ON CERTAIN $\mathfrak{F}$-PERFECT GROUPS

AYNUR ARIKAN

Communicated by Vasile Brînzănescu

We study minimal non-soluble p-groups and define other classes of groups which have no proper subgroup of finite index. We give some descriptions of these groups and present some of their properties.

AMS 2010 Subject Classification: 20F16, 20F19, 20F50.

Key words: $\mathfrak{F}$-perfect group, minimal non-soluble group, barely transitive group, residually nilpotent group, weak Fitting group, FC-group, McLain group, finitary permutation group.

1. **INTRODUCTION**

A group which has no proper subgroup of finite index is called $\mathfrak{F}$-perfect. Let $\mathfrak{X}$ be a class of groups. If a group G is not in $\mathfrak{X}$ but every proper subgroup of G is in $\mathfrak{X}$ then G is called a minimal non $\mathfrak{X}$-group and denoted usually by $MN\mathfrak{X}$. Clearly every perfect p-group for some prime p is $\mathfrak{F}$-perfect and the structure of many $MN\mathfrak{X}$-groups is investigated in perfect p-case.

In the present article we take $\mathfrak{S}$, the class of all soluble groups, as the class $\mathfrak{X}$, *i.e.* we consider certain $MN\mathfrak{S}$-groups. It is not known yet if locally finite $MN\mathfrak{S}$-p-groups exist. Such groups are mainly studied in [2–6] (in some general form) and given certain descriptions.

In Section 2, we give certain applications of Khukhro-Makarenko Theorem to $\mathfrak{F}$-groups. Also we provide a corollary to [17, Satz 6] which is used effectively in most of the cited articles and give certain applications of it. Finally, we define *Weak Fitting* groups and a useful class Ξ of groups and give certain results related to these notions.

In Section 3, we consider a property of generator subsets of some $MN\mathfrak{S}$-groups.

In the final section, we deal with the groups having FC-subgroups and define the class Θ of groups. We provide two points of view and consider such a group as a subgroup of $M(\mathbb{Q}, GF(p))$ for some prime p, the McLain groups, or represent as a finitary permutation group on an infinite set.

2. $\mathfrak{F}$-PERFECT GROUPS

First we use a result known as the Khukhro-Makarenko Theorem (in short KM-Theorem, [9–12]) to obtain certain useful results for $\mathfrak{F}$-perfect groups. The following lemma gives an idea about the derived length of finite index subgroups. Throughout the paper, we use $dl(G)$ to denote the derived length of a soluble group G.

Lemma 1. *Let G be an $\mathfrak{F}$-perfect group, and N, K be normal subgroups of G such that K contains N. If the factor group K/N is finite then it is central in G.*

Proof. Since K/N is finite, $C_G(K/N)$ has finite index in G. So we have $G = C_G(N/K)$ by hypothesis. In other words K/N is central in G. So the proof is complete. □

To obtain some results, for a proper subgroup S of a group G, we need to find a normal subgroups N of G such that $|N : N \cap S|$ is infinite. The following corollary provides an idea.

Corollary 2. *Let G be an $\mathfrak{F}$-perfect group and N be a soluble proper normal subgroup of G of derived length d. If S is a subgroup of G of derived length $\leq d-2$ then $|N : N \cap S|$ is infinite. In particular, for $i \geq d-2$, $N/N^{(i)}$ is infinite.*

Proof. Let $K := N \cap S$ and suppose that K has finite index in N. By KM-Theorem, N contains a characteristic subgroup C such that $dl(C) \leq d-2$ and N/C is finite. Then by Lemma 1, N/C is central in G, in particular, it is abelian. It follows that $dl(N) \leq d-1$, a contradiction. □

In the following corollary, N_g for $g \in N$ denotes the finite-index normal subgroup of N such that $g \notin N_g$.

Corollary 3. *Let G be an $\mathfrak{F}$-perfect group and N be a soluble residually finite proper normal subgroup of G of derived length d. Then for every $g \in N$, $dl(N_g) \geq d-1$.*

Let G be a group and H be a subgroup of G. If $|G : H|$ is infinite, $Core_G H = \bigcap_{g\in G} H^g = 1$ and for every proper subgroup K of G, $|K : K \cap H|$ is finite then G is called a *barely transitive* group and H is called a *point stabilizer*. Though the definition of barely transitive groups has permutation groups origin, we use the above abstract definition of these groups (see [13]).

