFP is reduced p.q.-Baer, and hence it is 2-primal and satisfies condition $(CZ2)$. Furthermore for every prime ideal P of a reduced ring R, we have $O(P) = \overline{O}(P)$. So we have the following.

Corollary 3.4. *[9, Corollary 1.7] Let R be a right p.q.-Baer ring with IFP and P a prime ideal of R. Then $O(P)$ is the unique minimal prime ideal of R contained in P and $O(P)$ is a completely prime ideal.*

If M_R is an indecomposable R-module of finite length, then $End(M_R)$ is a local ring, and its unique maximal ideal is nilpotent. Local rings with nilpotent maximal ideal are often called *completely primary* rings [23]. For instance, group algebras of finite p-groups over fields of characteristic p are another class of such completely primary rings.

Corollary 3.5. *Let R be a completely primary ring. If P is a prime ideal of R, then $\overline{O}(P)$ is the unique minimal prime ideal of R contained in P and $\overline{O}(P)$ is a completely prime ideal.*

Proof. We first show that R is generalized left p.q.-Baer. For every $a \in R$, if $Ra \neq R$ then $(Ra)^n = 0$ for some $n \in \mathbb{N}$. So $\ell_R(Ra)^n = R$ and if $Ra = R$ then $\ell_R(Ra) = 0$. So R is generalized left p.q.-Baer. Now it follows immediately from [4, Corollary 2.7(i)] and Corollary 3.3. $\square$

The following example shows that there exists a large class of generalized p.q.-Baer rings which are neither p.q.-Baer nor PP, but $\overline{O}(P) = P$ for every minimal prime ideal P of R. Hence by Corollary 3.3, every prime ideal of R contains a unique minimal prime ideal.

Recall that in [22, Lemma 3.1], Kist showed that $\overline{O}(P) = P$, for every minimal prime ideal P of any commutative ring.

Example 3.6. (i) Every commutative generalized p.q.-Baer ring is 2-primal and satisfies condition $(CZ2)$. So by Corollary 3.3, for every minimal prime ideal P of R, $\overline{O}(P) = P$. In particular, let n be an integer positive such that $p^2|n$ for some prime number p. Then, the ring of integers modulo n, $\mathbb{Z}_n$ and the polynomial ring, $\mathbb{Z}_n[x]$, are commutative generalized p.q.-Baer rings which are neither PP nor p.q.-Baer.

(ii) Let R be a reduced p.q.-Baer ring and A be one of the rings $R[x]/\langle x^n\rangle$ or $S(R,n)$. Then, the ring A is a 2-primal generalized p.q.-Baer ring which satisfies the $(CZ2)$ condition. Hence by Corollary 3.3, $\overline{O}(P)$ is a unique minimal prime ideal of A contained in P and $\overline{O}(P)$ is completely prime for every prime ideal P of A.

(iii) Let T be a commutative generalized p.q.-Baer ring with zero characteristic which is not p.q.-Baer. Let $S = \prod_{i=1}^{\infty} T_i$ where for each i, $T_i = T$. Let R be the subring of S generated by $\bigoplus_{i=1}^{\infty} T_i$ and $1 \in S$. Then, by [25, Lemma 4.9], R is a generalized p.q.-Baer ring which is 2-primal and satisfies the $(CZ2)$ condition, but it is neither generalized quasi-Baer, p.q.-Baer nor PP.

(iv) Assume that R is a 2-primal generalized left p.q.-Baer ring which satisfies condition the $(CZ2)$. By [25, Theorem 4.7], the ring eRe is a generalized left p.q.-Baer ring, for every idempotent $e \in R$. Hence by Corollary 3.3, $\overline{O}(P)$ is a unique minimal prime ideal of eRe such that $\overline{O}(P) \subseteq P$ and $\overline{O}(P)$ is completely prime for every prime ideal P of eRe.

4. DECOMPOSITIONS OF GENERALIZED P.Q.-BAER RINGS

In this section, various decompositions of generalized p.q.-Baer rings are determined. We consider the essentiality and splitting-off of $O(P)$ or P where P is a prime ideal of R.

Recall from [10] that a ring R is called *principally right FI-extending* if every principal right ideal is essential as a right R-module in a right ideal of R generated by an idempotent. Observe from [8, Corollary 1.11] that if R is semiprime, then R is right p.q.-Baer if and only if R is principally right FI-extending.

