k-CLEAN MONOMIAL IDEALS

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In this paper, we introduce the concept of k-clean monomial ideals as an extension of clean monomial ideals and present some homological and combinatorial properties of them. Using the hierarchal structure of k-clean ideals, we show that a $(d-1)$-dimensional simplicial complex is k-decomposable if and only if its Stanley-Reisner ideal is k-clean, where $k \leq d-1$. We prove that the classes of monomial ideals like Cohen-Macaulay ideals of codimension 2, monomial ideals of forest type without embedded prime ideal and symbolic powers of Stanley-Reisner ideals of matroid complexes are k-clean for all $k \geq 0$.

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INTRODUCTION

Let R be a Noetherian ring and M be a finitely generated R-module. It is well known that there exists a so called prime filtration

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset \ldots \subset M_{r-1} \subset M_r = M$$

that is such that $M_i/M_{i-1} \cong R/P_i$ for some $P_i \in \mathrm{Supp}(M)$. We call any such filtration of M a **prime filtration**. Set $\mathrm{Supp}(\mathcal{F}) = \{P_1, \ldots, P_r\}$. Let $\mathrm{Min}(M)$ denote the set of minimal prime ideals in $\mathrm{Supp}(M)$. If I is an ideal of R then we set $\min(I) = \mathrm{Min}(R/I)$. Dress [7] calls a prime filtration $\mathcal{F}$ of M **clean** if $\mathrm{Supp}(\mathcal{F}) = \mathrm{Min}(M)$. The module M is called clean, if M admits a clean filtration and R is clean if it is a clean module over itself.

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminate over a field K. Let Δ be a simplicial complex on the vertex set $[n] = \{1, 2, \ldots, n\}$. Dress [7] showed that Δ is (non-pure) shellable in the sense of Björner and Wachs [3], if and only if the Stanley-Reisner ring S/I_Δ is clean. The result of Dress is, in fact, the algebraic counterpart of shellability for simplicial complexes. Some subclasses of shellable complexes are k-decomposable simplicial complexes which were introduced by Billera and Provan [2] on pure simplicial complexes and then by Woodroofe [31] on not necessarily pure ones. Simon in [25] introduced “completed clean ideal trees” as an algebraic counterpart of pure k-decomposable complexes. Actually, in the sense of Simon, the Stanley-Reisner ideal of a k-decomposable complex is completed clean ideal tree.

Let $I \subset S$ be a monomial ideal. We call I Cohen-Macaulay (clean) if the quotient ring S/I has this property. In this paper, we define the concept of k-clean monomial ideals. The class of k-clean monomial ideals are, actually, subclass of clean monomial ideals. It is the aim of this paper to study the properties of k-clean monomial ideals and describe relations between these ideals and k-decomposable simplicial complexes. Moreover, some classes of k-clean monomial ideals are introduced. Also, some results of [1,16] are extended.

In Section 2, we introduce k-clean monomial ideals. We show that k-clean monomial ideals are clean and, also, every clean monomial ideal is k-clean for some $k \geq 0$ (see Theorem 2.4). In Section 3, we discuss some basic properties of k-clean ideals. Some homological invariants of k-clean monomial ideals like depth and Castelnuovo-Mumford regularity are described in this section. In the fourth section, we show that a $(d-1)$-dimensional simplicial complex Δ is k-decomposable if and only if its associated Stanley-Reisner ideal is k-clean, where $k \leq d$ (see Theorem 4.1). The last section is devoted to presenting some examples of k-clean monomial ideals. We show that irreducible monomial ideals and monomial complete intersection ideals are k-clean, for all $k \geq 0$ (see Theorems 5.1 and 5.2). Then by showing that Cohen-Macaulay monomial ideals of codimension 2 (see Theorem 5.4) are k-clean, we improve Proposition 1.4. of [16]. In Theorem 5.6, we show that a monomial ideal of forest type which has no embedded prime ideal is k-clean, for all $k \geq 0$. Finally, in Theorem 5.9, we show that symbolic powers of Stanley-Reisner ideals of matroid complexes are k-clean for all $k \geq 0$. In this way, we improve Theorem 2.1 of [1].

1. **PRELIMINARIES**

Let Δ be a simplicial complex of dimension $d-1$ with the vertex set $[n] := \{1, 2, \ldots, n\}$. Let K be a field. The Stanley-Reisner monomial ideal of Δ is denoted by I_Δ and it is a squarefree monomial ideal in the polynomial ring $S = K[x_1, \ldots, x_n]$ generated by the monomials $\mathbf{x}^F = \prod_{i \in F} x_i$ which F is a non-face in Δ. The quotient ring S/I_Δ is called the **face ring** or **Stanley-Reiner ring** of Δ. If $\mathcal{F}(\Delta) = \{F_1, \ldots, F_r\}$ is the set of maximal faces (facets) of Δ then we set $\Delta = \langle F_1, \ldots, F_r \rangle$.

For all undefined terms or notions on simplicial complexes we refer the reader to the books [13] or [27].

Given a simplicial complex Δ on $[n]$, the **link**, **star** and **deletion** of σ in Δ are defined, respectively, by

$$\begin{aligned}&\mathrm{link}_{\Delta}(\sigma)=\{F\in\Delta:\sigma\cap F=\emptyset,\sigma\cup F\in\Delta\},\\&\mathrm{star}_{\Delta}(\sigma)=\{F\in\Delta:\sigma\cup F\in\Delta\}\text{ and}\\&\Delta\backslash\sigma=\{F\in\Delta:\sigma\nsubseteq F\}.\end{aligned}$$

Moreover, the **Alexander dual** of Δ is defined as $\Delta^{\vee}=\{F\in\Delta:[n]\backslash F\notin\Delta\}$.

Let $I\subset S$ be a squarefree monomial ideal generated by monomials of degree at least 2. Then there exists a simplicial complex Δ on $[n]$ such that $I=I_{\Delta}$. The Alexander dual of I is defined $I^{\vee}=I_{\Delta^{\vee}}$.