In the following corollary $nc(S)$ denotes the nilpotency class of S.

Corollary 4. *Let G be an $\mathfrak{F}$-perfect group and N be a soluble (nilpotent) proper normal subgroup of G of derived length d (nilpotency class c). If S is* *a finite index subgroup of N then $dl(S) = d - 1$ or d ($nc(S) = c - 1$ or c). Furthermore, $dl(Core_N S) \geq d - 1$ ($nc(Core_N S) \geq c - 1$).*

Theorem 5. *Let G be locally finite perfect barely transitive group with a point stabilizer H and N be a proper normal subgroup of G whose derived length is d (nilpotency class is c). Then for every positive integer n and for every $x_1, \ldots, x_n \in G$,*

$$dl(N \cap H \cap H^{x_1} \cap \cdots \cap H^{x_n}) \geq d - 1 \quad (nc(N \cap H \cap H^{x_1} \cap \cdots \cap H^{x_n}) \geq c - 1).$$

Proof. Result follows by Corollary 4. □

Corollary 6. *Let G be a locally finite perfect barely transitive group with a point stabilizer H. Then for every positive integer n, H contains a proper soluble (nilpotent) subgroup K of finite exponent such that $dl(K) > n$ ($nc(K) > n$).*

Proof. By [8, Theorem 1] G is a p-group, in particular, G is locally nilpotent. Also by [13, Theorem 1.1(2)] every proper normal subgroup of G is nilpotent of finite exponent. Hence G is a union of a ascending chain of proper normal subgroups of finite exponent. This implies G has a proper normal subgroup of derived length (nilpotency class) $\geq n + 2$, since G is perfect. By Theorem 5, $dl(N \cap H) > n$ ($nc(N \cap H) > n$). Hence the proof is complete. □

The following lemma is a corollary to [17, Satz 6] which is useful to extend some results proved for Fitting groups to the groups which are product of residually nilpotent normal subgroups.

Lemma 7. *Let G be a residually nilpotent p-group, U be a finite subgroup of G and $a \in G \setminus U$. If G/L is infinite and elementary abelian for a normal subgroup L of G then G has a subgroup V such that $a \notin V$ and VL/L is infinite.*

Proof. Since G is residually nilpotent, we have that

$$\bigcap_{i=1}^{\infty} \gamma_i(G) = 1.$$

This yields

$$\bigcap_{i=1}^{\infty} (U\gamma_i(G)) = U(\bigcap_{i=1}^{\infty} \gamma_i(G)) = U$$

since U is finite. Hence there is an integer r such that $a \notin U\gamma_r(G)$, *i.e.* $a\gamma_r(G) \notin U\gamma_r(G)/\gamma_r(G)$. Since $G/\gamma_r(G)$ is nilpotent and

$$\frac{G/\gamma_r(G)}{L/\gamma_r(G)}$$

is an infinite elementary abelian, by [17, Satz 6] $G/\gamma_r(G)$ has a subgroup $V/\gamma_r(G)$ such that $a\gamma_r(G) \notin V/\gamma_r(G)$ and

$$\frac{V/\gamma_r(G)L/\gamma_r(G)}{L/\gamma_r(G)}$$

is infinite. Consequently, $a \notin V$ and and VL/L is infinite, as desired. $\square$

The following proof is almost the same with the proof of [13, Theorem 1.1].

LEMMA 8. *Let G be a countably infinite periodic locally soluble group which is not a product of two proper subgroups then G is a p-group for some prime p.*

Proof. Since G is locally soluble, it is locally finite by [15, 1.3.5]. Hence G has finite subgroups G_i for $i \geq 1$ such that $G = \bigcup_{i\geq 1} G_i$ and G_i is a proper subgroup of G_{i+1} for every $i \geq 1$. Let p be a prime which divides the order of an element of G. Since G_1 is soluble, it has a Hall$_{p'}$-subgroup and any two of them are conjugate. So we can construct inductively $P_n \in Syl_p G_n$ and $H_n \in Hall_{p'} G_n$ such that $P_n \leq P_{n+1}$, $H_n \leq H_{n+1}$ and $G_n = P_n H_n$. Now put

$$P := \bigcup_{i\geq 1} P_i \text{ and } H := \bigcup_{i\geq 1} H_i.$$

Hence we have that $G = PH$. By hypothesis either $G = P$ or $G = H$. But since p divides the order of an element in G we have that $G = P$ and hence G is a p-group. $\square$

