THE PADOVAN-CIRCULANT SEQUENCES AND THEIR APPLICATIONS

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In this paper, we define the generalized Padovan-circulant sequence and the Padovan-circulant sequences of the first, second, third and fourth kind by using the circulant matrices which are obtained from the characteristic polynomial of the Padovan sequence. Then we obtain miscellaneous properties of these sequences. Also, we consider the cyclic groups which are generated by the generating matrices and the auxiliary equations of the generalized Padovan-circulant sequence and the Padovan-circulant sequences of the first, second, third and fourth kind and then we study the orders of these cyclic groups. Furthermore, we extend the Padovan-circulant sequences of the first, second, third and fourth kind to groups and then we examine these sequences in finite groups. Finally, we obtain the lengths of the periods of the Padovan-circulant orbits of the first, second, third and fourth kind of the quarternion group Q_8 and the dihedral group D_n as applications of the results obtained.

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1. INTRODUCTION AND PRELIMINARIES

In [3], P.J. Davis defined the circulant matrix $C_n = [c_{ij}]_{n\times n}$associated with the numbers $c_0, c_1, \ldots, c_{n-1}$ as follows:

$$C_n = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & \cdots & c_3 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-3} & \cdots & c_0 & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}.$$

The $(n-1)$th degree polynomial $P(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$ is called the associated polynomial of the circulant matrix C_n.

For more information on the circulant matrix C_n, see [12, 17, 20].

The Padovan sequence is the sequence of integers $P(n)$ defined by initial values $P(0) = P(1) = P(2) = 1$ and recurrence relation

$$P(n) = P(n-2) + P(n-3).$$

The Padovan sequence is

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, \ldots.$$

It is easy to see that the characteristic polynomial of the Padovan sequence is

$$f(x) = x^3 - x - 1.$$

For more information on this sequence, see [11].

Let the $(n+k)$th term of a sequence be defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants.

In [13], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then by an inductive argument he obtained that

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

In Section 2, we define the recurrence sequences by the aid of the circulant matrix C_4 which is obtained by using the characteristic polynomial of the Padovan sequence. In [4–8, 16], the authors obtained the cyclic groups via some special matrices. In Section 3, we consider the multiplicative orders of the circulant matrix C_4 and the Padovan-circulant matrices of the first, second, third and fourth kind working to modulo m which are defined by the aid of the recurrence relations of the generalized Padovan-circulant sequence. Then we obtain the rules for the orders of the cyclic groups which are generated by reducing these matrices modulo m. The study of recurrence sequences in groups began with the earlier work of Wall [22] where the ordinary Fibonacci sequences in cyclic groups were investigated. The concept extended to some special linear recurrence sequences by several authors; see for example, [1, 2, 4, 7–10, 14, 16, 18, 19, 21]. In Section 4, we define the Padovan-circulant orbits of the first, second, third and fourth kind and then we study these sequences in finite groups. Also in this section, we obtain the lengths of the periods of the Padovan-circulant orbits of the first, second, third and fourth kind of the quarternion group Q_8 and the dihedral group D_n for generating pair (x, y).

2. THE PADOVAN-CIRCULANT SEQUENCES

We can write the following circulant matrix for the polynomial $f(x)$:

$$C_4 = \begin{bmatrix} -1 & -1 & 1 & 0 \\ -1 & 1 & 0 & -1 \\ 1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 1 \end{bmatrix}.$$

Define the generalized Padovan-circulant sequence by using the matrices C_4 as shown:

$$x_n = \begin{cases} x_{n-2} - x_{n-3} - x_{n-4}, & n \equiv 1 \bmod 4, \\ -x_{n-2} + x_{n-4} - x_{n-5}, & n \equiv 2 \bmod 4, \\ -x_{n-3} - x_{n-4} + x_{n-6}, & n \equiv 3 \bmod 4, \\ x_{n-4} - x_{n-5} - x_{n-6}, & n \equiv 0 \bmod 4, \end{cases} \quad \text{for } n > 4, \tag{2.1}$$

where $x_1 = x_2 = x_3 = 0$ and $x_4 = 1$.

For $n \geq 1$, by an inductive argument, we may write

$$x_{4n+1} - 2x_{4n-1} - x_{4n} = (-1)^n$$

and

$$x_{4n+2} + 2x_{4n} + x_{4n-1} = 1.$$

It is easy to show that

$$(C_4)^n = \begin{bmatrix} (-1)^n x_{4n+4} & x_{4n+3} & (-1)^n x_{4n+2} & x_{4n+1} \\ x_{4n+3} & x_{4n+4} & x_{4n+1} & x_{4n+2} \\ (-1)^n x_{4n+2} & x_{4n+1} & (-1)^n x_{4n+4} & x_{4n+3} \\ x_{4n+1} & x_{4n+2} & x_{4n+3} & x_{4n+4} \end{bmatrix}, \tag{2.2}$$

for $n \geq 0$, which can be proved by mathematical induction. Since $\det C_4 = 5$, we can write the Simpson formula for the generalized Padovan-circulant sequence as:

$$\left[(-1)^n (x_{4n+2})^2 + (-1)^{n+1} (x_{4n+4})^2\right]^2 + \left[(x_{4n+1})^2 - (x_{4n+3})^2\right]^2$$
$$+2(-1)^{n+1}\left[(x_{4n+2})^2 + (x_{4n+4})^2\right]\left[(x_{4n+1})^2 + (x_{4n+3})^2\right] +$$
$$8(-1)^n x_{4n+1}x_{4n+2}x_{4n+3}x_{4n+4} = (5)^n .$$

Define the Padovan-circulant sequences of the first, second, third and fourth kind by using (2.1) as shown, respectively:

(2.3) $x_n^1 = x_{n-2}^1 - x_{n-3}^1 - x_{n-4}^1$ for $n \geq 5$ where $x_1^1 = x_2^1 = x_3^1 = 0$ and $x_4^1 = 1$,

(2.4) $x_n^2 = -x_{n-2}^2 + x_{n-4}^2 - x_{n-5}^2$ for $n \geq 6$ where $x_1^2 = x_2^2 = x_3^2 = x_4^2 = 0$ and $x_5^2 = 1$,

(2.5) $x_n^3 = -x_{n-3}^3 - x_{n-4}^3 + x_{n-6}^3$ for $n \geq 7$ where $x_1^3 = x_2^3 = x_3^3 = x_4^3 = x_5^3 = 0$ and $x_6^3 = 1$,

and

(2.6) $x_n^4 = x_{n-4}^4 - x_{n-5}^4 - x_{n-6}^4$ for $n \geq 7$ where $x_1^4 = x_2^4 = x_3^4 = x_4^4 = x_5^4 = 0$ and $x_6^4 = 1$.