Definition 1.1 ([31]). Let Δ be a simplicial complex on vertex set $[n]$. Then a face $\sigma\in\Delta$ is called a **shedding face** if it satisfies the following property:

$$\text{no facet of } (\mathrm{star}_{\Delta}\sigma)\backslash\sigma \text{ is a facet of } \Delta\backslash\sigma.$$

Definition 1.2 ([31]). A $(d-1)$-dimensional simplicial complex Δ is recursively defined to be k**-decomposable** if either Δ is a simplex or else has a shedding face σ with $\dim(\sigma)\le k$ such that both $\mathrm{link}_{\Delta}\sigma$ and $\Delta\backslash\sigma$ are k-decomposable.

We consider the complexes $\{\}$ and $\{\emptyset\}$ to be k-decomposable for $k\ge -1$. Also k-decomposability implies to k'-decomposability for $k'\ge k$.

A 0-decomposable simplicial complex is called **vertex-decomposable**.

We say that the simplicial complex Δ is (non-pure) **shellable** if its facets can be ordered $F_1,F_2,\ldots,F_r$ such that, for all $r\ge 2$, the subcomplex $\langle F_1,\ldots,F_{j-1}\rangle\cap\langle F_j\rangle$ is pure of dimension $\dim(F_j)-1$ [3]. It was shown in [31] or [18] that a $(d-1)$-dimensional (not necessarily pure) simplicial complex Δ is shellable if and only if it is $(d-1)$-decomposable.

Let I be a monomial ideal of S. We denote by $G(I)$ the set of minimal monomial generators of I. Let $\min(I)$ be the set of minimal (under inclusion) prime ideals of S containing I.

For $\mathbf{a}\in\mathbb{N}^n$, set $\mathbf{x}^{\mathbf{a}}=\prod_{\mathbf{a}(i)>0}x_i^{\mathbf{a}(i)}$ and define the **support** of $\mathbf{a}$ by $\mathrm{supp}(\mathbf{a})=\{i:\mathbf{a}(i)>0\}$. We set $\mathrm{supp}(\mathbf{x}^{\mathbf{a}}):=\mathrm{supp}(\mathbf{a})$. Also, we define $\bar{\mathbf{a}}$ an n-tuple in $\{0,1\}^n$ with $\bar{\mathbf{a}}(i)=1$ if $\mathbf{a}(i)\neq 0$ and $\bar{\mathbf{a}}(i)=0$, otherwise. Set $\nu_i(\mathbf{x}^{\mathbf{a}}):=\mathbf{a}(i)$.

Let $u,v\in S$ be two monomials. We set $[u,v]=1$ if for all $i\in\mathrm{supp}(u)$, $x_i^{a_i}\nmid v$ and $[u,v]\neq 1$, otherwise.

For the monomial $u\in S$ and the monomial ideal $I\subset S$ set

$$I^u=\langle v\in G(I):[u,v]\neq 1\rangle \quad\text{and}\quad I_u=\langle v\in G(I):[u,v]=1\rangle.$$

Definition 1.3 ([23]). Let I be a monomial ideal with the minimal system of generators $\{u_1,\dots,u_r\}$. The monomial $v = x_1^{a_1}\dots x_n^{a_n}$ is called **shedding** if $I_v \neq 0$ and for each $u_i \in G(I_v)$ and each $l \in \mathrm{supp}(u)$ there exists $u_j \in G(I^v)$ such that $u_j : u_i = x_l$.

Definition 1.4 ([23]). Let I be a monomial ideal minimally generated with set $\{u_1,\dots,u_r\}$. We say I is a **k-decomposable** ideal if $r = 1$ or else has a shedding monomial v with $|\mathrm{supp}(v)| \leq k+1$ such that the ideals I^v and I_v are k-decomposable. (Note that since the number of minimal generators of I is finite, the recursion procedure will stop.)

A 0-decomposable monomial ideal is called **variable-decomposable**.

THEOREM 1.5 ([23, Theorem 2.10]). *Let Δ be a (not necessarily pure) $(d-1)$-dimensional simplicial complex on vertex set $[n]$. Then Δ is k-decomposable if and only if $I_{\Delta^\vee}$ is k-decomposable, where $k \leq d-1$.*

Definition 1.6 ([20]). A monomial ideal I is called **weakly polymatroidal** if for every two monomials $u = x_1^{a_1}\dots x_n^{a_n} >_{lex} v = x_1^{b_1}\dots x_n^{b_n}$ in $G(I)$ such that $a_1 = b_1,\dots,a_{t-1} = b_{t-1}$ and $a_t > b_t$, there exists $j > t$ such that $x_t(v/x_j) \in I$.

THEOREM 1.7 ([24, Theorem 4.33]). *Every weakly polymatroidal ideal I is variable-decomposable.*

2. k-CLEAN MONOMIAL IDEALS

In this section, we extend the concept of cleanness introduced by Dress [7]. Let $I \subset S$ be a monomial ideal. A prime filtration

$$\mathcal{F} : (0) = M_0 \subset M_1 \subset \dots \subset M_{r-1} \subset M_r = S/I$$

of S/I is called **multigraded**, if all M_i are multigraded submodules of S/I, and if there are multigraded isomorphisms $M_i/M_{i-1} \cong S/P_i(-\mathbf{a}_i)$ with some $\mathbf{a}_i \in \mathbb{Z}^n$ and some multigraded prime ideals P_i.

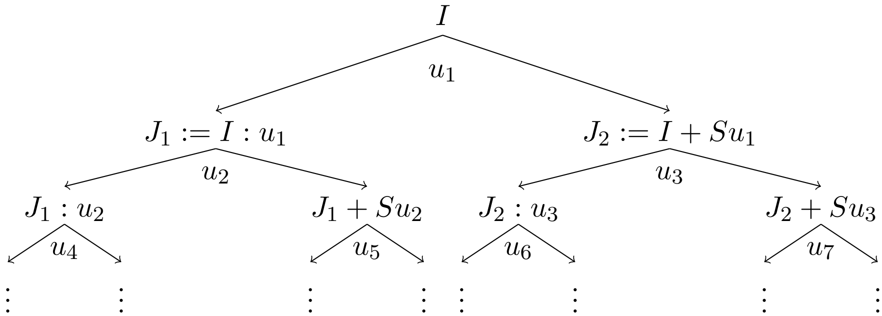
A multigraded prime filtration $\mathcal{F}$ of S/I is called **clean** if $\mathrm{Supp}(\mathcal{F}) \subseteq \min(I)$.

