GENERAL SOLUTIONS FOR THE AXIAL COUETTE FLOW OF RATE TYPE FLUIDS IN CYLINDRICAL DOMAINS

NIAT NIGAR, CONSTANTIN FETECAU and SHEHRAZ AKHTAR

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Axial Couette flow of Oldroyd-B fluids between two infinite coaxial circular cylinders is studied when the fluid motion is generated by the outer cylinder that is moving along its axis with an arbitrary time-dependent velocity. The corresponding solution for the motion through an infinite cylinder is obtained as a limiting case of previous general solution. Both solutions for the dimensionless fluid velocity satisfy all imposed initial and boundary conditions and can be easily particularized to give the similar solutions corresponding to Maxwell, second grade and Newtonian fluids performing the same motions. Finally, as a check of general results as well as to get some physical insight for oscillating motions of these fluids, three special cases with engineering applications are considered and different known results from the literature are recovered. The required time to reach the steady-state for sine oscillating motions is graphically determined. It is higher for second grade or Oldroyd-B fluids in comparison with Newtonian, respectively Maxwell fluids.

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1. INTRODUCTION

Flows between circular cylinders or through a cylinder are of interest both for academic researchers and industry. Research workers can study changes associated with variations of boundary conditions and geometry but exact solutions can be easier obtained for motions with symmetry along or about an axis. Exact solutions for motions of Newtonian fluids in cylindrical domains are provided by Batchelor [2] and Yih [22]. First exact solutions for motions of second grade, Maxwell or Oldroyd-B fluids in such domains seem to be those of Ting [16], Srivastava [15], respectively Waters and King [20].

During time, many exact solutions for such motions of non-Newtonian fluids have been established. Among the solutions corresponding to motions of Oldroyd-B fluids we remember those of Wood [21], Fetecau [4], Hayat *et al.* [8], Fetecau *et al.* [5] and McGinty *et al.* [11]. They correspond to different boundary conditions and some of them have been already extended to fractional Olroyd-B fluids [1, 9, 10, 18].

It is worth pointing out that the most part of previous solutions have been extended to motion problems with shear stress on the boundary [6, 13, 23]. In these cases, contrary to what is usually assumed, the force with which the cylinder is moved is given on the boundary. However, in both cases, general solutions for longitudinal motions of rate type fluids in cylindrical domains are lacking in the existing literature. Such solutions are very important from theoretical and practical point of view. They generate exact solutions for any motion with technical relevance of this type and the corresponding problem can be considered as being completely solved.

The purpose of this note is to provide general solutions for the axial Couette flow of Oldroyd-B fluids in cylindrical domains. These solutions, which are determined by means of integral transforms, can be easily reduced to the similar solutions for Maxwell, second grade and Newtonian fluids performing the same motions. Moreover, for validation as well as to get some physical signification that is absent in the literature for oscillating motions of non-Newtonian fluids between circular cylinders, three special cases are considered and different known results are recovered. Finally, the solutions corresponding to sine oscillating motions of Olroyd-B fluids are presented as a sum of steady-state (permanent) and transient solutions and the required time to reach the steady-state is graphically determined.

2. STATEMENT OF THE PROBLEM

Consider an incompressible Oldroyd-B fluid at rest between two infinite coaxial circular cylinders of radii R_1 and $R_2(> R_1)$. At time $t = 0^+$ the outer cylinder begins to slide along its axis with a velocity $Vf(t)$ where V is constant and the dimensionless function $f(.)$ is a piecewise continuous and $f(0) = 0$. Due to the shear, the fluid begins to move and its velocity is of the form $\mathbf{v} = v(r,t)e_z$ where e_z is the unit vector along the z direction of a fixed cylindrical coordinate system r, θ and z. For such a motion the continuity equation is satisfied.

Assuming that the extra-stress tensor $\mathbf{S}$, as well as the velocity $\mathbf{v}$, is a function of r and t only and neglecting the body forces, we can easy show that the constitutive and the motion equations reduce to the relevant partial differential equations [12, Eqs. $(22)_3$ and $(26)_2$]

$$(1)\quad \left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\tau(r,t)=\rho\frac{\partial v(r,t)}{\partial t},\quad \left(1+\lambda\frac{\partial}{\partial t}\right)\tau(r,t)=\mu(1+\lambda_r)\frac{\partial v(r,t)}{\partial r}.$$

In the above relations $\tau(r,t) = S_{rz}(r,t)$ is the non-trivial shear stress, λ and λ_r are the relaxation and retardation times, ρ and μ are the density respectively the viscosity of the fluid. In order to get the equation (1) we also assumed that there is no pressure gradient in the flow direction and took into consideration the fact that the fluid was at rest at the initial moment $t = 0$.