Note that the generating functions of the Padovan-circulant sequences of the first, second, third and fourth kind are as follows, respectively:

$$g^{(1)}(x) = \frac{x^3}{x^4 + x^3 - x^2 + 1},$$

$$g^{(2)}(x) = \frac{x^4}{x^5 - x^4 + x^2 + 1},$$

$$g^{(3)}(x) = \frac{x^5}{-x^6 + x^4 + x^3 + 1}$$

and

$$g^{(4)}(x) = \frac{x^5}{x^6 + x^5 - x^4 + 1}.$$

By(2.3), (2.4), (2.5) and (2.6), we can write the following companion matrices:

$$M_P^{(1)} = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$M_P^{(2)} = \begin{bmatrix} 0 & -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M_P^{(3)} = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$M_P^{(4)} = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The matrices $M_P^{(1)}$, $M_P^{(2)}$, $M_P^{(3)}$ and $M_P^{(4)}$ are said to be the Padovan-circulant matrices of the first, second, third and fourth kind.

By induction on n, we derive that

$$\left(M_P^{(1)}\right)^n = \begin{bmatrix} x^1_{n+4} & x^1_{n+5} & -x^1_{n+3} - x^1_{n+2} & -x^1_{n+3} \\ x^1_{n+3} & x^1_{n+4} & -x^1_{n+2} - x^1_{n+1} & -x^1_{n+2} \\ x^1_{n+2} & x^1_{n+3} & -x^1_{n+1} - x^1_{n} & -x^1_{n+1} \\ x^1_{n+1} & x^1_{n+2} & -x^1_{n} - x^1_{n-1} & -x^1_{n} \end{bmatrix} \text{ for } n > 1 \tag{2.7}$$

and

(2.8)

$$\left(M_P^{(2)}\right)^n = \begin{bmatrix} x^2_{n+5} & x^2_{n+6} & x^2_{n+3} - x^2_{n+2} & x^2_{n+4} - x^2_{n+3} & -x^2_{n+4} \\ x^2_{n+4} & x^2_{n+5} & x^2_{n+2} - x^2_{n+1} & x^2_{n+3} - x^2_{n+2} & -x^2_{n+3} \\ x^2_{n+3} & x^2_{n+4} & x^2_{n+1} - x^2_{n} & x^2_{n+2} - x^2_{n+1} & -x^2_{n+2} \\ x^2_{n+2} & x^2_{n+3} & x^2_{n} - x^2_{n-1} & x^2_{n+1} - x^2_{n} & -x^2_{n+1} \\ x^2_{n+1} & x^2_{n+2} & x^2_{n-1} - x^2_{n-2} & x^2_{n} - x^2_{n-1} & -x^2_{n} \end{bmatrix} \text{ for } n > 2$$

(2.9)

$$\left(M_P^{(3)}\right)^n = \begin{bmatrix} x^3_{n+6} & x^3_{n+7} & x^3_{n+8} & x^3_{n+3} - x^3_{n+5} & x^3_{n+4} & x^3_{n+5} \\ x^3_{n+5} & x^3_{n+6} & x^3_{n+7} & x^3_{n+2} - x^3_{n+4} & x^3_{n+3} & x^3_{n+4} \\ x^3_{n+4} & x^3_{n+5} & x^3_{n+6} & x^3_{n+1} - x^3_{n+3} & x^3_{n+2} & x^3_{n+3} \\ x^3_{n+3} & x^3_{n+4} & x^3_{n+5} & x^3_{n} - x^3_{n+2} & x^3_{n+1} & x^3_{n+2} \\ x^3_{n+2} & x^3_{n+3} & x^3_{n+4} & x^3_{n-1} - x^3_{n+1} & x^3_{n} & x^3_{n+1} \\ x^3_{n+1} & x^3_{n+2} & x^3_{n+3} & x^3_{n-2} - x^3_{n} & x^3_{n-1} & x^3_{n} \end{bmatrix} \text{ for } n > 2$$

(2.10)

$$\left(M_P^{(4)}\right)^n = \begin{bmatrix} x_{n+6}^4 & x_{n+7}^4 & x_{n+8}^4 & x_{n+9}^4 & -x_{n+5}^4 - x_{n+4}^4 & -x_{n+5}^4 \\ x_{n+5}^4 & x_{n+6}^4 & x_{n+7}^4 & x_{n+8}^4 & -x_{n+4}^4 - x_{n+3}^4 & -x_{n+4}^4 \\ x_{n+4}^4 & x_{n+5}^4 & x_{n+6}^4 & x_{n+7}^4 & -x_{n+3}^4 - x_{n+2}^4 & -x_{n+3}^4 \\ x_{n+3}^4 & x_{n+4}^4 & x_{n+5}^4 & x_{n+6}^4 & -x_{n+2}^4 - x_{n+1}^4 & -x_{n+2}^4 \\ x_{n+2}^4 & x_{n+3}^4 & x_{n+4}^4 & x_{n+5}^4 & -x_{n+1}^4 - x_{n}^4 & -x_{n+1}^4 \\ x_{n+1}^4 & x_{n+2}^4 & x_{n+3}^4 & x_{n+4}^4 & -x_{n}^4 - x_{n-1}^4 & -x_{n}^4 \end{bmatrix} \text{ for } n > 1$$

for $n \geq 1$. It is easy to see that $\det\left(M_P^{(1)}\right)^n = \det\left(M_P^{(4)}\right)^n = 1$ and $\det\left(M_P^{(2)}\right)^n = \det\left(M_P^{(3)}\right)^n = (-1)^n$.