LEMMA 9. *Let G be a periodic $MN\mathfrak{S}$ group which is not finitely generated. Then either G is p-group or generated by p-elements for some prime p.*

Proof. Clearly G is locally soluble. As in the proof of Lemma 8, $G = PH$. If $G = P$ then G is a p-group. If not, then also $H \neq G$ since H is a p'-group and G has an element of order divided by p. Now $G = P^H H$ and P^H is normal in G. Hence

$$P^H = \langle x^h : \ x \in P, \ h \in H \rangle = G,$$

i.e. G is generated by p-elements. $\square$

If a group G is a product of residually nilpotent normal subgroups then we say that G is a *Weak Fitting* group (WF-group in short). We also define the class Ξ of groups as follows:

Normal closure of every finitely generated subgroup of the group is residually nilpotent.

THEOREM 10. *Let G be a perfect p-group, where p is a prime. Suppose that G satisfies the following conditions:*

(i) *G is a Ξ-group,*

(ii) *G is not finitely generated,*

(iii) *every proper normal subgroup of G is soluble.*

If U is a finite subgroup of G and L is a proper soluble subgroup of G then

$$\bigcap_{y\in G\setminus L} \langle U, y\rangle = U.$$

Proof. Assume that the assertion is false. Then

$$\bigcap_{y\in G\setminus L} \langle U, y\rangle \neq U$$

for some finite subgroup U and a proper soluble subgroup L of G. Take

$$a \in (\bigcap_{y\in G\setminus L} \langle U, y\rangle) \setminus U.$$

Put $K = L\langle U, y\rangle^G$, then clearly K is soluble. Assume that $F^G = G$ for some finite subgroup F of G. Then G has elements $x_1, \ldots, x_n$ for some positive integer n such that $G = \prod_{i=1}^{n}\langle x_i^G\rangle$. But since G is a $MN\mathfrak{S}$-group, there is $1 \leq m \leq n$ such that $G = \langle x_m^G\rangle$ and hence $G = [G, x_m]$. But since G is locally nilpotent, this is a contradiction as in the proof of [4, Lemma 2.5]. Now we have that G has an ascending sequence of finite subgroups F_i for $i \geq 1$ such that

$$G = \bigcup_{i\geq 1} F_i^G$$

and $dl(F_i^G) < dl(F_{i+1}^G)$ for every $i \geq 1$, since G is perfect. Now G is locally finite and for every $i \geq 1$, $N_i := F_i^G$ is a soluble residually nilpotent subgroup of G. Since K is soluble and G is perfect, by Corollary 2 we have that there is a positive integer r such that $|N_r : N_r \cap K|$ is infinite. By [5, Lemma 2.11], N_r contains a normal subgroup M such that $M/(M \cap K)M'$ is infinite. Put $T := M\langle U, a\rangle$ then $T/(T\cap K)T'$ is infinite of finite exponent by [5, Lemma 2.13]. Put $S := (T \cap K)T'$ and $W/S := (T/S)^p$. Now T/W is infinite elementary abelian and T is residually finite. So by Lemma 7, T has a subgroup V such that $U \leq V$ and VW/W is infinite. Hence there is an element $z \in V \setminus K$ such that $a \notin \langle U, z\rangle$. But since $T \cap K \leq W$ and $L \leq K$ we have that $z \notin L$. But this is a contradiction. $\square$

The following result is a generalization of [6, Theorem 1.1(a)] and says that we may omit to impose to be a Ξ-group to all homomorphic images. The notion "$(*)$-triple" is defined in [6] as follows:

Let G be a group, L a proper subgroup, Y a subset, V a finitely generated subgroup of G and let $1 \neq w \in G$. If $w \notin V$ but $w \in \langle V, y\rangle$ for every $y \in Y \setminus L$, then (w, V, L) is called $(*)$-triple for Y. If $\bigcap_{y\in Y\setminus L}\langle V, y\rangle \neq V$, then there is $w \in (\bigcap_{y\in Y\setminus L}\langle V, y\rangle) \setminus V$ and hence (w, V, L) is an "$(*)$-triple" for Y and vice versa.