It is well known the fact that the present model contains as special cases Maxwell and Newtonian fluids for $\lambda_r = 0$, respectively $\lambda_r = \lambda = 0$. Furthermore, Eq. $(1)_2$ with $\lambda = 0$ can be easily written in the form of the governing equation corresponding to second grade fluids performing the same motion. Consequently, it is expected to get the solutions corresponding to these fluids as limiting cases of general solutions. Eliminating $\tau(r,t)$ between Eqs. (1) we obtain the governing equation

(2)
$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial v(r,t)}{\partial t} = \nu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t);\ r\epsilon(R_1,R_2),\ t>0,$$

for velocity. Here $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

The appropriate initial and boundary conditions are

$$v(r,0) = \left.\frac{\partial v(r,t)}{\partial t}\right|_{t=0} = 0,\ r\epsilon[R_1,R_2], \tag{3}$$

$$v(R_1,t) = 0,\ v(R_2,t) = Vf(t),\ t \geq 0. \tag{4}$$

By introducing the non-dimensional variables and functions

$$r^* = \frac{r}{R_2},\ t^* = \frac{\nu}{R_2^2}t,\ \lambda^* = \frac{\nu}{R_2^2}\lambda,\ \lambda_r^* = \frac{\nu}{R_2^2}\lambda_r,$$
$$R^* = \frac{R_1}{R_2},\ v^* = \frac{v}{V},\ f^*(t^*) = f\left(\frac{R_2^2}{\nu}t^*\right) \tag{5}$$

and dropping out the star notation, we attain to the next non-dimensional initial and boundary problem

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial v(r,t)}{\partial t} = \left(1+\lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t);\ r\epsilon(R,1),\ t>0, \tag{6}$$

$$v(r,0) = \left.\frac{\partial v(r,t)}{\partial t}\right|_{t=0} = 0,\ r\epsilon[R,1];\ v(R,t) = 0,\ v(1,t) = f(t),\ t \geq 0. \tag{7}$$

3. SOLUTION OF THE PROBLEM

In the following, the partial differential equation (6) with the initial and boundary conditions (7) will be solved by means of integral transforms.

Consequently, applying the Laplace transform to Eq. (6) and using the initial conditions (7) we find that the Laplace transform $\bar{v}(r,q)$ of $v(r,t)$ satisfies the ordinary differential equation

$$q(1+\lambda q)\bar{v}(r,q) = (1+\lambda_r q)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\bar{v}(r,q);\;\; r\epsilon(R,1), \tag{8}$$

with the boundary conditions

$$\bar{v}(R,q)=0,\;\; \bar{v}(1,q)=F(q). \tag{9}$$

Here $F(q)$ is the Laplace transform of the function $f(t)$ while q is the transform parameter. Of course, if $\bar{v}(r,q)$ satisfies Dirichlet's conditions in the domain [R,1] then at each point of the interval (R,1) where the function is continuous, its finite Hankel transform is given by [3]

$$\bar{v}_H(r_n,q) = \int_R^1 r\bar{v}(r,q)B_0(r,r_n)\mathrm{d}r. \tag{10}$$

Now, we multiply Eq. (8) by $rB_0(r,r_n)$ where $B_0(r,r_n) = J_0(rr_n)Y_0(Rr_n) - J_0(Rr_n)Y_0(rr_n)$, integrate from R to 1 and take into consideration the boundary conditions (7) and the result (see [3] and the fact that $B_{10}(1,r_n) = \frac{2}{\pi r_n}\frac{J_0(Rr_n)}{J_{0(r_n)}}$)

$$\int_R^1 rB_0(r,r_n)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\bar{v}(r,q)\mathrm{d}r = -r_n^2\bar{v}_H(r_n,q)+\frac{2}{\pi}\bar{v}(1,q)\frac{J_0(Rr_n)}{J_0(r_n)}, \tag{11}$$

in order to obtain

$$\bar{v}_H(r_n,q) = \frac{2}{\pi}F(q)\frac{\lambda_r q+1}{\lambda q^2+(1+\lambda_r r_n^2)q+r_n^2}\frac{J_0(Rr_n)}{J_0(r_n)}. \tag{12}$$

In the above relations $J_v(.)$ and $Y_v(.)$ are Bessel functions of the first and second kind of v order, r_n are the positive roots of transcendental equation $B_0(1,r)=0$ and $B_{10}(r,r_n) = J_1(rr_n)Y_0(Rr_n) - J_0(Rr_n)Y_1(rr_n)$. In order to obtain a suitable form of the velocity field, we write the second factor of the right part of Eq. (12) like

$$\frac{\lambda_r q+1}{\lambda q^2+(1+\lambda_r r_n^2)q+r_n^2} = \frac{\lambda_r}{\lambda}\frac{q+a_n}{(q+a_n)^2-b_n^2}+\frac{1-\lambda_r a_n}{\lambda b_n}\frac{b_n}{(q+a_n)^2-b_n^2}, \tag{13}$$

where $a_n = \frac{1+\lambda_r r_n^2}{2\lambda}$ and $b_n = \frac{\sqrt{(1+\lambda_r a_n)^2-4\lambda r_n^2}}{2\lambda}$.