It is clear that each of the eigenvalues of the matrices $M_P^{(1)}$, $M_P^{(2)}$, $M_P^{(3)}$ and $M_P^{(4)}$ are distinct. Let $\left\{\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)}, \alpha_4^{(1)}\right\}$, $\left\{\alpha_1^{(2)},\ \alpha_2^{(2)},\ \alpha_3^{(2)},\ \alpha_4^{(2)},\ \alpha_5^{(2)}\right\}$ and $\left\{\alpha_1^{(3)},\ \alpha_2^{(3)},\ \alpha_3^{(3)},\ \alpha_4^{(3)},\ \alpha_5^{(3)},\ \alpha_6^{(3)}\right\}$ be the sets of the eigenvalues of the matrices $M_P^{(1)}$, $M_P^{(2)}$ and $M_P^{(3)}$, respectively and let $V^{(k)}$ be a $(k+3) \times (k+3)$ Vandermonde matrix as follows:

$$V^{(k)} = \begin{bmatrix} \left(\alpha_1^{(k)}\right)^{k+2} & \left(\alpha_2^{(k)}\right)^{k+2} & \cdots & \left(\alpha_{k+2}^{(k)}\right)^{k+2} & \left(\alpha_{k+3}^{(k)}\right)^{k+2} \\ \left(\alpha_1^{(k)}\right)^{k+1} & \left(\alpha_2^{(k)}\right)^{k+1} & \cdots & \left(\alpha_{k+2}^{(k)}\right)^{k+1} & \left(\alpha_{k+3}^{(k)}\right)^{k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{(k)} & \alpha_2^{(k)} & \cdots & \alpha_{k+2}^{(k)} & \alpha_{k+3}^{(k)} \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

where $1 \leq k \leq 3$.

Suppose that

$$W_k^i = \begin{bmatrix} \left(\alpha_1^{(k)}\right)^{n+k+3-i} \\ \left(\alpha_2^{(k)}\right)^{n+k+3-i} \\ \vdots \\ \left(\alpha_{k+3}^{(k)}\right)^{n+k+3-i} \end{bmatrix}$$

and $V_j^{(k,i)}$ is a $(k+3) \times (k+3)$ matrix obtained from $V^{(k)}$ by replacing the jth column of $V^{(k)}$ by W_k^i for all $1 \leq k \leq 3$.

Then we can give the Binet formulas for the Padovan-circulant sequences of the first, second and third kind with the following Theorem.

Theorem 2.1. *Let x_n^k be the nth term of the sequence of the kth kind for all $1 \leq k \leq 3$. Then*

$$m_{ij}^{(k,n)} = \frac{\det V_j^{(k,i)}}{\det V^{(k)}}$$

where $\left(M_P^{(k)}\right)^n = \left[m_{ij}^{(k,n)}\right]$ *such that* $1 \leq k \leq 3$.

Proof. Since the eigenvalues of the matrix $M_P^{(k)}$ are distinct, the matrix $M_P^{(k)}$ is diagonalizable. Let

$$D^{(1)} = diag\left(\alpha_1^{(1)},\ \alpha_2^{(1)},\ \alpha_3^{(1)}, \alpha_4^{(1)}\right),$$

$$D^{(2)} = diag\left(\alpha_1^{(2)},\ \alpha_2^{(2)},\ \alpha_3^{(2)},\ \alpha_4^{(2)},\ \alpha_5^{(2)}\right)$$

and

$$D^{(3)} = diag\left(\alpha_1^{(3)},\ \alpha_2^{(3)},\ \alpha_3^{(3)},\ \alpha_4^{(3)},\ \alpha_5^{(3)},\ \alpha_6^{(3)}\right),$$

then it is readily seen that $M_P^{(k)}V^{(k)} = V^{(k)}D^{(k)}$. Since the matrix $V^{(k)}$ is invertible, $\left(V^{(k)}\right)^{-1} M_P^{(k)}V^{(k)} = D^{(k)}$. Thus, the matrix $M_P^{(k)}$ is similar to $D^{(k)}$. So we get $\left(M_P^{(k)}\right)^n V^{(k)} = V^{(k)}\left(D^{(k)}\right)^n$ for $n \geq 1$. Then we write the following linear system of equations for $n \geq 1$:

$$\begin{aligned}
&m_{i1}^{(k,n)}\left(\alpha_1^{(k)}\right)^{k+2} + m_{i2}^{(k,n)}\left(\alpha_1^{(k)}\right)^{k+1} + \cdots + m_{ik+2}^{(k,n)} = \left(\alpha_1^{(k)}\right)^{n+k+3-i}\\
&m_{i1}^{(k,n)}\left(\alpha_2^{(k)}\right)^{k+2} + m_{i2}^{(k,n)}\left(\alpha_2^{(k)}\right)^{k+1} + \cdots + m_{ik+2}^{(k,n)} = \left(\alpha_2^{(k)}\right)^{n+k+3-i}\\
&\vdots\\
&m_{i1}^{(k,n)}\left(\alpha_{k+3}^{(k)}\right)^{k+2} + m_{i2}^{(k,n)}\left(\alpha_{k+3}^{(k)}\right)^{k+1} + \cdots + m_{ik+2}^{(k,n)} = \left(\alpha_{k+3}^{(k)}\right)^{n+k+3-i}.
\end{aligned}$$

So, we obtain that

$$m_{ij}^{(1,n)} = \frac{\det V_j^{(1,i)}}{\det V^{(1)}} foreachi, j = 1, 2, 3, 4,$$

$$m_{ij}^{(2,n)} = \frac{\det V_j^{(2,i)}}{\det V^{(2)}} foreachi, j = 1, 2, 3, 4, 5$$

and

$$m_{ij}^{(3,n)} = \frac{\det V_j^{(3,i)}}{\det V^{(3)}} foreachi, j = 1, 2, 3, 4, 5, 6. \quad \square$$