Corollary 11. *Let G be an infinitely generated periodic Ξ-group with all proper subgroup soluble and has $(*)$-triple for G then G is soluble.*

As an application of Lemma 7 we may give the following results which are extended from "Fitting" case (in [1]) to "Weakly Fitting" case. There are some further such results in the literature which can be extended in such a way, but we do not include them here.

Let $F(x_1, x_2, \dots)$ be a free group of countable rank and define $\phi_0(x_1) = x_1$, and for $i \geq 1$

$$\phi_i(x_1, \dots, x_{2^i}) = [\phi_{i-1}(x_1, \dots, x_{2^{i-1}}), \phi_{i-1}(x_{2^{i-1}+1}, \dots, x_{2^i})].$$

Lemma 12. *Let G be a WF-p-group. If for every proper subgroup of G is contained in a proper normal subgroup of G and for every given sequence of elements $x_1, x_2, \dots, x_n, \dots$ there is a non-negative integer d such that*

$$\phi_d(x_1, \dots, x_{2^d}) = 1,$$

then $G' \neq G$.

Proof. Just consider $\langle y, a, U\rangle^G$ as a residually nilpotent subgroup of G and follow Lemma 7 in the proof of [1, Lemma 2.3]. $\square$

Now we can reformulate the main theorem and its corollary in [1] as follows:

Theorem 13. *Let G be a WF-p-group. If for every proper subgroup H of G, $H^{(n)}$ is hypercentral for some non-negative integer n and $H^G \neq G$ then $G' \neq G$.*

Corollary 14. *Let G be a WF-p-group and suppose that every proper subgroup of G is contained in a proper normal subgroup of G then the following hold.*

(i) *If every proper subgroup of G is soluble then G is soluble.*

(ii) *If every proper subgroup of G is hypercentral then G is hypercentral.*

3. GENERATING SUBSETS

Now we give a property of some generator sets for $MN\mathfrak{S}$-p-groups.

Theorem 15. *Let G be a locally finite $MN\mathfrak{S}$-p-group. Then G has a generating subset X such that X contains proper generating subsets Y and Z for G such that $X \setminus Y$, $X \setminus Z$ are infinite and $Y \cap Z = \emptyset$.*

Proof. Since G is nonabelian, X has elements $x_{1,1},\ x_{1,2}$ such that

$$\phi_1(x_{1,1}, x_{1,2}) \neq 1.$$

Clearly $\langle X \setminus \{x_{1,1}, x_{1,2}\}\rangle = G$ and hence there exit $y_{1,1}, y_{1,2} \in X \setminus \{x_{1,1}, x_{1,2}\}$ such that

$$\phi_1(y_{1,1}, y_{1,2}) \neq 1$$

and $\{x_{1,1}, x_{1,2}\} \cap \{y_{1,1}, y_{1,2}\} = \emptyset$.

Now assume that we have found elements

$$x_{r,1}, \dots, x_{r,2^r} \in X \setminus \{x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{2,3}, x_{2,4}, \dots, x_{r-1,1}, \dots, x_{r-1,2^{r-1}},$$
$$y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}, \dots, y_{r-1,1}, \dots, y_{r-1,2^{r-1}}\}$$

for some

$$y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}, \dots, y_{r-1,1}, \dots, y_{r-1,2^{r-1}} \in$$
$$X \setminus \{x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{2,3}, x_{2,4}, \dots, x_{r,1}, \dots, x_{r,2^{r-1}}\}$$

and

$$y_{r,1}, \dots, y_{r,2^r} \in X \setminus \{x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{2,3}, x_{2,4}, \dots, x_{r-1,1}, \dots, x_{r-1,2^{r-1}},$$
$$y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}, \dots, y_{r-1,1}, \dots, y_{r-1,2^{r-1}}, x_{r,1}, \dots, x_{r,2^r}\}$$

such that

$$\phi_r(x_{r,1}, \dots x_{r,2^r}) \neq 1 \text{ and } \phi_r(y_{r,1}, \dots y_{r,2^r}) \neq 1$$

for $r \geq 1$ and $\{x_{r,1}, \dots x_{r,2^r}\} \cap \{y_{r,1}, \dots y_{r,2^r}\} = \emptyset$.