Definition 2.1. Let $I \subset S$ be a monomial ideal. A non unit monomial $u \notin I$ is called a **cleaner** monomial of I if $\min(I + Su) \subseteq \min(I)$.

Definition 2.2. Let $I \subset S$ be a monomial ideal. We say that I is **k-clean** whenever I is a prime ideal or I has no embedded prime ideals and there exists a cleaner monomial $u \notin I$ with $|\mathrm{supp}(u)| \leq k+1$ such that both $I : u$ and $I + Su$ are k-clean.

We recall the concept of ideal tree from [25]:

Let $I \subset S$ be a k-clean monomial ideal. By the definition, there are cleaner monomials $u_1, u_2, \ldots$ with $|\mathrm{supp}(u_i)| \leq k+1$ decomposing I. Therefore we obtain the rooted, finite, directed and binary tree $\mathcal{T}$:



$\mathcal{T}$ is called the **ideal tree** of I and the number of all cleaner monomials appeared in $\mathcal{T}$ is called the **length** of $\mathcal{T}$. We denote the length of $\mathcal{T}$ by $l(\mathcal{T})$.

We define the k**-cleanness length** of the k-clean monomial ideal I by

$$l(I) = \min\{l(\mathcal{T}) : \mathcal{T} \text{ is an ideal tree of } I\}.$$

Example 2.3. Consider the monomial ideal

$$I = (x_1x_2x_4, x_1x_2x_5, x_1x_2x_6, x_1x_3x_5, x_1x_3x_6, x_1x_4x_5, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6)$$

and

$$J = (x_1x_2, x_1x_3, x_1x_4)$$

of the polynomial ring $S = K[x_1, \ldots, x_6]$. I and J are, respectively, 1-clean and 0-clean and have ideal trees $\mathcal{T}_1$ and $\mathcal{T}_2$ such that the cleaner monomials appeared in $\mathcal{T}_1$ and $\mathcal{T}_2$ are, respectively, $x_2x_3, x_1x_4, x_1x_5, x_2x_4, x_2x_5, x_2, x_1, x_3x_6, x_3$ and x_1.

Theorem 2.4. *Every k-clean monomial ideal I is clean. Also, every clean monomial ideal is k-clean for some $k \geq 0$.*

Proof. Let I be a k-clean monomial ideal. We use induction on the k-cleanness length of I. Let I be not prime and there exists a cleaner monomial $u \notin I$ of multidegree $\mathbf{a}$ with $|\mathrm{supp}(u)| \leq k+1$ such that both $I : u$ and $I + Su$ are k-clean. By induction, $I : u$ and $I + Su$ are clean. Let

$$\mathcal{F}_1 : I + Su = J_0 \subset J_1 \subset \ldots \subset J_r = S$$

and

$$\mathcal{F}_2 : 0 = \frac{L_0}{I : u} \subset \frac{L_1}{I : u} \subset \ldots \subset \frac{L_s}{I : u} = \frac{S}{I : u}.$$

be clean prime filtrations and let $(L_i/I : u)/(L_{i-1}/I : u) \cong S/Q_i(-\mathbf{a}_i)$ where Q_i are prime ideals. It is known that the multiplication map $\varphi : S/I : u(-\mathbf{a}) \xrightarrow{.u} I + Su/I$ is an isomorphism. Restricting φ to $L_i/I : u$ yields a monomorphism $\varphi_i : L_i/I : u \xrightarrow{.u} I + Su/I$. Set $H_i/I := \varphi_i(L_i/I : u)$. Hence $H_i/I \cong (L_i/I : u)(-\mathbf{a})$. It follows that

$$\frac{H_i}{H_{i-1}} \cong \frac{H_i/I}{H_{i-1}/I} \cong \frac{(L_i/I : u)(-\mathbf{a})}{(L_{i-1}/I : u)(-\mathbf{a})} \cong \frac{S}{Q_i}(-\mathbf{a} - \mathbf{a}_i).$$

Therefore we obtain the following prime filtration induced from $\mathcal{F}_2$:

$$\mathcal{F}_3 : I = H_0 \subset H_1 \subset \ldots \subset H_s = I + Su.$$

By adding $\mathcal{F}_1$ to $\mathcal{F}_3$ we obtain the following prime filtration

$$\mathcal{F} : I = H_0 \subset H_1 \subset \ldots \subset H_s = I + Su \subset J_1 \subset \ldots \subset J_r = S.$$

Finally, $\mathrm{Supp}(\mathcal{F}) = \mathrm{Supp}(\mathcal{F}_1) \cup \mathrm{Supp}(\mathcal{F}_2) \subset \min(I+Su) \cup \min(I : u) \subseteq \min(I)$ and therefore I is clean.

To prove the second assertion, suppose that I is a clean monomial ideal. If I is prime then we are done. Suppose that I is not prime and let

$$\mathcal{F} : (0) = M_0 \subset M_1 \subset \ldots \subset M_{r-1} \subset M_r = S/I$$

be a clean prime filtration of S/I with $M_i/M_{i-1} \cong S/P_i(-\mathbf{a}_i)$. We use induction on the length of the prime filtration $\mathcal{F}$. Since that $\mathrm{Ass}(S/I) \subseteq \mathrm{Supp}(\mathcal{F}) \subseteq \min(I)$, we have $\mathrm{Ass}(S/I) = \min(I)$. Hence I has no embedded prime ideal. It follows from Proposition 10.1. of [15] that there is a chain of monomial ideals $I = I_0 \subset I_1 \subset \ldots \subset I_r = S$ and monomials u_i of multidegree $\mathbf{a}_i$ such that $I_i = I_{i-1} + Su_i$ and $I_{i-1} : u_i = P_i$. Since that $I + Su_1$ has a clean filtration, it is k-clean, by induction hypothesis, where $|\mathrm{supp}(u_1)| \leq k + 1$. On the other hand, $I + Su_1/I \cong S/P_1$. Therefore $\min(I + Su_1) = \{P_1\} \subset \min(I)$. This means that I is k-clean. □