If we choose

$$V^{(4)}=\begin{bmatrix}\left(\alpha_1^{(4)}\right)^5 & \left(\alpha_2^{(4)}\right)^5 & \cdots & \left(\alpha_5^{(4)}\right)^5 & \left(\alpha_6^{(4)}\right)^5\\ \left(\alpha_1^{(4)}\right)^4 & \left(\alpha_2^{(4)}\right)^4 & \cdots & \left(\alpha_5^{(4)}\right)^4 & \left(\alpha_6^{(4)}\right)^4\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \alpha_1^{(4)} & \alpha_2^{(4)} & \cdots & \alpha_5^{(4)} & \alpha_6^{(4)}\\ 1 & 1 & \cdots & 1 & 1\end{bmatrix}$$

and

$$W_4^i=\begin{bmatrix}\left(\alpha_1^{(4)}\right)^{n+6-i}\\ \left(\alpha_2^{(4)}\right)^{n+6-i}\\ \vdots\\ \left(\alpha_6^{(4)}\right)^{n+6-i}\end{bmatrix},$$

then we obtain the Binet formula for the Padovan-circulant sequence of the fourth kind as follows:

$$m_{ij}^{(4,n)}=\frac{\det V_j^{(4,i)}}{\det V^{(4)}}\text{ for each } i,j=1,2,3,4,5,6$$

such that $\left(M_P^{(4)}\right)^n=\left[m_{ij}^{(4,n)}\right]$.

3. THE CYCLIC GROUPS *VIA* THE MATRICES C_4, $M_P^{(1)}$, $M_P^{(2)}$, $M_P^{(3)}$ AND $M_P^{(4)}$

For a given matrix $A=[a_{ij}]$ of integers, $A \pmod m$ means that the entries of A are reduced modulo m. Let $\langle A\rangle_m=\{(A)^n \pmod m \mid n\geq 0\}$. If $\gcd(\det A,m)=1$, $\langle A\rangle_m$ is the cyclic group. We denote cardinal of the set $\langle A\rangle_m$by $|\langle A\rangle_m|$. Since $\det C_4=5$, it is clear that the set $\langle C_4\rangle_m$ is a cyclic group for every positive integer m such that $\gcd(5,m)=1$. Similarly, the sets $\left\langle M_P^{(1)}\right\rangle_m$, $\left\langle M_P^{(2)}\right\rangle_m$, $\left\langle M_P^{(3)}\right\rangle_m$ and $\left\langle M_P^{(4)}\right\rangle_m$ are cyclic groups for every positive integer m.

Now we consider the cyclic groups which are generated by the matrices C_4, $M_P^{(1)}$, $M_P^{(2)}$, $M_P^{(3)}$ and $M_P^{(4)}$.

THEOREM 3.1. *Let p be a prime and let $\langle G\rangle_{p^\varepsilon}$ be any of the cyclic groups of $\langle C_4\rangle_{p^\varepsilon}$, $\left\langle M_P^{(1)}\right\rangle_{p^\varepsilon}$, $\left\langle M_P^{(2)}\right\rangle_{p^\varepsilon}$, $\left\langle M_P^{(3)}\right\rangle_{p^\varepsilon}$ and $\left\langle M_P^{(4)}\right\rangle_{p^\varepsilon}$ such that $\varepsilon\in N$. If*

u is the largest positive integer such that $\left|\langle G\rangle_p\right| = \left|\langle G\rangle_{p^u}\right|$, then $\left|\langle G\rangle_{p^v}\right| = p^{v-u}\cdot\left|\langle G\rangle_p\right|$ for every $v\geq u$. In particular, if $\left|\langle G\rangle_p\right| \neq \left|\langle G\rangle_{p^2}\right|$, then $\left|\langle G\rangle_{p^v}\right| = p^{v-1}\cdot\left|\langle G\rangle_p\right|$ for every $v\geq 2$.

Proof. Let us consider the cyclic group $\left\langle M_P^{(1)}\right\rangle_{p^\varepsilon}$. Suppose that a is a positive integer and $\left|\left\langle M_P^{(1)}\right\rangle_{p^\varepsilon}\right|$ is denoted by $k(p^\varepsilon)$. If $\left(M_P^{(1)}\right)^{k\left(p^{a+1}\right)} \equiv I \pmod{p^{a+1}}$, then $\left(M_P^{(1)}\right)^{k\left(p^{a+1}\right)} \equiv I \pmod{p^a}$ where I is a 4×4 identity matrix. Thus we obtain that $k(p^a)$ divides $k\left(p^{a+1}\right)$. On the other hand, writing $\left(M_P^{(1)}\right)^{k(p^a)} = I + \left(m_{ij}^{(a)}\cdot p^a\right)$, by the binomial theorem, we obtain

$$\left(M_P^{(1)}\right)^{k(p^a)\cdot p} = \left(I+\left(m_{ij}^{(a)}\cdot p^a\right)\right)^p = \sum_{i=0}^{p}\binom{p}{i}\left(m_{ij}^{(a)}\cdot p^a\right)^i \equiv I \pmod{p^{a+1}},$$

which yields that $k\left(p^{a+1}\right)$ divides $k(p^a)\cdot p$. Then, we have that $k\left(p^{a+1}\right) = k(p^a)$ or $k\left(p^{a+1}\right) = k(p^a)\cdot p$. It is clear that $k\left(p^{a+1}\right) = k(p^a)\cdot p$ holds if and only if there is a $m_{ij}^{(a)}$ which is not divisible by p. Since u is the largest positive integer such that $k(p) = k(p^u)$, $k(p^u) \neq k\left(p^{u+1}\right)$. There is an $m_{ij}^{(u+1)}$ which is not divisible by p. Therefore, we get that $k\left(p^{u+1}\right) \neq k\left(p^{u+2}\right)$. To complete the proof we may use an inductive method on u.