Since

$$\langle X \setminus \{x_{1,1}, \dots, x_{r,2^r}, y_{1,1}, \dots, y_{r,2^r}\}\rangle = G,$$

as above there are elements

$$x_{r+1,1}, \dots, x_{r+1,2^{r+1}} \in X \setminus \{x_{1,1}, \dots, x_{r,2^r}, y_{1,1}, \dots, y_{r,2^r}\}$$

and

$$y_{r+1,1}, \dots, y_{r+1,2^{r+1}} \in X \setminus \{x_{1,1}, \dots, x_{r,2^r}, y_{1,1}, \dots, y_{r,2^r}, x_{r+1,1}, \dots, x_{r+1,2^{r+1}}\}$$

such that

$$\phi_{r+1}(x_{r+1,1}, \dots, x_{r+1,2^{r+1}}) \neq 1$$
$$\phi_{r+1}(y_{r+1,1}, \dots, y_{r+1,2^{r+1}}) \neq 1.$$

Clearly

$$\{x_{r+1,1}, \dots, x_{r+1,2^{r+1}}\} \cap \{y_{r+1,1}, \dots, y_{r+1,2^{r+1}}\} = \emptyset.$$

Put $Y := \langle x_{i,1}, \dots, x_{i,2^i} \ :\ \ i \geq 1\rangle$ and $Z := \langle y_{i,1}, \dots, y_{i,2^i} \ :\ \ i \geq 1\rangle$. Since Y and Z are nonsoluble we have $\langle Y\rangle = G = \langle Z\rangle$. Also $(X \setminus Y) \supseteq Z$, $(X \setminus Z) \supseteq Y$ and so $X \setminus Y$ and $X \setminus Z$ are infinite and $Y \cap Z = \emptyset$. Hence the proof is complete. $\square$

In Theorem 15, if $Z(G) = 1$ then we may assume that the elements appearing in commutators are all distinct. It can be accomplished by considering commutator centralizers $C_G([x, y])$ which are infinite by [5, Lemma 2.4]. For example, if we consider the commutator $[[x, y], [x, z]]$ with repeating element x, and $y \neq z$. Then we can choose an element $u \in C_G([x, y])$ such that ux is distinct from y and z. Hence $[[ux, y], [x, z]] = [[x, y], [x, z]]$, but ux, x, y, z are all distinct.

Corollary 16. *Let G be a locally finite $MN\mathfrak{S}$-p-group. Then G has a sequence of descending generating subsets*

$$X_1 > X_2 > \cdots > X_n > \ldots \text{ and } Y_1 > Y_2 > \cdots > Y_n > \ldots$$

for G such that for each $i \geq 1$, $X_{i+1} \setminus X_i$ and $Y_{i+1} \setminus Y_i$ are infinite, and $X_1 \cap Y_1 = \emptyset$.

Proof. The result follows by taking a generating subset for G, applying the procedure in Theorem 15 and considering that $\langle X \setminus E \rangle = G$ for every generator subset X for G and finite subset F of X. $\square$

4. *FC*-SUBGROUPS

Having certain FC-subgroups in groups may provide some opportunities to figure out the structure of the groups (see [6, Theorem 1.1(b)], for example). In this section, we give certain useful results.

The following is a slightly generalized form of [14, Theorem 2.4] (see [14] for the definition of the notion "locally degree preserving").

Theorem 17. *Let G be a perfect locally finite p-group for some prime p. If there exists $a \in G \setminus Z(G)$ such that $\langle a^G, g\rangle$ is an FC-group for every $g \in G$ then there exists a locally degree-preserving embedding of an epimorphic image of G into $M(\mathbb{Q}, GF(p))$ for some prime p.*

Proof. Since G is perfect, considering $G/Z(G)$ we may assume that G has trivial center. Since $N := \langle a^G \rangle$ is an FC-group and G has no proper subgroup of finite index, the socle of N, say S, is infinite and elementary abelian. Now the action of $\overline{G} := G/C_G(S)$ on S via conjugation gives rise to a faithful representation of $\overline{G}$ over the field $GF(p)$. By [16, Theorem B(ii)], $\overline{G}$ is unipotent. Let $g \in G$ then $S\langle g \rangle$ is an FC-group. Hence

$$|S : C_S(g)| \leq |S\langle g\rangle : C_{S\langle g\rangle}(g)| < \infty.$$

Therefore the representation is finitary linear. By [14, Theorem 2.3], there exists a locally degree-preserving embedding of an epimorphic image of G into $M(\mathbb{Q}, GF(p))$. $\square$