3. SOME PROPERTIES OF k-CLEAN MONOMIAL IDEALS

THEOREM 3.1. *Let $I \subset S$ be k-clean. Then for all monomial $u \in S$, $I : u$ is k-clean.*

Proof. We use induction on the k-cleanness length of I. If I is prime then $I : u$ is prime, too and we have nothing to prove. Assume that I is not prime. Suppose v is a cleaner monomial of I with $|\mathrm{supp}(v)| \leq k + 1$ and $I : v$ and $I + (v)$ are k-clean. We consider two cases:

Case 1. Let $v|u$. Then $I : u = (I : v) : u/v$ and by induction hypothesis $I : u$ is k-clean.

Case 2. Let $v \nmid u$. We show that $v/\gcd(u,v)$ is a cleaner monomial of $I : u$. We have

$$(I:u)+(\frac{v}{\gcd(u,v)}) = (I+(v)):u \text{ and } (I:u) : \frac{v}{\gcd(u,v)} = (I:v) : \frac{u}{\gcd(u,v)}.$$

By induction, $(I : u) + (\frac{v}{\gcd(u,v)})$ and $(I : u) : \frac{v}{\gcd(u,v)}$ are k-clean. Since $\min(I + (v)) \subset \min(I)$, by some elementary computations, we obtain that $\min((I + (v)) : u) \subset \min(I : u)$. Therefore $v/\gcd(u,v)$ is a cleaner monomial of $I : u$. □

Theorem 3.2. *The radical of each k-clean monomial ideal is k-clean.*

Proof. Let $I = (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_r})$ be a k-clean monomial ideal with cleaner monomial $u = \mathbf{x}^{\mathbf{b}}$ with $|\mathrm{supp}(u)| \le k+1$. We use induction on the k-cleanness length of I. Denote the radical of I by $\sqrt{I}$. By induction hypothesis, $\sqrt{I + Su}$ and $\sqrt{I : u}$ are k-clean. Let $v = \mathbf{x}^{\mathrm{supp}(u)}$ and let w be the product of variables x_i with $i \in \mathrm{supp}(u)$ and $\mathbf{a}_j(i) > \mathbf{b}(i) > 0$ for some $1 \le j \le r$. $\sqrt{I} + Sv$ is k-clean, because $\sqrt{I} + Sv = \sqrt{I + Su}$. Also, $\sqrt{I} : v = (\sqrt{I : u}) : w$ and so $\sqrt{I} : v$ is k-clean, by Theorem 3.1. On the other hand, $\min(\sqrt{I} + Sv) \subset \min(\sqrt{I + Su}) = \min(I + Su) \subset \min(I) = \min(\sqrt{I})$ and so v is a cleaner monomial of $\sqrt{I}$. □

Let $u = x_{i_1}^{a_1} \ldots x_{i_t}^{a_t} \in S$. The **polarization** of u is defined by

$$u^p = x_{i_1 1} \ldots x_{i_1 a_1} \ldots x_{i_t 1} \ldots x_{i_t a_t}.$$

If $I \subset S$ is a monomial ideal. The polarization of I is a monomial ideal of $S^p = K[x_{ij} : x_{ij} | u^p$ for some $u \in G(I)]$ given by $I^p = (u^p : u \in G(I))$.

Define the K-algebra homomorphism $\pi : S^p \to S$ by $\pi(x_{ij}) = x_i$.

Theorem 3.3. *Let I be a monomial ideal with no embedded prime ideal. If I^p is k-clean then I is k-clean, too.*

Proof. We use induction on the k-cleanness length of I^p. If I is a prime ideal then we have nothing to prove. Suppose that I is not prime. Let u be a cleaner monomial of I^p with $|\mathrm{supp}(u)| \le k + 1$ and let $I^p : u$ and $I^p + (u)$ be k-clean. We claim that $\pi(u)$ is a cleaner monomial of I. Note that

$$I : \pi(u) = \pi(I^p : u) \text{ and } I + (\pi(u)) = \pi(I^p + (u)).$$

By induction hypothesis, $I : \pi(u)$ and $I+(\pi(u))$ are k-clean. Since $|\mathrm{supp}(\pi(u))| \le |\mathrm{supp}(u)| \le k + 1$, it remains to show that $\pi(u)$ is a cleaner monomial of I. Let $P \in \min(I + (\pi(u)))$. Hence there exists $Q \in \min(I^p + (u))$ such that $P = \pi(Q)$. Since $Q \in \min(I^p)$, it follow that $P \in \min(I)$, as desired. □

Lemma 3.4. *Let $I \subset S$ be a k-clean monomial ideal with cleaner monomial u. Then u^p is a cleaner monomial of I^p.*

Proof. Let $Q \in \min(I^p + (u^p))$. Then $Q \in \mathrm{Ass}(S^p/I^p + (u^p))$. By Corollary 2.6 of [9], $\pi(Q) \in \mathrm{Ass}(S/I + (u)) = \min(I + (u)) \subset \min(I)$. Again, by Proposition 2.3 of [9], $Q \in \min(I^p)$, as desired. □

The following theorem describes projective dimension and Castelnuovo-Mumford regularity of k-clean monomial ideals.

THEOREM 3.5. *Let $I \subset S$ be a k-clean monomial ideal with the cleaner monomial u. Then*

(i) $\mathrm{pd}(S/I) = \max\{\mathrm{pd}(S/I + (u)), \mathrm{pd}(S/I : u)\}$;

(ii) $\mathrm{reg}(S/I) = \max\{\mathrm{reg}(S/I + (u)), \mathrm{reg}(S/I : u) + \deg(u)\}$.