There are similar proofs for the cyclic groups $\langle C_4\rangle_{p^\varepsilon}$, $\left\langle M_P^{(2)}\right\rangle_{p^\varepsilon}$, $\left\langle M_P^{(3)}\right\rangle_{p^\varepsilon}$ and $\left\langle M_P^{(4)}\right\rangle_{p^\varepsilon}$.

THEOREM 3.2. *Let $\langle G\rangle_m$ be any the cyclic groups of $\langle C_4\rangle_m$, $\left\langle M_P^{(1)}\right\rangle_m$, $\left\langle M_P^{(2)}\right\rangle_m$, $\left\langle M_P^{(3)}\right\rangle_m$ and $\left\langle M_P^{(4)}\right\rangle_m$ and let $m = \prod_{i=1}^{t} p_i^{e_i}$, $(t\geq 1)$ where p_i's are distinct primes. Then $\left|\langle G\rangle_m\right| = \operatorname{lcm}\left[\left|\langle G\rangle_{p_1^{e_1}}\right|, \left|\langle G\rangle_{p_2^{e_2}}\right|, \ldots, \left|\langle G\rangle_{p_t^{e_t}}\right|\right]$.*

Proof. Let us consider the cyclic group $\left\langle M_P^{(2)}\right\rangle_m$, where m is a positive integer. Let $\left|\left\langle M_P^{(2)}\right\rangle_{p_i^{e_i}}\right| = \lambda_i$ for $1\leq i\leq t$ and let $\left|\left\langle M_P^{(2)}\right\rangle_m\right| = \lambda$. Then by (2.8), we have

$$x_{\lambda_i+5}^2 \equiv 1 \bmod p_i^{e_i},$$
$$x_{\lambda_i+u}^2 \equiv -1 \bmod p_i^{e_i} \text{ for } u=-2,-1,0$$
$$x_{\lambda_i+v}^2 \equiv 0 \bmod p_i^{e_i} \text{ for } v=1,2,3,4,6$$

and

$$\begin{aligned} x_{\lambda+5}^{2} &\equiv 1 \bmod m, \\ x_{\lambda+u}^{2} &\equiv -1 \bmod m \text{ for } u=-2,-1,0 \\ x_{\lambda+v}^{2} &\equiv 0 \bmod m \text{ for } v=1,2,3,4,6, \end{aligned}$$

which implies that $\lambda_i | \lambda$ for all values of i. Thus it is verified that $\left|\left\langle M_P^{(2)}\right\rangle_m\right|$ equals the least common multiple of $\left|\left\langle M_P^{(2)}\right\rangle_{p_i^{e_i}}\right|$'s.

There are similar proofs for the cyclic groups $\langle C_4\rangle_{p^\varepsilon}$, $\left\langle M_P^{(1)}\right\rangle_{p^\varepsilon}$, $\left\langle M_P^{(3)}\right\rangle_{p^\varepsilon}$ and $\left\langle M_P^{(4)}\right\rangle_{p^\varepsilon}$. □

It is well-known that a sequence is periodic if, after certain points, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence.

Reducing the generalized Padovan-circulant sequence and the Padovan-circulant sequences of the first, second, third and fourth kind by a modulus m, we can get the repeating sequences, respectively denoted by

$$\{x_n(m)\} = \{x_1(m),\ x_2(m),\ x_3(m), \ldots, x_j(m), \ldots\}$$

and

$$\left\{x_n^k(m)\right\} = \left\{x_1^k(m),\ x_2^k(m),\ x_3^k(m), \ldots, x_j^k(m), \ldots\right\},$$

where $x_j(m) = x_j\,(\text{mod}m)$, $x_j^k(m) = x_j^k\,(\text{mod}m)$ and $1 \le k \le 4$. They have the same recurrence relation as in (2.1), (2.3), (2.4), (2.5) and (2.6), respectively.

THEOREM 3.3. *For* $1 \le k \le 4$, *the sequences* $\left\{x_n^k(m)\right\}$ *are simply periodic for every positive integer m. Similarly, the sequence* $\{x_n(m)\}$ *is a simply periodic sequence if* $\gcd(5,m)=1$.

Proof. Let us consider the generalized Padovan-circulant sequence and let $\gcd(5,m)=1$. Suppose that $S = \{(s_1,s_2,s_3,s_4) | 0 \le s_i \le m-1\}$. Then we have $|S| = m^4$. Since there are m^4 distinct 4-tuples of elements of $\mathbb{Z}_m$, at least one of the 4-tuples appears twice in the sequence $\{x_n(m)\}$. Thus, the subsequence following this 4-tuple repeats; that is the sequence $\{x_n(m)\}$ is periodic. So if $x_{i+4}(m) \equiv x_{j+4}(m)$, $x_{i+3}(m) \equiv x_{j+3}(m)$, $x_{i+2}(m) \equiv x_{j+2}(m)$, $x_{i+1}(m) \equiv x_{j+1}(m)$ and $i > j$, then $i \equiv j \bmod 4$. From the definition, we can easily derive

$$x_i(m) \;\equiv\; x_j(m), x_{i-1}(m) \equiv x_{j-1}(m), \ldots, x_{i-j+2}(m) \equiv x_2(m),$$

$x_{i-j+1}(m) \equiv x_1(m)$.

Thus it is verified that the sequence $\{x_n(m)\}$ is simply periodic.

There are similar proofs for the sequences $\left\{x_n^k(m)\right\}$, $(1 \leq k \leq 4)$. $\square$

We denote the periods of the sequences $\{x_n(m)\}$, $\left\{x_n^k(m)\right\}$, $(1 \leq k \leq 4)$ by $l_P(m)$, $l_P^k(m)$, respectively.

Then, we have the following useful results from (2.2), (2.7), (2.8), (2.9) and (2.10), respectively.