Theorem 17 directs our attention to McLain groups for some perfect groups (of course if such groups exist). For example,

Corollary 18. *Let G be a locally finite $MN\mathfrak{S}$-p-group for some prime p. If there exists $a \in G \setminus Z(G)$ such that $\langle a^G, g\rangle$ is an FC-group for every $g \in G$ then there exists a locally degree-preserving embedding of an epimorphic image of G into $M(\mathbb{Q}, GF(p))$ for some prime p.*

Proof. Clearly G is perfect and has no proper subgroup of finite index. So the result follows by Theorem 17. $\square$

Define the class Θ of groups as follows:

The normal closure of every two generated subgroup of the group is an FC-group.

Theorem 19. *Let G be an infinitely generated locally finite $\mathfrak{F}$-perfect-p-group for some prime p with trivial center. If G is a Θ-group then G has an epimorphic image which can be represented as a finitary permutation group.*

Proof. Assume that for every $1 \neq a \in G$, $C_G(\langle a^G\rangle)$ is abelian. Let c, d be nontrivial elements of G. Since G is locally nilpotent and $\langle c, d\rangle^G$ is an FC-group, $\langle c, d\rangle^G$ is hypercentral and hence $Z(\langle c, d\rangle^G) \neq 1$. Let $z \in Z(\langle c, d\rangle^G)$ then $\langle c, d\rangle^G \leq C_G(\langle z^G\rangle)$. This implies that $\langle c, d\rangle^G$ and hence G is abelian, a contradiction.

Consequently, G has an element $1 \neq a \in G$ such that $C_G(\langle a^G\rangle)$ is non-abelian. Now put $C := C_G(\langle a^G\rangle)$. By [18, Theorem 1.9] $C/Z(G)$ is residually finite and thus C has a proper normal subgroup L of finite index. Put $\Omega := \{a^g :\ g \in G\}$ and let G act on Ω via conjugation. Then Ω is infinite, since G has no proper subgroup of finite index and $Z(G) = 1$. Let $1 \neq b \in G$ and put $B := \langle b^G\rangle$. Since $\langle a, b\rangle^G \neq G$ is an FC-group, $|B : C_B(a)|$ and $\{[b, a^x] :\ x \in G\}$ are finite. Since $Z(G) = 1$ and $L \leq C_G([g, a])$, we also have that

$$|C_G([g, a]) : C_G([g, a]) \cap C_G(a)| < \infty$$

for every $g \in B \setminus C_B(a)$. If a^x and a^y are conjugates of a in G then

$$|C_G(a^y) : C_G(a^y) \cap C_G(a^x)| < \infty,$$

since $L \leq C_G(a^x) \cap C_G(a^y)$ *i.e.* $C_G(a^x)$ and $C_G(a^y)$ commensurable. By [7, Lemma 4], *supp*(b) is finite. Therefore G acts on Ω as a finitary permutation group and $G/C_G(\langle a^G\rangle)$ is isomorphic to a subgroup of $FSym(\Omega)$. $\square$

Theorem 19 gives another point of view to some $MN\mathfrak{S}$-groups (if they exist).

Corollary 20. *Let G be an infinitely generated locally finite $MN\mathfrak{S}$-p-group for some prime p. If $G \in \Theta$ then G has an epimorphic image which can be represented as a finitary permutation group.*

Proof. Since $Z(G/Z(G)) = 1$, the result follows by Theorem 19. □

Lemma 21. *Let G be an infinitely generated locally finite $MN\mathfrak{S}$-p-group for some prime p. Suppose that G has a noncentral element a such that for every $b \in G$, $\langle a, b\rangle^G$ is an FC-group. Then for every $b \in G$, $dl(\langle b\rangle^G) \leq dl(C_G(a)) + 1$, i.e, there is a bound on the derived length of the normal closure of every element of G.*