Proof. (i) Without loss of generality we may assume that $I \subset \mathfrak{m}^2$. By Corollary 1.6.3. of [13], $\mathrm{pd}(S/I) = \mathrm{pd}(S^p/I^p)$ and $\mathrm{reg}(S/I) = \mathrm{reg}(S^p/I^p)$. Let Δ be a simplicial complex with $I_\Delta = I^p$. By Lemma 3.4, u^p is a cleaner monomial of I^p. Let $u^p = \mathbf{x}^\sigma$ for some $\sigma \in \Delta$. Therefore Δ is a k-decomposable simplicial complex with shedding monomial σ, by Theorem 4.1. Now it follows from Theorem 2.8 of [21] that

$$\begin{aligned}\mathrm{pd}(S/I) = \mathrm{pd}(S^p/I_\Delta) = &\ \max\{\mathrm{pd}(S^p/I_{\Delta\setminus\sigma}), \mathrm{pd}(S^p/J_{\mathrm{link}_\Delta\sigma})\}\\ = &\ \max\{\mathrm{pd}(S^p/(I+(u))^p), \mathrm{pd}(S^p/(I:u)^p)\}\\ = &\ \max\{\mathrm{pd}(S/I+(u)), \mathrm{pd}(S/I:u)\}\end{aligned}$$

where $J_{\mathrm{link}_\Delta\sigma}$ is the Stanley-Reisner ideal of $\mathrm{link}_\Delta\sigma$ considered as a complex on $V(\Delta)\setminus\sigma$.

(ii) follows by a similar argument from Theorem 2.8 of [21] and Theorem 4.1. □

Remark 3.6. The concept of sequentially Cohen-Macaulayness was introduced in [27] for finitely generated (graded) modules. We specially recall this concept for the quotient rings. Let $I \subset S$ be a monomial ideal. We say that I is sequentially Cohen-Macaulay if there exists a finite filtration

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset \ldots \subset M_r = S/I$$

of submodules of S/I with these properties that M_i/M_{i-1} is Cohen-Macaulay and

$$\dim(M_1/M_0) \le \dim(M_2/M_1) \le \ldots \le \dim(M_r/M_{r-1}.$$

It was proven in [15] that cleanness implies sequentially Cohen-Macaulayness. Therefore the class of k-clean monomial ideals is contained in the class of sequentially Cohen-Macaulay monomial ideals. In particular, since that every unmixed sequentially Cohen-Macaulay monomial ideal is Cohen-Macaulay, we conclude that the unmixed k-clean monomial ideals are Cohen-Macaulay.

4. A VIEW TOWARD k-DECOMPOSABLE SIMPLICIAL COMPLEXES

In this section, we prove the main result of this paper. In fact, we show that a squarefree k-clean monomial ideal is Stanley-Reisner ideal of a k-decomposable simplicial complex, and *vice versa*.

THEOREM 4.1. *Let Δ be a $(d-1)$-dimensional simplicial complex. Then $\sigma \in \Delta$ is a shedding face of Δ if and only if $\mathbf{x}^\sigma$ is a cleaner monomial of I_Δ.*
In particular, Δ is k-decomposable if and only if I_Δ is k-clean, where $0 \le k \le d-1$.

Proof. We first show that σ is a shedding face of Δ if and only if $\min(I_\Delta + (\mathbf{x}^\sigma)) \subseteq \min(I_\Delta)$. Since that Stanley-Reisner rings are reduced, it follows that

$$\min(I_\Delta) = \{P_{F^c} : F \in \mathcal{F}(\Delta)\}$$

and

$$\min(I_\Delta + (\mathbf{x}^\sigma)) = \{P_{F^c} : F \in \mathcal{F}(\Delta \backslash \sigma)\}.$$

Let σ be the shedding face of Δ. To show that $\mathbf{x}^\sigma$ is a cleaner monomial of I_Δ, it suffices to prove $\mathcal{F}(\Delta\backslash\sigma) \subseteq \mathcal{F}(\Delta)$. Suppose, on the contrary, that $F \in \mathcal{F}(\Delta\backslash\sigma)$ and $F \subsetneq G$ with $G \in \mathcal{F}(\Delta)$. This implies that $\sigma \subset G$ and so $G \in \mathrm{star}_\Delta\sigma$. On the other hand, since F is a facet of $\Delta\backslash\sigma$, it follows that there is $t \in \sigma$ such that $\sigma\backslash\{t\} \subset F$. We claim that $G = F\dot{\cup}\{t\}$. The inclusion "$\supseteq$" is clear. For the converse inclusion, if $s \in G\backslash(F \cup \{t\})$ for some s, then $\sigma \nsubseteq F \cup \{s\}$ and so $F \cup \{s\} \in \mathcal{F}(\Delta\backslash\sigma)$, a contradiction. Therefore $G = F\dot{\cup}\{t\}$ and it follows that $F \in \mathcal{F}((\mathrm{star}_\Delta\sigma)\backslash\sigma)$. But this contradicts the assumption that σ is a shedding face of Δ. Hence $\mathbf{x}^\sigma$ is a cleaner monomial.

Let Δ be k-decomposable with the shedding face $\sigma \in \Delta$. By the first part, $\mathbf{x}^\sigma$ is a cleaner monomial of I_Δ. To show that I_Δ is k-clean, we use induction on the number of the facets of Δ. If Δ is a simplex then the assertion is trivial. So assume that $|\mathcal{F}(\Delta)| > 1$. It is easy to check that $J_{\mathrm{link}_\Delta\sigma} = I_\Delta : \mathbf{x}^\sigma$ and $I_{\Delta\backslash\sigma} = I_\Delta + (\mathbf{x}^\sigma)$. By induction hypothesis, $\mathrm{link}_\Delta\sigma$ and $\Delta\backslash\sigma$ are k-decomposable if and only if $I_\Delta : \mathbf{x}^\sigma$ and $I_\Delta + (\mathbf{x}^\sigma)$ are k-clean. Therefore I_Δ is k-clean.