COROLLARY 3.4. *Let p be a prime. Then*

i. *If $p \neq 5$, then $l_P(p) = 4 \cdot \left|\langle C_4 \rangle_p\right|$.*

ii. *$l_P^k(p) = \left|\langle M_P^k \rangle_p\right|$ for $1 \leq k \leq 4$.*

Let p be a prime and let

$$A_1(p^\varepsilon) = \left\{x^n \,(\mathrm{mod} p^\varepsilon) : n \in \mathbb{Z},\ x^4 = x^2 - x - 1\right\},$$

$$A_2(p^\varepsilon) = \left\{x^n \,(\mathrm{mod} p^\varepsilon) : n \in \mathbb{Z},\ x^5 = -x^3 + x - 1\right\},$$
$$A_3(p^\varepsilon) = \left\{x^n \,(\mathrm{mod} p^\varepsilon) : n \in \mathbb{Z},\ x^6 = -x^3 - x^2 + 1\right\}$$

and

$$A_4(p^\varepsilon) = \left\{x^n \,(\mathrm{mod} p^\varepsilon) : n \in \mathbb{Z},\ x^6 = x^2 - x - 1\right\}$$

such that $\varepsilon \geq 1$. Then, it is clear that the sets $A_1(p^\varepsilon)$, $A_2(p^\varepsilon)$, $A_3(p^\varepsilon)$ and $A_4(p^\varepsilon)$ are cyclic groups.

Now we can give a relationship between the characteristic equations of the Padovan-circulant sequences of the first, second, third and fourth kind and the periods $l_P^1(m)$, $l_P^2(m)$, $l_P^3(m)$ and $l_P^4(m)$ by the following Corollary.

COROLLARY 3.5. *Let p be a prime and let $\varepsilon \in N$. Then, the cyclic groups $A_1(p^\varepsilon)$, $A_2(p^\varepsilon)$, $A_3(p^\varepsilon)$ and $A_4(p^\varepsilon)$ are isomorphic to the cyclic groups $\left\langle M_P^{(1)} \right\rangle_{p^\varepsilon}$, $\left\langle M_P^{(2)} \right\rangle_{p^\varepsilon}$, $\left\langle M_P^{(3)} \right\rangle_{p^\varepsilon}$ and $\left\langle M_P^{(4)} \right\rangle_{p^\varepsilon}$.*

4. THE PADOVAN-CIRCULANT SEQUENCES OF THE FIRST, SECOND THIRD AND FOURTH KIND IN GROUPS

Let G be a finite j-generator group and let X be the subset of $\underbrace{G \times G \times G \times \cdots \times G}_{j}$ such that $(x_1, x_2, \ldots, x_j) \in X$ if, and only if, G is generated by x_1, x_2, $\ldots$, x_j. We call $(x_1, x_2, \ldots, x_j)$ a generating j-tuple for G.

Definition 4.1. For a generating j-tuple $(x_1, x_2, \ldots, x_j) \in X$, we define the Padovan-circulant orbits of the first, second, third and fourth kind as follows, respectively:

$$a^1_{n+4} = \left(a^1_n\right)^{-1} \left(a^1_{n+1}\right)^{-1} \left(a^1_{n+2}\right)$$

for $n \geq 1$, with initial conditions

$$\begin{cases} a^1_1 = (x_1)^{-1},\ a^1_2 = x_2,\ a^1_3 = x_3,\ a^1_4 = x_4 & \text{if } j = 4, \\ a^1_1 = x_1,\ a^1_2 = (x_1)^{-1},\ a^1_3 = x_2,\ a^1_4 = x_3 & \text{if } j = 3, \\ a^1_1 = (x_1)^{-2},\ a^1_2 = x_1,\ a^1_3 = (x_1)^{-1},\ a^1_4 = x_2 & \text{if } j = 2, \end{cases}$$

$$a^2_{n+5} = \left(a^2_n\right)^{-1} \left(a^2_{n+1}\right) \left(a^2_{n+3}\right)^{-1}$$

for $n \geq 1$, with initial conditions

$$\begin{cases} a^2_1 = (x_1)^{-1},\ a^2_2 = x_2,\ a^2_3 = x_3,\ a^2_4 = x_4,\ a^2_5 = x_5 & \text{if } j = 5, \\ a^2_1 = (x_1)^{-1},\ a^2_2 = (x_1)^{-1},\ a^2_3 = x_2,\ a^2_4 = x_3,\ a^2_5 = x_4 & \text{if } j = 4, \\ a^2_1 = (x_1)^{-1},\ a^2_2 = (x_1)^{-1},\ a^2_3 = (x_1)^{-1},\ a^2_4 = x_2,\ a^2_5 = x_3 & \text{if } j = 3, \\ a^2_1 = e,\ a^2_2 = (x_1)^{-1},\ a^2_3 = (x_1)^{-1},\ a^2_4 = (x_1)^{-1},\ a^2_5 = x_2 & \text{if } j = 2, \end{cases}$$

$$a^3_{n+6} = \left(a^3_n\right) \left(a^3_{n+2}\right)^{-1} \left(a^3_{n+3}\right)^{-1}$$

for $n \geq 1$, with initial conditions

$$\begin{cases} a^3_1 = x_1,\ a^3_2 = x_2,\ a^3_3 = x_3,\ a^3_4 = x_4,\ a^3_5 = x_5,\ a^3_6 = x_6 & \text{if } j = 6, \\ a^3_1 = e,\ a^3_2 = x_1,\ a^3_3 = x_2,\ a^3_4 = x_3,\ a^3_5 = x_4,\ a^3_6 = x_5 & \text{if } j = 5, \\ a^3_1 = x_1,\ a^3_2 = e,\ a^3_3 = x_1,\ a^3_4 = x_2,\ a^3_5 = x_3,\ a^3_6 = x_4 & \text{if } j = 4, \\ a^3_1 = x_1,\ a^3_2 = x_1,\ a^3_3 = e,\ a^3_4 = x_1,\ a^3_5 = x_2,\ a^3_6 = x_3 & \text{if } j = 3, \\ a^3_1 = x_1,\ a^3_2 = x_1,\ a^3_3 = x_1,\ a^3_4 = e,\ a^3_5 = x_1,\ a^3_6 = x_2 & \text{if } j = 2, \end{cases}$$