Proof. Put $W_b := \langle a, b\rangle$ then $|W_b : C_{W_b}(a)|$ is finite. By KM-Theorem W_b has a characteristic subgroup R_b of finite index such that $dl(R_b) \leq dl(C_G(a))$. Put $\overline{G} := G/R_b$. Since $\overline{G}/C_{\overline{G}}(\overline{W_b})$ is finite and G has no proper subgroup of finite index, we have that $\overline{G} = C_{\overline{G}}(\overline{W_b})$. This implies in particular that $W_b' \leq R_b$ and thus $dl(W_b) \leq dl(R_b) + 1 \leq dl(C_G(a)) + 1$. In particular, $dl(\langle b\rangle^G) \leq dl(C_G(a)) + 1$. Hence the proof is complete. □

REFERENCES

[1] A. Arıkan and T. Özen, *On a generalization of groups with all subgroups subnormal.* Rend. Semin. Mat. Univ. Padova **112** (2004), 71–76.

[2] A. Arıkan, S. Sezer and H. Smith, *On locally finite minimal non-solvable groups.* Cent. Eur. J. Math. **8** (2010), *2*, 266–273.

[3] A. Arıkan and A. Arıkan, *On Fitting p-groups with all proper subgroups satisfying a commutator law.* J. Algebra **352** (2012), 341–346.

[4] A. Arıkan, *Characterization of minimal non-solvable Fitting p-groups.* J. Group Theory **11** (2008), 95–103.

[5] A.O. Asar, *Locally nilpotent p-groups whose proper subgroups are hypercentral of nilpotent-by-Chernikov.* J. Lond. Math Soc. **61** (2000), 412–422.

[6] A.O. Asar, *On infinitely generated groups whose proper subgroups are solvable.* J. Algebra **399** (2014), 870–886.

[7] V.V. Belyaev, *On the question of existence of minimal non-FC-groups.* Sib. Math. J. **39** (1998), *6*, 1093–1095; translation from Sibirsk. Mat. Zh. **39** (1998), *6*, 1267–1270.

[8] V.V. Belyaev and M. Kuzucuoğlu, *Locally finite barely transitive groups.* Algebra Logic **42** (2003), *3*, 147–152; translation from Algebra Logika **42** (2003), *3*, 261–270.

[9] E.I. Khukhro and N. Yu. Makarenko, *Large characteristic subgroups satisfying multilinear commutator identities.* J. Lond. Math. Soc. **75** (2007), 635–646.

[10] E.I. Khukhro, A.A. Klyachko, N. Yu. Makarenko and Yu. B. Mel'nikova, *Automorphism invariance and identities.* Bull. Lond. Math. Soc. **41** (2009), 804–816.

[11] A.A. Klyachko and Yu. B. Mel'nikova, *A short proof of the Khukhro-Makarenko Theorem on large characteristic subgroups with laws.* Sb. Math. **200** (2009), *5*, 661–664; translation from Mat. Sb. **200** (2009), *5*, 33–36.

[12] A.A. Klyachko and M.V. Milentyeva, *Large and symmetric: The Khukhro-Makarenko theorem on laws–without laws.* J. Algebra **424** (2015), 222–241.

[13] M. Kuzucoglu, *Barely transitive permutation groups.* Arch. Math. **55** (1990), 521–532.

[14] F. Leinen and O. Puglisi, *Unipotent finitary linear groups.* J. Lond. Math. Soc. **48** (1993), 59–76.

[15] J.C. Lennox and D.J.S. Robinson, *The Theory of Infinite Soluble Groups.* Oxford Math. Monogr. Clarendon Press, Oxford, 2004.

[16] U. Meierfrankenfeld, R.E. Phillips and O. Puglisi, *Locally solvable finitary linear groups.* J. Lond. Math. Soc. **47** (1993), 31–40.

[17] M. Möhres. *Torsionsgruppen, deren Untergruppen alle subnormal sind.* Geom. Dedicata **31** (1989), 237–244.

[18] M.J. Tomkinson, *FC-groups.* Research Notes in Mathematics **96**, Pitman Advanced Publ. Program, London–Boston–Melbourne, 1984.

Received 20 November 2016

Gazi University,
Science Faculty,
Department of Mathematics,
06500 Teknikokullar, Ankara, Turkey
yalincak@gazi.edu.tr