PROPOSITION 4.1. *Let R be a generalized right (principally) quasi-Baer ring with $\mathcal{S}_\ell(R) = \boldsymbol{B}(R)$. Then for every (principal) ideal I of R, there exists* *a positive integer n such that the ideal I^n is left essential in an ideal of R generated by a central idempotent.*

Proof. There exists a central idempotent $c \in \mathcal{S}_\ell(R)$ and a positive integer n such that $r_R(I^n) = cR$. Put $e = 1-c$, then $e \in \mathcal{S}_r(R)$ and $r_R(I^n) = (1-e)R$, $I^n \subseteq Re$, as $I^n \subseteq \ell_R(r_R(I^n)) = \ell_R(cR) = R(1-c)$. Also assume that J is a left ideal in Re such that $I^n \cap J = 0$. Then $I^n J = 0$. So $J \subseteq r_R(I^n) = (1-e)R$. Consequently, $J \subseteq (1-e)R \cap eR = 0$. Therefore $J = 0$ and thus I^n is left essential in eR. $\square$

THEOREM 4.2. *Let R be a generalized right quasi-Baer ring with $\mathcal{S}_\ell(R) = \boldsymbol{B}(R)$. Then there exists a positive integer n such that $R = A \oplus B$ (ring direct sum), where $Soc(A)$ is left essential in A and*

$$Soc(R) = (Soc(R))^n \oplus Soc(B).$$

Proof. By Proposition 4.1, there exists a central idempotent $e \in R$ and a positive integer n such that $(Soc(R))^n$ is left essential in eR. Now assume that $A = eR$ and $B = (1-e)R$. Then we have $R = A \oplus B$, $(Soc(R))^n = Soc(A)$ and $Soc(R) = (Soc(R))^n \oplus Soc(B)$, and the proof is complete. $\square$

COROLLARY 4.3. *[6, Proposition 2.3] Let R be a semiprime quasi-Baer ring. Then $R = A \oplus B$ (ring direct sum), where $Soc(A)$ is right and left essential in A and $Soc(B) = 0$.*

Proof. This result is a consequence of [8, Proposition 1.17] and Theorem 4.2. $\square$

LEMMA 4.4. *Let R be a generalized right p.q.-Baer ring. If P is a prime ideal of R, then:*

$$\overline{O}(P) = \{a \in R \mid a^k Re = 0 \ \textit{for some} \ e \in \mathcal{S}_\ell(R) \setminus P, \ \textit{positive integer } k\}.$$

Proof. Let $a \in \overline{O}(P)$. Then there exists a positive integer k such that $a^k \in O(P)$. So there exists $b \in R \backslash P$ such that $a^k Rb = 0$. Since R is generalized right p.q.-Baer, there exists $e \in \mathcal{S}_\ell(R)$ and a positive integer n, such that $r_R(a^k R)^n = eR$. Since $b = eb$, $e \in R \setminus P$, as $b \in R \setminus P$. Hence $a^{nk} Re = 0$. Thus $\overline{O}(P) \subseteq \{a \in R \mid a^k Re = 0$ for some $e \in \mathcal{S}_\ell(R) \setminus P$, positive integer k$\}$. The reverse containment is obvious. $\square$

Now we derive a condition which ensures that for a prime ideal P, $\overline{O}(P)$ is an ideal of R. Observe that for every prime ideal P of R, $nil(R/O(P)) = \overline{O}(P)/P$. Therefore if $\overline{O}(P)$ is an ideal of R, then $nil(R/O(P))$ is an ideal of the ring $R/O(P)$.

PROPOSITION 4.5. *Let R be a generalized right p.q.-Baer ring. If $nil(R)$ is an ideal, then for every prime ideal P of R, $\overline{O}(P)$ is an ideal of R and so $nil(R/O(P))$ is an ideal of $R/O(P)$.*

Proof. The proof is similar to that of [9, Proposition 2.1] with the use of Lemma 4.4. □

PROPOSITION 4.6. *Let P be a prime ideal of a generalized left p.q.-Baer ring R. Then P is left essential in R or $P = Re$ for some idempotent $e \in R$.*