$$a^4_{n+6} = \left(a^4_n\right)^{-1} \left(a^4_{n+1}\right)^{-1} \left(a^4_{n+2}\right)$$

for $n \geq 1$, with initial conditions

$$\begin{cases} a^4_1 = (x_1)^{-1},\ a^4_2 = x_2,\ a^4_3 = x_3,\ a^4_4 = x_4,\ a^4_5 = x_5,\ a^4_6 = x_6 & \text{if } j{=}6, \\ a^4_1 = x_1,\ a^4_2 = (x_1)^{-1},\ a^4_3 = x_2,\ a^4_4 = x_3,\ a^4_5 = x_4,\ a^4_6 = x_5 & \text{if } j{=}5, \\ a^4_1 = (x_1)^{-2},\ a^4_2 = x_1,\ a^4_3 = (x_1)^{-1},\ a^4_4 = x_2,\ a^4_5 = x_3,\ a^4_6 = x_4 & \text{if } j{=}4, \\ a^4_1 = (x_1)^{3},\ a^4_2 = (x_1)^{-2},\ a^4_3 = x_1,\ a^4_4 = (x_1)^{-1},\ a^4_5 = x_2,\ a^4_6 = x_3 & \text{if } j{=}3, \\ a^4_1 = (x_1)^{-5},\ a^4_2 = (x_1)^{3},\ a^4_3 = (x_1)^{-2},\ a^4_4 = x_1,\ a^4_5 = (x_1)^{-1},\ a^4_6 = x_2 & \text{if } j{=}2, \end{cases}$$

We denote the Padovan-circulant orbits of the first, second, third and fourth kind by $P^1_{(x_1,\ldots,x_j)}(G)$, $P^2_{(x_1,\ldots,x_j)}(G)$, $P^3_{(x_1,\ldots,x_j)}(G)$and $P^4_{(x_1,\ldots,x_j)}(G)$, respectively.

Theorem 4.1. *The Padovan-circulant orbits of the first, second, third and fourth kind of a finite group are simply periodic.*

Proof. Let us consider the Padovan-circulant orbit of the third kind $P^3_{(x_1,\ldots,x_j)}(G)$. Suppose that n is the order of G. Since there are n^6 distinct triples of elements of G, at least one of the triples appears twice in the sequence $P^3_{(x_1,\ldots,x_j)}(G)$. Thus, consider the subsequence following this triple. Because of the repeating, the sequence is periodic. Since the sequence $P^3_{(x_1,\ldots,x_j)}(G)$ is periodic, there exist natural numbers u and v, with $u \geq v$, such that

$$a^3_{u+1} = a^3_{v+1},\ a^3_{u+2} = a^3_{v+2},\ \ldots,\ a^3_{u+6} = a^3_{v+6}.$$

By the defining relation of the Padovan-circulant orbit of the third kind, we know that

$$\left(a^3_{n+6}\right)\left(a^3_{n+3}\right)\left(a^3_{n+2}\right) = a^3_n.$$

Therefore, $a^3_u = a^3_v$, and hence,

$$a^3_{u-v+1} = a^3_1,\ a^3_{u-v+2} = a^3_2,\ \ldots,\ a^3_{u-v+6} = a^3_6,$$

which implies that the orbit $P^3_{(x_1,\ldots,x_j)}(G)$ is simply periodic.

There are similar proofs for the orbits $P^1_{(x_1,\ldots,x_j)}(G)$, $P^2_{(x_1,\ldots,x_j)}(G)$ and $P^4_{(x_1,\ldots,x_j)}(G)$. $\square$

We denote the length of the period of the orbit $P^k_{(x_1,\ldots,x_j)}(G)$ by $LP^k_{(x_1,\ldots,x_j)}(G)$ for $1 \leq k \leq 4$. From the definitions of the Padovan-circulant orbits of the first, second, third and fourth kind it is clear that the lengths of the periods of these sequences in a finite group depend on the chosen generating set and the order in which the assignments of $x_1, x_2, \ldots, x_j$ are made.

We will now address the lengths of the periods of the Padovan-circulant orbits of the first, second, third and fourth kind of the quarternion group Q_8 and the dihedral group D_n for generating pair (x, y).

THEOREM 4.2. *Consider the quaternion group Q_8 where*

$$Q_8 = \left\langle x, y : x^4 = e,\ y^2 = x^2,\ y^{-1}xy = x^{-1} \right\rangle.$$

Then $LP^1_{(x,y)}(Q_8) = 14$*,* $LP^2_{(x,y)}(Q_8) = 30$*,* $LP^3_{(x,y)}(Q_8) = 62$ *and* $LP^4_{(x,\,y)}(Q_8) = 62$*.*

Proof. We prove this by direct calculation. Let us consider the Padovan-circulant orbit of the first kind. The orbit $P^1_{(x,y)}(Q_8)$ is

$$x^2,\ x,\ x^3,\ y,\ e,\ y,\ xy^3,\ e,\ x,\ x^3,\ y,\ x^2,\ y,\ xy,\ x^2,\ x,\ x^3,\ y,\ e,\ \ldots,$$

which has period 14.

There are similar proofs for the orbits $P^2_{(x,y)}(Q_8)$, $P^3_{(x,y)}(Q_8)$ and $P^4_{(x,\,y)}(Q_8)$. $\square$

THEOREM 4.3. *Consider the dihedral group D_n where*

$$D_n = \left\langle x, y : x^2 = y^2 = (xy)^n = e \right\rangle .$$

Then

$$LP^1_{(x,y)}(D_n) = \begin{cases} \frac{7n}{2}, & n \equiv 0 \bmod 4, \\ 7n, & n \equiv 2 \bmod 4, \\ 14n, & \text{otherwise}, \end{cases}$$

$$LP^2_{(x,y)}(D_n) = \begin{cases} \frac{15n}{2} \cdot \alpha, & n \equiv 0 \bmod 4, \\ 15n \cdot \alpha, & n \equiv 2 \bmod 4, \\ 30n \cdot \alpha, & \text{otherwise} \end{cases}$$

and

$$LP^3_{(x,y)}(D_n) = LP^4_{(x,y)}(D_n) = \begin{cases} \frac{31n}{2} \cdot \beta, & n \equiv 0 \bmod 4, \\ 31n \cdot \beta, & n \equiv 2 \bmod 4, \\ 62n \cdot \beta, & \text{otherwise}, \end{cases}$$

where $\alpha, \beta \in N$.