Introducing Eq. (13) into (12), applying the inverse Laplace transform and using the convolution theorem, we obtain

$$v_H(r_n,t) = \frac{2}{\pi}\frac{J_0(Rr_n)}{J_0(r_n)}\int_0^t f(t-s)\left[\frac{\lambda_r}{\lambda}\mathrm{ch}(b_n s)+\frac{1-\lambda_r a_n}{\lambda b_n}\mathrm{sh}(b_n s)\right]e^{-a_n s}\mathrm{d}s. \tag{14}$$

Integrating by parts into above equation, we find that

$$v_H(r_n,t)=\frac{2}{\pi}\frac{J_0(Rr_n)}{r_n^2J_0(r_n)}f(t)-\frac{2}{\pi}\frac{J_0(Rr_n)}{r_n^2J_0(r_n)}\int_0^t f'(t-s)c_n(s)\mathrm{d}s, \tag{15}$$

where $c_n(s)=\left[\mathrm{ch}(b_ns)+\frac{1-\lambda_r r_n^2}{2\lambda b_n}\mathrm{sh}(b_ns)\right]\exp\left(-a_ns\right)$.

Applying the inverse Hankel transform and using the identities

$$v(r,t)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2J_0^2(r_n)B_0(r,r_n)}{J_0^2(Rr_n)-J_0^2(r_n)}v_H(r_n,t),$$

$$\pi\sum_{n=1}^{\infty}\frac{J_0(r_n)J_0(Rr_n)}{J_0^2(Rr_n)-J_0^2(r_n)}B_0(r,r_n)=\frac{1}{\ln R}\ln\left(\frac{R}{r}\right), \tag{16}$$

we find for dimensionless velocity $v(r,t)$ the simple expression

$$v(r,t)=\frac{1}{\ln R}\ln\left(\frac{R}{r}\right)f(t)+\pi\sum_{n=1}^{\infty}\frac{J_0(r_n)J_0(Rr_n)}{J_0^2(r_n)-J_0^2(Rr_n)}B_0(r,r_n)\int_0^t f'(t-\tau)c_n(s)\mathrm{d}s, \tag{17}$$

which clearly satisfies the initial and boundary conditions (7). Moreover, taking the limits of Eq. (17) for $\lambda_r\to 0$, $\lambda\to 0$ or both λ and $\lambda_r\to 0$, the solutions corresponding to Maxwell, second grade or Newtonian fluids performing the same motion are obtained. The solution corresponding to second grade fluids, for instance, is

$$v_{SG}(r,t)=\tfrac{1}{\ln R}\ln\left(\tfrac{R}{r}\right)f(t)+\pi\sum_{n=1}^{\infty}\frac{J_0(r_n)J_0(Rr_n)}{J_0^2(r_n)-J_0^2(Rr_n)}\frac{B_0(r,r_n)}{1+\lambda_r r_n^2}$$
$$\int_0^t f'(t-s)\exp\left(-\tfrac{r_n^2s}{1+\lambda_r r_n^2}\right)\mathrm{d}s. \tag{18}$$

4. FLOW THROUGH AN INFINITE CIRCULAR CYLINDER

Let us consider the same fluid at rest in an infinite circular cylinder of radius R. After time $t=0$ the cylinder is moving along its axis with the same velocity $Vf(t)$. The fluid is gradually moved and the governing equation for velocity has the same form (2). The initial conditions are the same as before and the boundary condition and the limitation condition at $r=0$ are

$$v(R,t)=Vf(t),\quad |v(0,t)|<\infty;\quad t\geq 0. \tag{19}$$

The non-dimensional velocity field corresponding to this motion

$$v(r,t)=f(t)-2\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_nJ_1(r_n)}\int_0^t f'(t-s)c_n(s)\mathrm{d}s, \tag{20}$$

can be easily obtained following the same way as before and using the associate finite Hankel transform [14, Sec. 14]. The sum into Eq. (20) is taken over all positive roots r_n of the equation $J_0(r)=0$. Of course, these new roots are the limit of the previous roots when $R\to 0$ but we kept the same notation for writing simplicity.