Proof. Let P be a prime ideal of R that is not left essential. Then there exists a nonzero element $a \in R$ such that $P \cap Ra = 0$. Since R is generalized left p.q.-Baer, there exists a positive integer n such that $\ell_R(Ra)^n = Re$ for some idempotent $e \in R$. It is clear that $P \subseteq \ell_R(Ra)^n$. Let $x \in \ell_R(Ra)^n$. So $x(Ra)^n = 0$, and since $a \notin P$, we have $x \in P$. Hence $P = \ell_R(Ra)^n = Re$. □

PROPOSITION 4.7. *Let P be a prime ideal of a generalized left p.q.-Baer ring R. Then $N(P)$ is left essential in P or $O(P) = Re$ for some idempotent $e \in R$.*

Proof. Assume $N(P)$ is not left essential in P. Then there exists a nonzero principal left ideal Ra of R such that $Ra \subseteq P$ and $N(P) \cap Ra = 0$. Since R is generalized left p.q.-Baer, there exists a positive integer n such that $\ell_R(Ra)^n = Re$ for some idempotent $e \in \mathcal{S}_r(R)$. Thus $O(P) \subseteq N(P) \subseteq \ell_R(Ra)^n = Re$. If $e \notin P$, then $aRe \subseteq P(R)$. Thus $a \in N(P)$, a contradiction. Hence $e \in P$. Since $eR(1-e) = 0$, it follows that $e \in O(P)$. Therefore $O(P) = Re$. □

LEMMA 4.8. *Let P be a prime ideal in a generalized right p.q.-Baer ring R. If $nil(R)$ is an ideal, then either:*

(i) *$\overline{O}(P)$ is right essential in P; or*

(ii) *$\overline{O}(P) \subseteq eR \subseteq P$ for some left semicentral $e \in R$. If e is central, then $\overline{O}(P) = eR$.*

Proof. Assume $\overline{O}(P)$ is not right essential in P. Then there exists a nonzero principal right ideal $X = aR$ of R such that $X \subseteq P$ and $\overline{O}(P) \cap X = 0$. There exists an idempotent $e \in R$ such that $r_R(X^n) = eR$ for some positive integer n. Since $\overline{O}(P) \cap X^n = 0$ so $X^n\overline{O}(P) = 0$, it follows that $\overline{O}(P) \subseteq r_R(X^n) = eR$. We have $a^nRe = 0$, since $X^n e = 0$. So, if $e \in R \setminus P$, then $a^n \in O(P)$. Hence $a \in \overline{O}(P)$. Then $a \in \overline{O}(P) \cap X = 0$, a contradiction. So $e \in P$. Then $eR \subseteq P$. Therefore $\overline{O}(P) \subseteq eR \subseteq P$. Since $e \in S_\ell(R)$, $(1-e)Re = 0$. If e is central, then $e \in O(P)$, since $1 - e \in R \setminus P$. Hence $eR \subseteq \overline{O}(P)$. It follows that $\overline{O}(P) = eR$, and the result follows. □

Let I be an ideal of a ring R and P a prime ideal of R. Observe that $I \cap P$ is a prime ideal of I. Define

$$O_I(P) = \{x \in I \mid xIs = 0 \text{ for some } s \in I \setminus P\}.$$

Theorem 4.9. *Let R be an abelian generalized right p.q.-Baer ring. If $nil(R)$ is an ideal and P is a prime ideal of R, then either:*
(i) $\overline{O}(P)$ is right essential in R; or
(ii) $R = T \oplus P$(ring direct sum) for some ideal T of R, and $O(P) = P$.

Proof. We use the method in the proof of [9, Theorem 2.4]. Assume P is not right essential in R. Then there exists a principal right ideal $X \neq 0$ such that $P \cap X = 0$. There exists an idempotent $e \in R$ such that $r_R(X^n) = eR$ for some positive integer n. Since $X^n P = 0$, $P \subseteq eR$. Observe $(1-e)Re = 0 \subseteq P$. If $1 - e \in P$ then $1 - e \in eR$, therefore $e = 1$. Since $X^n e = 0$, $X^n = 0 \subseteq P$. Thus $X \subseteq P$, it follows that $X \cap P = X = 0$, a contradiction. Thus $e \in P$ yields $P = eR$. Since $PR(1 - e) = 0$, it follows that $O(P) = P = eR$. Now, put $T = (1 - e)R$. Then, we have $R = T \oplus P$.