Proof. Firstly, let us consider the Padovan-circulant orbit of the first kind. The orbit $P^1_{(x,y)}(D_n)$ is

$$\begin{aligned} &x_1 = e,\ x_2 = x,\ x_3 = x,\ x_4 = y,\ \ldots, \\ &x_{15} = e,\ x_{16} = (yx)^3 y,\ x_{17} = (yx)^3 y,\ x_{18} = y(xy)^4,\ \ldots, \\ &x_{14i+1} = e,\ x_{14i+2} = (yx)^{4i-1} y,\ x_{14i+3} = (yx)^{4i-1} y,\ x_{14i+4} = y(xy)^{4i},\ \ldots. \end{aligned}$$

So we need the smallest integer i such that $4i = n \cdot \mu_1$ for $\mu_1 \in N$.
If $n \equiv 0$ mod 4, $i = \frac{n}{4}$. Thus, $LP^1_{(x,y)}(D_n) = 14 \cdot \frac{n}{4} = \frac{7n}{2}$.
If $n \equiv 2$ mod 4, $i = \frac{n}{2}$. Thus, $LP^1_{(x,y)}(D_n) = 14 \cdot \frac{n}{2} = 7n$.
If $n \equiv 1$ mod 4 or $n \equiv 3$ mod 4, $i = n$. Thus, $LP^1_{(x,y)}(D_n) = 14n$.

Secondly, let us consider the Padovan-circulant orbit of the third kind. The orbit $P^3_{(x,y)}(D_n)$ is in the following form:

$$\begin{aligned} &x_1 = x,\ x_2 = x,\ x_3 = x,\ x_4 = e,\ x_5 = x,\ x_6 = y,\ \ldots, \\ &x_{63} =, (yx)^{167} y\ x_{64} = (xy)^{312} x,\ x_{65} = (yx)^{599} y, \\ &x_{66} = (yx)^{188},\ x_{67} = (xy)^{564} x,\ x_{62i+6} = (yx)^{1248} y,\ \ldots, \\ &x_{62i\cdot\beta+1} = (yx)^{u_1\cdot 4i-1} y,\ x_{62i\cdot\beta+2} = (xy)^{u_2\cdot 4i} x,\ x_{62i\cdot\beta+3} = (yx)^{u_3\cdot 4i-1} y, \\ &x_{62i\cdot\beta+4} = (yx)^{u_4\cdot 4i},\ x_{62i\cdot\beta+5} = (xy)^{u_5\cdot 4i} x,\ x_{62i\cdot\beta+6} = (yx)^{u_6\cdot 4i} y,\ \ldots. \end{aligned}$$

where $u_1, u_2, u_3, u_5, u_6 \in N$ and u_4 is a positive odd integer such that

$$\gcd(u_1, u_2, u_3, u_4, u_5, u_6) = 1.$$

So we need the smallest integer i such that $4i = n \cdot \mu_2$ for $\mu_2 \in N$.
If $n \equiv 0$ mod 4, $i = \frac{n}{4}$. Thus, $LP^3_{(x,y)}(D_n) = 62 \cdot \frac{n}{4} \cdot \beta = \frac{31n}{2} \cdot \beta$.
If $n \equiv 2$ mod 4, $i = \frac{n}{2}$. Thus, $LP^3_{(x,y)}(D_n) = 62 \cdot \frac{n}{2} \cdot \beta = 31n \cdot \beta$.
If $n \equiv 1$ mod 4 or $n \equiv 3$ mod 4, $i = n$. Thus, $LP^3_{(x,y)}(D_n) = 62n \cdot \beta$.

There are similar proofs for the orbits $P^2_{(x,y)}(D_n)$ and $P^4_{(x,\,y)}(D_n)$. $\square$

5. CONCLUSIONS

In Section 2, we have defined the generalized Padovan-circulant sequence and the Padovan-circulant sequences of the first, second, third and fourth kind and then, we have obtained the relationships among the elements of the sequences and the generating matrices of the sequences. Also, we have given the Simpson formula of the generalized Padovan-circulant sequence. Furthermore, we have obtained the generating functions and the Binet-type formulas for the Padovan-circulant sequences of the first, second, third and fourth kind.

In Section 3, we have studied the generalized Padovan-circulant sequence and the Padovan-circulant sequences of the first, second, third and fourth kind modulo m. Also, we have obtained the cyclic groups which are generated by reducing the multiplicative orders of the generating matrices and the auxiliary equations of these sequences modulo m and then, we have studied the orders of these cyclic groups.

In Section 4, we have extended the Padovan-circulant sequences of the first, second, third and fourth kind to groups. Then we have redefined these sequences by the means of the elements of the groups and we have examined them in finite groups. Finally, we have obtained the lengths of the periods of the Padovan-circulant sequences of the first, second, third and fourth kind in the quarternion group Q_8 and the dihedral group D_n.

6. FURTHER WORK

There are many open problems in this area. Below are a few of them:

• Does there exist a relationship among the Padovan sequence and the considered sequences in this paper?

• Does there exist a formula for calculating the periods $l_P(m)$, $l_P^1(m)$, $l_P^2(m)$, $l_P^3(m)$ and $l_P^4(m)$?

• What general theories can be obtained regarding the lengths of the periods of the Padovan-circulant orbits of the first, second, third and fourth kind of a general group? For example, does there exist a decision process to determine whether, or not, a given group has finite length.

• Let us consider infinite groups such that the lengths of the periods of the Padovan-circulant orbits of the first, second, third and fourth kind of these groups are finite. To find these lengths it would be useful to have a program. This would possibly rely on using the Knuth-Bendix method, see [15].

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