Moreover, it is worth pointing out that the solution (20) can be also obtained as a limiting case of the solution (17) when $R\to 0$. In order to prove that, we firstly use the equivalence

$$B_0(1,r_n)=0 \;\Leftrightarrow\; J_0(r_n)=\frac{J_0(Rr_n)}{Y_0(Rr_n)}Y_0(r_n), \tag{21}$$

which shows that the roots of the equation $J_0(r)=0$ are the limit of the roots of the equation $B_0(1,r_n)=0$ when $R\to 0$.

Secondly, using the previous result and the identity

$$J_0(r_n)Y_1(r_n)-J_1(r_n)Y_0(r_n)=-\frac{2}{\pi r_n}, \tag{22}$$

we can show that

$$\lim_{R\to 0}\frac{J_0(r_n)J_0(Rr_n)}{J_0^2(Rr_n)-J_0^2(r_n)}B_0(r,r_n)=-\frac{2J_0(rr_n)}{\pi r_nJ_1(r_n)}. \tag{23}$$

Making $\lambda\to 0$ into Eq. (20) the solution

$$v_{SG}(r,t)=f(t)-2\sum_{n=1}^{\infty}\frac{1}{1+\lambda_r r_n^2}\frac{J_0(rr_n)}{r_nJ_1(r_n)}\int_0^t f'(t-s)\exp\left(-\frac{r_n^2 s}{1+\lambda_r r_n^2}\right)\mathrm{d}s, \tag{24}$$

corresponding to second grade fluid is obtained. For $\lambda_r=0$ into Eq. (24), the general solution for Newtonian fluids is obtained.

5. SPECIAL CASES

The general expressions given by Eqs. (17) and (20) can generate exact solutions for any motion with technical relevance of this type.

Consequently, the motion of Newtonian, second grade, Maxwell and Oldroyd-B fluids between two infinite coaxial cylinders induced by the outer cylinder that slides along its axis with a given velocity or the motion through an infinite circular cylinder performing the same motion is completely solved. However, in order to certify the accuracy of general results, as well as to bring to light their theoretical and practical value and some physical insight for some motions with technical relevance, three special cases are considered.

5.1. CASE $f(t) = H(t)$ (UNIFORM TRANSLATION OF THE CYLINDER)

Taking $f(t) = H(t)$, the Heaviside unit step function, into Eqs. (17) and (20) we find the dimensionless velocity fields

$$v(r,t) = \frac{\ln(R/r)}{\ln R} H(t) + \pi H(t) \sum_{n=1}^{\infty} \frac{J_0(r_n) J_0(R r_n)}{J_0^2(r_n) - J_0^2(R r_n)} B_0(r, r_n) c_n(t), \tag{25}$$

$$v(r,t) = H(t) \left(1 - 2 \sum_{n=1}^{\infty} \frac{J_0(r r_n)}{r_n J_1(r_n)} c_n(t) \right), \tag{26}$$

corresponding to motions induced by an infinite circular cylinder that slides along its axis with a constant velocity V. In order to determine the solutions (25) and (26) we used the known results

$$H'(t) = \delta(t), \quad \int_0^t \delta(t-s) f(s) \mathrm{d}s = f(t). \tag{27}$$

Both solutions (25) and (26) are written as a sum of steady solutions

$$v_s(r,\infty) = \frac{\ln(R/r)}{\ln R} \quad as \quad r \epsilon [R,1]; \quad v_s(r,\infty) = 1 \quad as \quad r \epsilon [0,1] \tag{28}$$

and the corresponding transient solutions. In the second case, at large times, the whole system is moving as a solid body.

5.2. CASE $f(t) = H(t) t^{\alpha}$ (RAMP-TYPE TRANSLATION OF THE CYLINDER)

By now replacing $f(t)$ by $H(t) t^{\alpha}$ $(\alpha > 0)$ into Eqs. (17) and (20), we obtain the solutions corresponding to motions due to a slowly $(\alpha < 1)$, constantly $(\alpha = 1)$ or highly $(\alpha > 1)$ accelerating translation of the cylinder.