Now assume P is right essential in R. If $\overline{O}(P)$ is not right essential in P, Lemma 4.8 yields $\overline{O}(P) = eR$ and $\overline{O}(P) \subseteq P$, where e is a central idempotent. Let $S = (1 - e)R$. Since e is a central idempotent, by [25, Theorem 4.7] the ring S is also a generalized right p.q.-Baer ring. Since $R = eR \oplus S$, it follows that $P = \overline{O}(P) \oplus (S \cap P) = \overline{O}(P) \oplus P_1$, where $P_1 = S \cap P$ is a prime ideal of S. Since $\overline{O}_S(P_1) \subseteq \overline{O}(P) = eR$, $\overline{O}_S(P_1) = 0$. We claim that the ring S is a prime ring. Let X, Y be principal right ideals of S such that $XY = 0$. Since S is generalized right p.q.-Baer ring, $r_S(X^n) = fS$ for some central idempotent $f \in S$ and positive integer n, so $Y \subseteq r_S(X^n) = fS$. If $f \in S \setminus P_1$, since $X^n S f = 0$ then $X \subseteq \overline{O}_S(P_1) = 0$. Hence $X = 0$. If $f \in P_1$ so $1 - f \in S \setminus P_1$. Since $fS(1 - f) = 0$ it follows that $f \in \overline{O}_S(P_1) = 0$. So $f = 0$ and hence $Y = 0$. Thus the ring S is a prime ring so $P_1 = S \cap P = 0$. It follows that $\overline{O}(P) = P$, a contradiction. Therefore $\overline{O}(P)$ is right essential in P and hence $\overline{O}(P)$ is right essential in R, and the proof is complete. □

Proposition 4.10. *Let R be a generalized left p.q.-Baer ring. If every essential left ideal of R is an essential extension of an ideal of R (e.g., if R has essential left socle), then the left singular ideal $Z_\ell(R)$ is nil.*

Proof. Let $x \in Z_\ell(R)$, then $\ell_R(x)$ is left essential in R. So there exists an ideal I of R such that $I \subseteq \ell_R(x)$ and I is a left essential in R. Since R is generalized left p.q.-Baer, there exists an idempotent $e \in R$ such that $\ell_R(Rx)^n = Re$ for some positive integer n. Since $0 = Ix = IRx$, we have $I(Rx)^n = 0$ and that $I \subseteq \ell_R(Rx)^n = Re$. We show that $x^n = 0$. Assume to the contrary that $x^n \neq 0$. So $e \neq 1$ and $I \cap R(1-e) \subseteq Re \cap R(1-e) = 0$. So $I \cap R(1 - e) = 0$ and $R(1 - e) \neq 0$, a contradiction, since I is left essential. Therefore $x^n = 0$, and the result follows. □

A ring R is called *orthogonally finite* if there are no infinite sets of orthogonal idempotents in R. By [24, Proposition 6.59] for any ring R, the following are equivalent:

(1) R satisfies ACC on right direct summands;
(2) R satisfies DCC on left direct summands;
(3) R has no infinite set of nonzero orthogonal idempotents.

PROPOSITION 4.11. *Let R be a generalized left p.q.-Baer ring with IFP. If R is orthogonally finite, then for every right annihilator L there exists an idempotent $e \in R$ such that $L = eR \oplus (L \cap (1-e)R)$ and $L \cap (1-e)R$ is nil.*

Proof. If L is nil then there is nothing to prove. Assume to the contrary that L is not nil. So there exists some $x \in L$ which is not nilpotent. Since R is generalized left p.q.-Baer, $\ell_R(Rx)^n = Rf$ for some positive integer n and some idempotent $f \neq 1$. Since R satisfy IFP, it is easy to show that $\ell_R(Rx)^n = \ell_R(x^n)$. Thus $\ell_R(L) \subseteq \ell_R(x^n) = Rf$ and hence $(1-f)R = r_R(Rf) \subseteq r_R(\ell_R(L)) = L$. So L has a nonzero idempotent element. Now let e be a nonzero idempotent in L such that $r_R(e)$ is minimal among all right annihilators of idempotent elements of L. Then, by a similar argument as that in [24, Theorem 7.55], one can show that $L \cap (1-e)R$ is nil. Since $R = eR \oplus (1-e)R$ and $eR \subseteq L$, it is easy to show that $L = eR \oplus (L \cap (1-e)R)$. □

PROPOSITION 4.12. *Let R be a commutative generalized p.q.-Baer ring. Then $R/nil(R)$ is a commutative p.q.-Baer ring.*