The reverse directions of both parts follow easily in similar arguments. □

Remark 4.2. Note that a k-clean monomial ideal need not be k'-clean for $k' < k$. Consider the monomial ideal $I \subset K[x_1, \ldots, x_6]$ with the minimal generator set

$$G(I) = \{x_1x_2x_4, x_1x_2x_5, x_1x_2x_6, x_1x_3x_5, x_1x_3x_6, x_1x_4x_5, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6\}.$$

I is the Stanley-Reisner ideal of the simplicial complex

$$\Delta = \langle 124, 125, 126, 135, 136, 145, 236, 245, 256, 345, 346\rangle$$

on [6]. It was shown in [25] that Δ is shellable but not vertex-decomposable. It follows from Theorem 4.1 that I is clean but not 0-clean. To see more examples of clean ideals which are not 0-clean we refer the reader to [11, 22].

Remark 4.3. Let I be a clean monomial ideal and $\dim(S/I) = d$. By Theorem 2.4, I is k-clean for some $k \geq 0$ with cleaner monomial u. It follows from Theorem 3.2 that $\sqrt{I}$ is k-clean with cleaner monomial $v = \mathbf{x}^{\mathrm{supp}(u)}$. Let $I_\Delta = \sqrt{I}$ for some simplicial complex Δ on $[n]$. By Theorem 4.1, we have $|\mathrm{supp}(u)| = |\mathrm{supp}(v)| \leq \dim(\Delta) + 1 = d$. Therefore I is $(d-1)$-clean.

On the other hand, every k-clean monomial ideal is also $(k+1)$-clean. This means that the k-cleanness is a hierarchical structure. Therefore we have the following implications:

$$0\text{-clean} \Rightarrow 1\text{-clean} \Rightarrow \ldots \Rightarrow (d-1)\text{-clean} \Leftrightarrow \text{clean}.$$

In Remark 4.2 we implied that above implications are strict.

Corollary 4.4. *Let $I \subset S$ be a squarefree monomial ideal generated by monomials of degree at least 2. Then I is k-clean if and only if $I^\vee$ is k-decomposable.*

Proof. Let Δ be a simplicial complex on $[n]$ such that $I = I_\Delta$. The assertion follows from Theorems 4.1 and 1.5. □

5. SOME CLASSES OF k-CLEAN IDEALS

In this section, we introduce some classes of k-clean monomial ideals.

5.1. IRREDUCIBLE MONOMIAL IDEALS

Theorem 5.1. *Every irreducible monomial ideal is 0-clean.*

Proof. Let I be a irreducible monomial ideal. We want to show that I is 0-clean. By Theorem 1.3.1. of [13], I is generated by pure powers of the variables. Without loss of generality we may assume that $I = (x_1^{a_1}, \ldots, x_m^{a_m})$ with $a_i \neq 0$ for all i. We use induction on $\sum_{i=1}^m a_i$. If $\sum_{i=1}^m a_i = m$, then I is prime and we are done. Suppose that $\sum_{i=1}^m a_i > m$. So we can assume that $a_1 > 1$. We have

$$I : x_1 = (x_1^{a_1-1}, x_2^{a_2}, \ldots, x_m^{a_m}) \text{ and } I + (x_1) = (x_1, x_2^{a_2}, \ldots, x_m^{a_m}).$$

By induction hypothesis, $I : x_1$ and $I+(x_1)$ are 0-clean. Clearly, x_1 is a cleaner monomial and so the proof is completed. □

5.2. MONOMIAL COMPLETE INTERSECTION IDEALS

THEOREM 5.2. *Let $I \subset S$ be a monomial complete intersection ideal. Then S/I is* 0*-clean.*

Proof. Let $G(I) = \{M_1, \ldots, M_r\}$. By the assumption $M_1, \ldots, M_r$ is a regular sequence. Hence $\gcd(M_i, M_j) = 1$ for all $i \neq j$. If I is a primary ideal then we are done, by Theorem 5.1. Suppose that I is not primary. We use induction on n the number of variables. Let $|\mathrm{supp}(M_1)| > 1$ and let $\nu_1(M_1) = a$. Then

$$I : x_1^a = (M_1/x_1^a, M_2, \ldots, M_r) \text{ and } I + (x_1^a) = (x_1^a, M_2, \ldots, M_r).$$

Since that $(M_1/x_1^a, M_2, \ldots, M_r)$ and $(x_1^a, M_2, \ldots, M_r)$ are complete intersection monomial ideals with the number of variables less that n, we deduce that $I : x_1^a$ and $I+(x_1^a)$ are 0-clean, by induction hypothesis. Set $J := (M_2, \ldots, M_r)$. Since that

$$\min(I + (x_1^a)) = \{P + (x_1) : P \in \min(J)\}$$

and

$$\min(I) = \{P + (x_i) : P \in \min(J) \text{ and } x_i | M_1\}.$$

we conclude that $\min(I+(x_1^a)) \subset \min(I)$ and so x_1^a is a cleaner monomial. □

5.3. COHEN-MACAULAY MONOMIAL IDEALS OF CODIMENSION 2

Proposition 2.3 from [14] says that if $I \subset S$ is a squarefree monomial ideal with 2-linear resolution, then after suitable renumbering of the variables, one has the following property:

if $x_ix_j \in I$ with $i \neq j$, $k > i$ and $k > j$, then either x_ix_k or x_jx_k belongs to I.

Let I has a 2-linear resolution and the monomials in $G(I)$ be ordered by the lexicographical order induced by $x_n > x_{n-1} > \ldots > x_1$. Let $u = x_sx_t > v = x_ix_j$ be squarefree monomials in $G(I)$ with $s < t$ and $i < j$. We have $t \geq j$. If $t = j$, then $x_s(v/x_i) = u \in G(I)$. If $t > j$ then by the above property either $x_ix_t \in G(I)$ or $x_jx_t \in G(I)$. This immediately implies the following lemma.