Of course, of a special interest is the case $\alpha = 1$ corresponding to the motion induced by ramp-type translation of the cylinder [17]. More exactly, after time $t = 0$ the cylinder slides along its axis with a constantly accelerating velocity V_t and the corresponding solutions

$$(29)\quad v(r,t) = H(t)\left\{\frac{\ln(R/r)}{\ln R}t + \pi\sum_{n=1}^{\infty}\frac{J_0(r_n)J_0(Rr_n)B_0(r,r_n)}{r_n^2[J_0^2(Rr_n) - J_0^2(r_n)]}\left[1 - d_n(t)\right]\right\},$$

$$(30)\quad v(r,t) = H(t)\left\{t - 2\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^3 J_1(r_n)}\left[1 - d_n(t)\right]\right\},$$

are just the dimensionless forms of the solutions (17) and (20) obtained by Fetecau *et al.* [5] by a different technique. Into above relations $d_n(t) = \left\{\text{ch}(b_n t) + \frac{1+(\lambda_r - 2\lambda)r_n^2}{2\lambda b_n}\text{sh}(b_n t)\right\} e^{(-a_n t)}$.

As expected, the starting solutions (29) and (30) are presented as sum of large-time solutions

$$(31)\quad v_{LT}(r,t) = H(t)\left\{\frac{\ln(R/r)}{\ln R}t + \pi\sum_{n=1}^{\infty}\frac{J_0(r_n)J_0(Rr_n)}{r_n^2[J_0^2(r_n) - J_0^2(Rr_n)]}B_0(r,r_n)\right\},$$

$$(32)\quad v_{LT}(r,t) = H(t)\left\{t - 2\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^3 J_1(r_n)}\right\},$$

and the transient solutions that contain exponential functions and tend to zero for $t \to \infty$.

5.3. CASE $f(t) = H(t)\sin(\omega t)$ (OSCILLATING TRANSLATION OF THE CYLINDER)

Let us now assume that the outer cylinder oscillates along its axis after the time $t = 0^+$. The outcome motion can correspond to $f(t) = H(t)\sin(\omega t)$, $f(t) = H(t)\cos(\omega t)$ or a combination of them but we consider here only one case. By replacing $f(t)$ with $H(t)\sin(\omega t)$ into Eq. (17) and using again Eqs. (27), we find the starting solution

$$v(r,t) = H(t)\left\{\frac{\ln(R/r)}{\ln R}\sin(\omega t) + \omega\pi\sum_{n=1}^{\infty}\frac{J_0(r_n)J_0(Rr_n)}{J_0^2(r_n) - J_0^2(Rr_n)}\right.$$

$$(33)\quad \left. B_0(r,r_n)\int_0^t \cos(\omega s)c_n(t-s)\mathrm{d}s\right\},$$

The similar solution corresponding to the motion through an infinite circular cylinder, as it results from Eq. (20), is

$$(34)\quad v(r,t) = H(t)\left\{\sin(\omega t) - 2\omega\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n J_1(r_n)}\int_0^t \cos(\omega s)c_n(t-s)\mathrm{d}s\right\}.$$

Usually, the starting solutions for oscillating motions of fluids are important for those who want to eliminate the transients from their experiments. More exactly, they want to know the required time after which the fluid flows according to steady-state (permanent) solutions. In order to solve this problem, we must write the transient solutions (33) and (34) as sums between steady-state and transient solutions. To do that, we firstly evaluate the integral from these relations, namely

$$\int_0^t \cos(\omega s)c_n(t-s)\mathrm{d}s = \omega\frac{c_n-\lambda(r_n^2-\lambda\omega^2)}{c_n^2\omega^2+(r_n^2-\lambda\omega^2)^2}\sin(\omega t)+\frac{(1+\lambda\lambda_r\omega^2)r_n^2}{c_n^2\omega^2+(r_n^2-\lambda\omega^2)^2}\cos(\omega t)$$
$$-\left\{\frac{(1+\lambda\lambda_r\omega^2)r_n^2}{c_n^2\omega^2+(r_n^2-\lambda\omega^2)^2}\mathrm{ch}(b_nt)+r_n^2\frac{c_n(1-\lambda\lambda_r\omega^2)-2\lambda(r_n^2-\lambda\omega^2)}{[c_n^2\omega^2+(r_n^2-\lambda\omega^2)^2]b_n}\mathrm{sh}(b_nt)\right\}\exp(-a_nt),$$

where $c_n = 1+\lambda_r r_n^2$.