Proof. Let $\overline{R} = R/nil(R)$ and $\overline{0} \neq \overline{x} \in \overline{R}$. Since $R/nil(R)$ is a reduced ring, $\ell_{\overline{R}}(\overline{x}) = \ell_{\overline{R}}(\overline{x}^k)$ for every positive integer k. Since R is generalized p.q.-Baer, there exists a positive integer n such that $\ell_R(x^n) = Re$ for some idempotent $e \in R$. We show that $\ell_{\overline{R}}(\overline{x}) = \overline{R}\overline{e}$. Let $\overline{r} \in \ell_{\overline{R}}(\overline{x})$, then $rx \in nil(R)$. So there is a positive integer m such that $(rx)^m = 0$. So $r^m \in \ell_R(x^{mn}) = \ell_R(x^n)$ by [25, Lemma 2.6]. Since $\ell_R(x^n) = Re$, $r^m e = r^m$, which implies that $r^m(1-e) = 0$. So $(r(1-e))^m = 0$ and that $r(1-e) \in nil(R)$. Thus $(r - re) \in nil(R)$. Therefore $\overline{r}\,\overline{e} = \overline{r}$, it follows that $\ell_{\overline{R}}(\overline{x}) \subseteq \overline{R}\overline{e}$. Also since $ex^n = 0$, it implies that $(ex)^n = 0$. So $ex \in nil(R)$ and this means $\overline{e} \in \ell_{\overline{R}}(\overline{x})$, so $\overline{R}\overline{e} \subseteq \ell_{\overline{R}}(\overline{x})$, and the proof is complete. □

The final result of this paper provides a lattice connection between the generalized right p.q.-Baer and generalized right quasi-Baer conditions.

THEOREM 4.13. *Let R be a generalized right p.q.-Baer ring. If R satisfies any of the following conditions, then R is generalized right quasi-Baer.*

(*i*) *For any nonzero right ideal I of R, $I \cap C(R)$ is a nonzero finitely generated right ideal of $C(R)$ such that for every positive integer n, $I^n \cap C(R) \subseteq (I \cap C(R))^n$.*

(*ii*) *R is right Noetherian with* $\mathcal{S}_\ell(R) = \boldsymbol{B}(R)$.

Proof. Let I be a nonzero right ideal of R. If $r_R(I^n) = 0$, for some positive integer n, then we are done. So we assume that $r_R(I^n) \neq 0$ for every positive integer n.

(*i*) By the hypothesis, $I \cap C(R)$ is a nonzero finitely generated right ideal of $C(R)$. Since by [25, Proposition 2.2(iii)] $C(R)$ is generalized p.q.-Baer, by [25, Proposition 2.8], $r_{C(R)}(I \cap C(R))^n = eC(R)$ for some idempotent $e \in C(R)$ and positive integer n. Now we prove $r_R(I^n) = eR$. If $I^n e \neq 0$ then $I^n e$ is a nonzero right ideal of R so by hypothesis $0 \neq I^n e \cap C(R) \subseteq (I \cap C(R))^n$. Let $0 \neq x \in I^n e \cap C(R)$. Then $x = ye$ for some $y \in I^n$. Since $x \in (I \cap C(R))^n$, $xe = 0$. Hence $x = 0$, which is a contradiction. It follows that $eR \subseteq r_R(I^n)$, and then $r_R(I^n) = R \cap r_R(I^n) = (eR \oplus (1-e)R) \cap r_R(I^n) = eR \oplus ((1-e)R) \cap r_R(I^n))$. Now assume $(1-e)R \cap r_R(I^n) \neq 0$. Then $(1-e)R \cap r_R(I^n)$ is a nonzero right ideal of R. Thus, by hypothesis, $0 \neq (1-e)R \cap r_R(I^n) \cap C(R) = (1-e)R \cap r_{C(R)}(I^n) \subseteq (1-e)R \cap r_{C(R)}(I \cap C(R))^n = (1-e)R \cap eR = 0$; which is also a contradiction. Hence$(1-e)R \cap r_R(I^n) = 0$, it follows that $r_R(I^n) = eR$. Therefore R is a generalized right quasi-Baer ring.

(*ii*) Assume that I is a nonzero right ideal of R, since R is a right Noetherian so by [25, Proposition 2.8] $r_R(I^k) = eR$, for some idempotent $e \in R$ and positive integer k. Thus R is generalized right quasi-Baer. □

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