LEMMA 5.3. *If I is a squarefree monomial ideal generated in degree 2 which has a linear resolution, then after suitable renumbering of the variables, I is weakly polymatroidal.*

THEOREM 5.4. *Let $I \subset S$ be a monomial ideal which is Cohen-Macaulay and of codimension* 2*. Then S/I is* 0*-clean.*

Proof. Since I has no embedded prime ideals, if we show that I^p is 0-clean then it follows from Theorem 3.3 that I is 0-clean. Let Δ be a simplicial complex with $I_\Delta = I^p$. Since I is Cohen-Macaulay, by Corollary 1.6.3. of [13], I_Δ is Cohen-Macaulay, too. In particular, $I_\Delta^\vee$ has linear resolution, by the Eagon-Reiner theorem [8]. It follows from Lemma 5.3 and Theorems 1.7 and 4.1 that $I_\Delta = I^p$ is 0-clean, as desired. □

5.4. MONOMIAL IDEALS OF FOREST TYPE WHICH HAVE NO EMBEDDED PRIME IDEAL

We recall some notions from [26]:

Let I be a monomial ideal with $G(I) = \{u_1, \dots, u_r\}$. A variable x_i is called a **free variable** of I if there exists a $1 \leq t \leq r$ such that $x_i | u_t$ and $x_i \nmid u_j$ for any $j \neq t$. A monomial u_t is called a **leaf** of $G(I)$ if u_t is the only generator of I, or there exists a $1 \leq j \leq r$, $j \neq t$ such that $\gcd(u_t, u_i) | \gcd(u_t, u_j)$ for all $i \neq t$. In this case u_j is called a **branch** of u_t. We say that I is a **monomial ideal of forest type** if any subset of $G(I)$ has a leaf. A simplicial complex Δ is a **simplicial forest** in sense of [10] if $I(\Delta)$ is a monomial ideal of forest type.

LEMMA 5.5. *Let $I \subset S$ be a monomial ideal and u a monomial in S which is regular over S/I. If I is k-clean then $I + (u)$ and $I : u$ are k-clean.*

Proof. Since u is regular over S/I, we have $I : u = I$. This implies that $I : u$ is k-clean. It remains to show that $I + (u)$ is k-clean.

If I is a prime ideal then, by using induction on the $|\mathrm{supp}(u)|$, it is easily verified that $I + (u)$ is k-clean. So suppose that I is not prime. We use induction on the k-cleanness length of I. Let v be a cleaner monomial of I with $|\mathrm{supp}(v)| \leq k + 1$. We claim that v is a cleaner monomial of $J = I + (u)$. Since u is regular over S/I and $\min(I + (v)) \subseteq \min(I)$, we have $\gcd(u, w) = 1$ for all $w \in G(I) \cup \{v\}$. It follows that $J : v = (I : v) + (u)$ and u is regular on $S/I + (v)$ and $S/I : v$. Now, by induction hypothesis, $J : v = (I : v) + (u)$ and $J + (v) = (I + (v)) + (u)$ are k-clean.

Now let $P \in \min(J + (v))$. Then there exists x_i with $x_i | u$ such that $x_i \in P$. We have $P \backslash x_i \in \min(I + (v))$ and so $P \backslash x_i \in \min(I)$. It follows that $P \in \min(J)$, as desired. □

THEOREM 5.6. *Let $I \subset S$ be a monomial ideal of forest type which has no embedded prime ideal. Then I is 0-clean.*

Proof. Our argument uses an idea from the proof presented in [26, Theorem 3.4.]. We use induction on n the number of variables. Let I be minimally generated by $u_1, \dots, u_r$. Let x_i be a free variable of I. Then there exists $1 \le j \le r$ such that $x_i | u_j$. Let $\nu_i(u_j) = a$ and set $u' = u_j / x_i^a$. It is clear that

$$I : x_i^a = (u_1, \dots, u_{r-1}, u') \text{ and } I + (x_i^a) = (u_1, \dots, u_{r-1}, x_i^a).$$

By Lemma 3.1 of [26], $I : x_i^a = (u_1, \dots, u_{r-1}, u')$ is a monomial ideal of forest type. Furthermore, the minimal prime ideals of $(u_1, \dots, u_{r-1}, u')$ are exactly the prime minimal ideals of I which does not contain x_i. Therefore $I : x_i^a$ has no embedded prime ideal and so it is 0-clean, by induction. On the other hand, $(u_1, \dots, u_{r-1})$ is a monomial ideal of forest type and it has no embedded prime ideal. It follows from induction hypothesis that $(u_1, \dots, u_{r-1})$ is 0-clean. Finally, Lemma 5.5 obtains that $I + (x_i^a)$ is 0-clean.

Note that

$$\min(I + (x_i^a)) = \{Q + (x_i) : Q \in \min((u_1, \dots, u_{r-1}))\}$$

and

$$\min(I) = \{Q + (x_j) : Q \in \min((u_1, \dots, u_{r-1})),\ x_j | u_r\}.$$

Since $x_i | u_r$, it follows that $\min(I + (x_i^a)) \subseteq \min(I)$. Therefore x_i^a is a cleaner monomial. □

The **nonface complex** or the **Stanley-Reisner complex** of I is denoted by $\delta_{\mathcal{N}}(I)$ and it is the simplicial complex over a set of vertices $\{v_1, \dots, v_n\}$ defined by

$$\delta_{\mathcal{N}}(I) = \{\{v_{i_1}, \dots, v_{i_s}\} : x_{i_1} \dots x_{i_s} \notin I\}.$$

Let $I(\Delta)$ be the facet ideal of a simplicial complex Δ. Set $\Delta_N := \delta_{\mathcal{N}}(I(\Delta))$.

COROLLARY 5.7. *Let Δ be a simplicial forest. Then Δ_N is vertex-decomposable.*

Proof. Since $I(\Delta)$ is a monomial ideal of forest type, $I_{\Delta_N} = I(\Delta)$ is 0-clean, by Theorem 5.6. It follows from Theorem 4.1 that Δ_N is vertex decomposable. □

Remark 5.8. In [26, Theorem 3.4.], it was shown that every monomial ideal of forest type is pretty clean. A clean monomial ideal is a pretty clean ideal which has no embedded prime ideal. Hence it follows from [26, Theorem 3.4.] that every monomial ideal of forest type with no embedded prime ideal is clean. Theorem 5.6 improves this result.