Introducing this last result into above relations, we find the steady-state solutions

$$v_p(r,t) = \left\{\frac{1}{\ln R}\ln\left(\frac{R}{r}\right)+\pi\omega^2\sum_{n=1}^{\infty}\frac{c_n-\lambda(r_n^2-\lambda\omega^2)}{c_n^2\omega^2+(r_n^2-\lambda\omega^2)^2}\frac{J_0(r_n)J_0(Rr_n)}{J_0^2(r_n)-J_0^2(Rr_n)}B_0(r,r_n)\right\}\sin(\omega t)+$$
$$\pi\omega\cos(\omega t)\sum_{n=1}^{\infty}\frac{(1+\lambda\lambda_r\omega^2)r_n^2}{c_n^2\omega^2+(r_n^2-\lambda\omega^2)^2}\frac{J_0(r_n)J_0(Rr_n)}{J_0^2(r_n)-J_0^2(Rr_n)}B_0(r,r_n), \tag{35}$$

$$v_p(r,t) = \left\{1-2\omega^2\sum_{n=1}^{\infty}\frac{c_n-\lambda(r_n^2-\lambda\omega^2)}{c_n^2\omega^2+(r_n^2-\lambda\omega^2)^2}\frac{J_0(rr_n)}{r_nJ_1(r_n)}\right\}\sin(\omega t)$$
$$-\ 2\omega\cos(\omega t)\sum_{n=1}^{\infty}\frac{1+\lambda\lambda_r\omega^2}{c_n^2\omega^2+(r_n^2-\lambda\omega^2)^2}\frac{r_nJ_0(rr_n)}{J_1(r_n)}, \tag{36}$$

respectively, the transient solutions

$$v_t(r,t) = -\omega\pi\sum_{n=1}^{\infty}\frac{r_n^2J_0(r_n)J_0(Rr_n)}{J_0^2(r_n)-J_0^2(Rr_n)}B_0(r,r_n)\left\{\frac{1+\lambda\lambda_r\omega^2}{c_n^2\omega^2+(r_n^2-\lambda\omega^2)^2}\right.$$
$$\left.\mathrm{ch}(b_nt)+\frac{c_n(1-\lambda\lambda_r\omega^2)-2\lambda(r_n^2-\lambda\omega^2)}{[c_n^2\omega^2+(r_n^2-\lambda\omega^2)^2]b_n}\mathrm{sh}(b_nt)\right\}\exp(-a_nt), \tag{37}$$

$$v_t(r,t) = 2\omega\sum_{n=1}^{\infty}\frac{r_nJ_0(rr_n)}{J_1(r_n)}\left\{\frac{1+\lambda\lambda_r\omega^2}{c_n^2\omega^2+(r_n^2-\lambda\omega^2)^2}\mathrm{ch}(b_nt)\right.$$
$$\left.+\frac{c_n(1-\lambda\lambda_r\omega^2)-2\lambda(r_n^2-\lambda\omega^2)}{[c_n^2\omega^2+(r_n^2-\lambda\omega^2)^2]b_n}\mathrm{sh}(b_nt)\right\}\exp(-a_nt), \tag{38}$$

corresponding to the two motions between two infinite circular cylinders or through an infinite circular cylinder.

For validation, it is worth pointing out that making $\lambda_r = 0$ into Eqs. (37) and (38) we recover the dimensionless forms of the transient solutions corresponding to the same motions of Maxwell fluids obtained by Vieru *et al.* [19, Eqs. (23) and (30)] by a different technique. Furthermore, as it results from Fig. 1 the diagrams of steady-state solutions (35) and (36) are identical to those corresponding to the solutions

$$(39)\quad v_p(r,t) = \text{Im}\left\{\frac{I_0(r\sqrt{\gamma})K_0(R\sqrt{\gamma}) - K_0(r\sqrt{\gamma})I_0(R\sqrt{\gamma})}{I_0(\sqrt{\gamma})K_0(R\sqrt{\gamma}) - K_0(\sqrt{\gamma})I_0(R\sqrt{\gamma})}e^{i\omega t}\right\},$$

$$(40)\quad v_p(r,t) = \text{Im}\left\{\frac{I_0(r\sqrt{\gamma})}{I_0(\sqrt{\gamma})}e^{i\omega t}\right\},\ \gamma = \omega\frac{(\lambda_r - \lambda)\omega + i(1 + \lambda\lambda_r\omega^2)}{1 + (\lambda_r\omega)^2},$$

which are the dimensionless forms of the steady-state solutions obtained by Fetecau *et al.* [7, Eqs. (32) and $(37)_2$] where λ_2 and λ_4 have to be zero while $\lambda_1 = \lambda$ and $\lambda_3 = \lambda_r$. Into Figs. 1, the corresponding roots r_n have been approximated by $n\pi/(R-1)$, respectively $(4n-1)\pi/4$.

6. CONCLUSIONS

Axial Coutte flow of Oldroyd-B fluids between two infinite circular coaxial cylinders is completely solved when the fluid motion is induced by the outer cylinder that is moving along its axis. General solutions for the dimensionless velocity of the fluid are presented under integral and series form in terms of standard Bessel functions of zero order. They satisfy all imposed initial and boundary conditions and can easily be particularized to give the similar solutions for Maxwell, second grade and Newtonian fluids performing the same motion. Furthermore, they can generate exact solutions for any motion with technical relevance of this type and are easily reduced to the solutions corresponding to the fluid motion through an infinite circular cylinder that is moving along its symmetry axis.

For validation, as well as to obtain some physical insight for some oscillating motions, three special cases are considered and different known results from the existing literature are recovered. In the case of constantly accelerating translation of the cylinder, for instance, our solutions (29) and (30) are just the dimensionless forms of the solutions (17) and (20) that have been obtained in [5] by a different technique. Moreover, the solutions corresponding to oscillating motions of the cylinder are presented as a sum of steady-state and transient solutions. Figs. 1 show that steady-state solutions (35) and (36) are

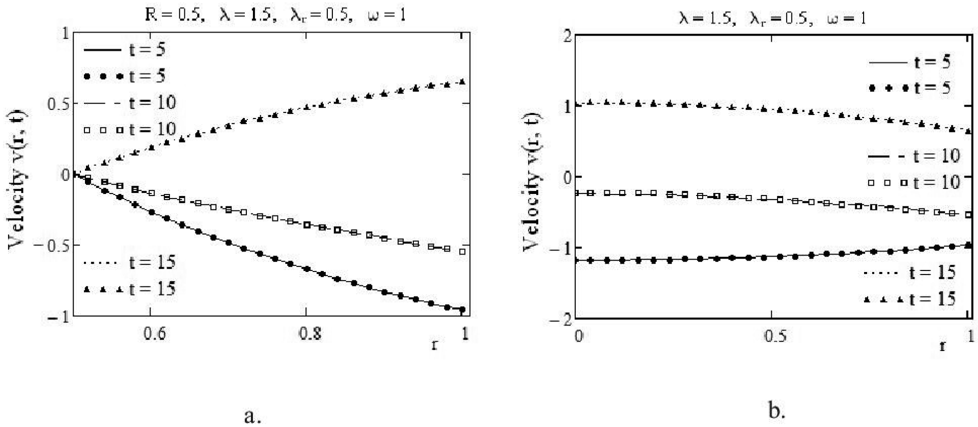

Fig. 1 – Profiles of permanent dimensionless velocities (35) and (39), respectively (36) and (40), for different values of the time t.

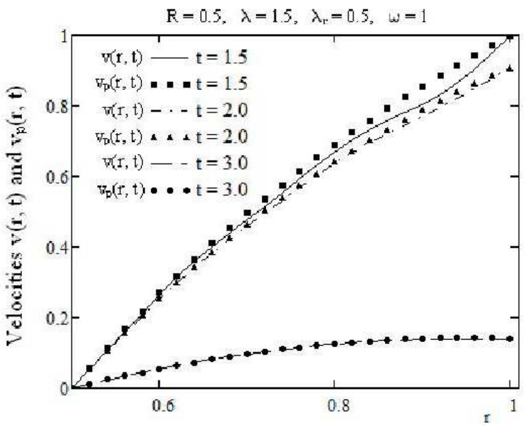

Fig. 2 – Variations of starting and permanent dimensionless velocities (33), respectively (35), for different values of the time t.

equivalent to those obtained by Fetecau *et al.* [7, Eqs. (32) and $(37)_2$] by a different technique.

Practically speaking, an important problem regarding the technical relevance of starting solutions is to determine the required time to get the steady-state. More exactly, to determine the time after which the fluid flows according to steady-state solutions. This time, for longitudinal oscillations of Oldroyd-B fluids between two infinite circular cylinders, is first time determined here for different values of the relaxation or retardation time λ or λ_r and of the frequency ω of oscillations. In Fig. 2, for comparison, the diagrams of starting and steady-state dimensionless velocities (33) and(35) against r are presented for the same values of pertinent parameters. At small values of the time t the difference between them is significant but it rapidly disappears and the required dimensionless time to reach the steady-state is small enough.

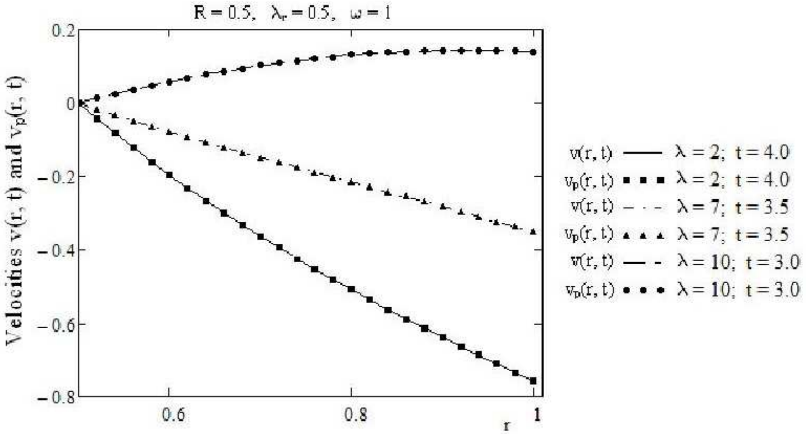

Fig. 3 – Required time to reach the steady-state for the motion due to sine oscillations of the outer cylinder for different values of the relaxation time λ.