5.5. SYMBOLIC POWERS OF STANLEY-REISNER IDEALS OF MATROID COMPLEXES

Let Δ be a simplicial complex and let $I_\Delta^{(m)}$ denote the mth symbolic power of I_Δ. Minh and Trung [19] and Varbaro [30] independently proved that Δ is a matroid if and only if $I_\Delta^{(m)}$ is Cohen-Macaulay for all $m \in \mathbb{N}$. Later, in [28], Terai and Trung showed that Δ is a matroid if and only if $I_\Delta^{(m)}$ is Cohen-Macaulay for some integer $m \geq 3$. Recently, Bandari and Soleyman Jahan [1] proved that if Δ is a matroid, then $I_\Delta^{(m)}$ is clean for all $m \in \mathbb{N}$. In this section, we improve this result by showing that if Δ is a matroid, then $I_\Delta^{(m)}$ is 0-clean for all $m \in \mathbb{N}$.

THEOREM 5.9. *Let Δ be a matroid complex with $I = I_\Delta$. Then for all $m \geq 1$, $I^{(m)}$ is 0-clean.*

Proof. Let $\Delta = \langle F_1, \dots, F_t \rangle$. Then $I = I_\Delta = \bigcap_{i=1}^t P_{F_i^c}$ and $(I_\Delta)^{(m)} = \bigcap_{i=1}^t (P_{F_i^c})^{(m)}$. Since Δ is a matroid and I is Cohen-Macaulay, it follows that $I^{(m)}$ has no embedded prime ideal. Therefore if we show that $(I^{(m)})^p$ is 0-clean then the proof is completed, by Theorem 3.3.

In [1] the authors introduced an ordering on the variables of S^p and showed that $((I^{(m)})^p)^\vee$ has linear quotients with respect to this ordering. We improve this result by considering the same ordering to show that $((I^{(m)})^p)^\vee$ is weakly polymatroidal. Then by Theorem 1.7 and Corollary 4.4, $(I^{(m)})^p$ is 0-clean. We use some notations of the proof of [1, Theorem 2.1.]. It is known that Δ^c is a matroid. Let $\dim(\Delta^c) = r - 1$. We set $J = ((I^{(m)})^p)^\vee$. Then

$$G(J) = \{x_{i_1,j_1} x_{i_2,j_2} \dots x_{i_r,j_r} : \{i_1, \dots, i_r\} \text{ is a facet of } \Delta^c\}$$

where $1 \leq j_l \leq m$ and $\sum_{l=1}^r j_l \leq m + r - 1$.

Consider the order $<$ on the variables of S^α by setting $x_{i,j} > x_{i',j'}$ if either $j < j'$, or $j = j'$ and $i < i'$. Let $u, v \in G(J)$ with $u = x_{i_r,j_r} \dots x_{i_2,j_2} x_{i_1,j_1} > v = x_{i'_r,j'_r} \dots x_{i'_2,j'_2} x_{i'_1,j'_1}$ such that $x_{i_l,j_l} = x_{i'_l,j'_l}$ for all $l > t$ and $x_{i_t,j_t} > x_{i'_t,j'_t}$. We have two cases:

Case 1. $x_{i_t} | x_{i'_r} \dots x_{i'_{t+1}} x_{i'_t}$. Let $i'_l = i_t$. It is clear that $j_t < j'_l$. In particular, $x_{i_t,j_t}(v/x_{i'_l,j'_l}) \in G(J)$.

Case 2. $x_{i_t} \nmid x_{i'_r} \dots x_{i'_{t+1}} x_{i'_t}$. Since $I_{\Delta^\vee}$ is matroidal, it follows from [12, Lemma 3.1.] that there exists $i'_l \notin \{i_1, \dots, i_r\}$ such that $x_{i_t}(x_{i'_r} \dots x_{i'_1}/x_{i'_l}) \in I_{\Delta^\vee}$. Therefore

$$x_{i'_r,j'_r} \dots x_{i'_{l-1},j'_{l-1}} x_{i'_{l+1},j'_{l+1}} \dots x_{i_t,j_t} x_{i'_{t-1},j'_{t-1}} \dots x_{i'_1,j'_1} \in G(J).$$

Therefore J is weakly polymatroidal, as desired. □

It follows from Theorem 5.9 that we can add the condition "0-cleanness of $I_\Delta^{(m)}$ for all $m > 0$" to [1, Corollary 2.3.]:

Corollary 5.10. *Let Δ be a pure simplicial complex and $I = I_\Delta \subset S$. Then the following conditions are equivalent:*

(i) *Δ is a matroid;*
(ii) *$S/I^{(m)}$ is 0-clean for all integer $m > 0$;*
(iii) *$S/I^{(m)}$ is clean for some integer $m > 0$;*
(iv) *$S/I^{(m)}$ is clean for some integer $m \geq 3$;*
(v) *$S/I^{(m)}$ is Cohen-Macaulay for some integer $m \geq 3$;*
(vi) *$S/I^{(m)}$ is Cohen-Macaulay for all integer $m > 0$.*

Cowsik and Nori in [6] proved that for any homogeneous radical ideal I in the polynomial ring S, all the powers of I are Cohen-Macaulay if and only if I is a complete intersection. We call the simplicial complex Δ **complete intersection** if I_Δ is a complete intersection ideal. Therefore the simplicial complex Δ is a complete intersection if and only if I_Δ^m is Cohen-Macaulay for any $m \in \mathbb{N}$ ([29, Theorem 3]). We improve this result in the following. By the fact that if I_Δ^m is Cohen-Macaulay then I_Δ^m is equal to the mth symbolic power $I_\Delta^{(m)}$ of I_Δ we have

Corollary 5.11. *Let Δ be a pure simplicial complex and $I = I_\Delta \subset S$. Then the following conditions are equivalent:*

(i) *Δ is a complete intersection;*
(ii) *S/I^m is 0-clean for all integer $m > 0$;*
(iii) *S/I^m is clean for some integer $m > 0$;*
(iv) *S/I^m is clean for some integer $m \geq 3$;*
(v) *S/I^m is Cohen-Macaulay for some integer $m \geq 3$;*
(vi) *S/I^m is Cohen-Macaulay for all integer $m > 0$.*

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