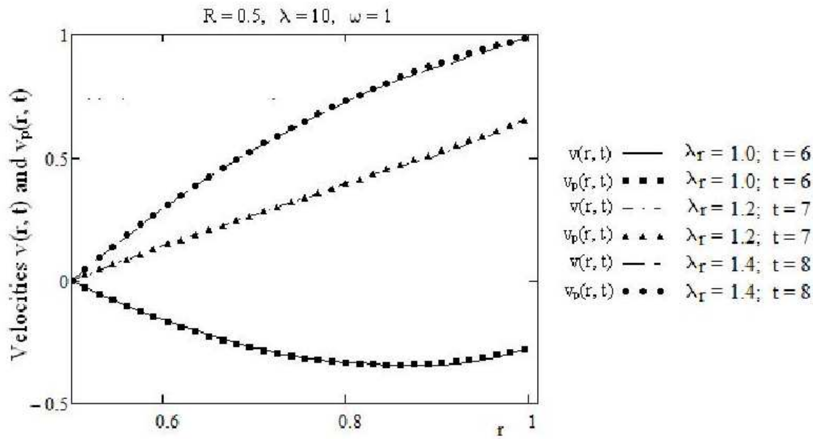

Fig. 4 – Required time to reach the steady-state for the motion due to sine oscillations of the outer cylinder for different values of λ_r.

Fig. 3 and 4 clearly show that the required time to reach steady-state in such motions of Oldroyd-B fluids is decreasing function with respect to λ and increases for increasing values of λ_r. Consequently, as expected, the effects of the two material parameters λ and λ_r on the fluid motion are opposite and the main conclusions are:

• The steady-state for such oscillating motions of fluids is rather obtained for Newtonian and Maxwell fluids in comparison to second grade, respectively Oldroyd-B fluids.
• It is also later obtained for Newtonian and second grade fluids in comparison to Maxwell, respectively Oldroyd-B fluids.

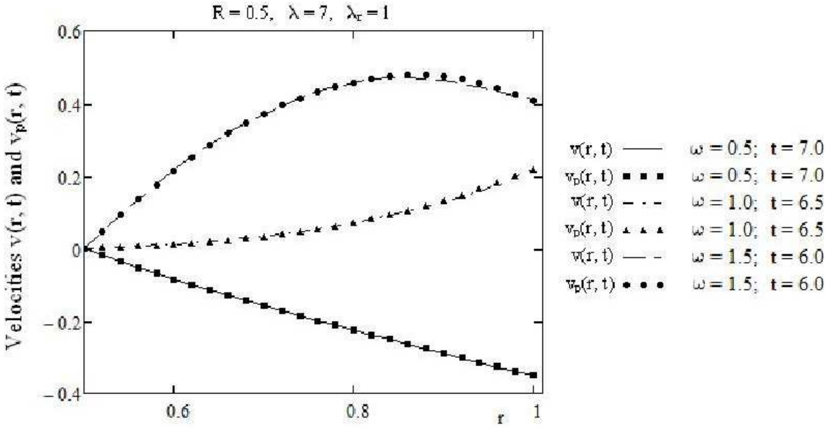

Fig. 5 – Required time to reach the steady-state for the motion due to sine oscillations of the outer cylinder for different values of the frequency ω.

Finally, for completion, the influence of the frequency omega of oscillations is brought to light by means of Fig. 5. The required time to reach the steady-state, as it results from this figure, is a decreasing function with regard to ω. It is also worth pointing out that all present results are in accord with those obtained in [7] for similar motions through an infinite circular cylinder but they are the first results of this type for oscillating motions of fluids between two infinite coaxial circular cylinders.

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GC University,
Abdus Salam School of Mathematical Sciences,
Lahore 54600, Pakistan
nigarjavedyash@gmail.com

Academy of Romanian Scientists,
050094 Bucharest, Romania
and
Technical University Gheorghe Asachi,
Department of Mathematics,
Iaşi, Romania

Khawaja Fareed University of Engineering,
and Information Technology,
Rahim Yar Khan, Pakistan