ON THE PROPERTIES OF (p, q)-FIBONACCI AND (p, q)-LUCAS QUATERNIONS

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Based on generalized Lucas numbers, (p, q)-Lucas quaternions are introduced. Moreover, some identities such as Catalan identity, d'Ocagne's identity etc., involving (p, q)-Fibonacci and (p, q)-Lucas quaternions are established.

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1. INTRODUCTION

As usual, generalized Fibonacci sequence $\{\mathcal{F}_n\}_{n\geq 1}$ is recursively defined as

$$\mathcal{F}_{n+1} = p\mathcal{F}_n + q\mathcal{F}_{n-1}$$

with initial conditions $\mathcal{F}_0 = 0$ and $\mathcal{F}_1 = 1$. On the other hand, the recurrence expression for generalized Lucas sequence $\{\mathcal{L}_n\}_{n\geq 1}$ is given by

$$\mathcal{L}_{n+1} = p\mathcal{L}_n + q\mathcal{L}_{n-1},$$

with initials $\mathcal{L}_0 = 2$ and $\mathcal{L}_1 = p$. The closed forms popularly known as Binet formulas for these two sequences are given by the expressions

$$\mathcal{F}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } \mathcal{L}_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{p+\sqrt{p^2+4q}}{2}$ and $\beta = \frac{p-\sqrt{p^2+4q}}{2}$ with $\Delta = p^2 + 4q > 0$. Moreover, it can be observed that, $\alpha^n = \alpha\mathcal{F}_n + q\mathcal{F}_{n-1}$ and $\beta^n = \beta\mathcal{F}_n + q\mathcal{F}_{n-1}$.

The Irish mathematician and physicist William Rowan Hamilton introduced quaternions as an extension of complex numbers. According to him, a quaternion q is a hyper-complex number represented by an equation

$$q = ae_0 + be_1 + ce_2 + de_3,$$

where $a, b, c, d \in \mathbb{R}$ and the set $\{e_0, e_1, e_2, e_3\}$ forms a standard orthonormal basis in $\mathbb{R}^4$. The collection of quaternions are usually denoted by $\mathbb{H}$ and constitute a non-commutative field called a skew field that extends the complex field $\mathbb{C}$. It is well-known that, the standard basis vectors e_0, e_1, e_2, e_3 satisfy the multiplication rule as per the following composition table:

TABLE 1

The multiplication table of basis for $\mathbb{H}$

$*$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	-1	e_3	$-e_2$
e_2	e_2	$-e_3$	-1	e_1
e_3	e_3	e_2	$-e_1$	-1

Horadam, in [4], defined Fibonacci and Lucas quaternions by the equations

$$QF_n = F_n e_0 + F_{n+1} e_1 + F_{n+2} e_2 + F_{n+3} e_3,$$

and

$$QL_n = L_n e_0 + L_{n+1} e_1 + L_{n+2} e_2 + L_{n+3} e_3,$$

where F_n and L_n denote n-th Fibonacci number and n-th Lucas number respectively.

Fibonacci and Lucas quaternions are generalized in many ways. For details of these works, one can go through [1–3, 6–8]. A recent generalization for Fibonacci quaternion is due to İpek [6]. He introduced (p, q)-Fibonacci quaternion sequence $\{\mathcal{F}_n(p, q)\}_{n \geq 0}$ which is defined recursively by the expression:

$$\mathcal{QF}_n = \mathcal{F}_n e_0 + \mathcal{F}_{n+1} e_1 + \mathcal{F}_{n+2} e_2 + \mathcal{F}_{n+3} e_3 = \sum_{s=0}^{3} \mathcal{F}_{n+s} e_s,$$

where $\mathcal{F}_n$ and $\mathcal{QF}_n$ denote the n-th (p, q)-Fibonacci number and the n-th (p, q)-Fibonacci quaternion respectively. In [6], İpek has established the recurrence relation for (p, q)-Fibonacci quaternions which is given by the equation

$$\mathcal{QF}_{n+1} = p\mathcal{QF}_n + q\mathcal{QF}_{n-1}, \ n \geq 1 \tag{1.1}$$

and derived some identities including Binet formula, generating functions and certain bionomial sums involving (p, q)-Fibonacci quaternions.

In this article, we first introduce (p, q)-Lucas quaternions and then derive some new identities such as Catalan identity, d'Ocagne's identity etc., for both these quaternions.

Definition 1.1. The (p,q)-Lucas quaternion sequence $\{\mathcal{L}_n(p,q)\}_{n\geq 0}$ is defined recursively by

$$\mathcal{QL}_n = \mathcal{L}_n e_0 + \mathcal{L}_{n+1} e_1 + \mathcal{L}_{n+2} e_2 + \mathcal{L}_{n+3} e_3 = \sum_{s=0}^{3} \mathcal{L}_{n+s} e_s,$$

where $\mathcal{L}_n$ denotes generalized Lucas number.

In view of the above recursive definition, it can be easily observed that (p,q)-Lucas quaternions has recurrence relation of the form

$$\mathcal{QL}_{n+1} = p\mathcal{QL}_n + q\mathcal{QL}_{n-1}, \; n \geq 1.$$

For $\alpha = \frac{p+\sqrt{\Delta}}{2}$, it can be seen that

$$\begin{aligned}\alpha\mathcal{QF}_n + q\mathcal{QF}_{n-1} &= \alpha\sum_{s=0}^{3} \mathcal{F}_{n+s} e_s + q\sum_{s=0}^{3} \mathcal{F}_{n-1+s} e_s \\ &= \sum_{s=0}^{3} (\alpha\mathcal{F}_{n+s} + q\mathcal{F}_{n+s-1}) e_s.\end{aligned}$$

Since $\alpha^n = \alpha\mathcal{F}_n + q\mathcal{F}_{n-1}$ [5] and setting $A = \sum_{s=0}^{3} \alpha^s e_s$, the above expression reduces to

$$\alpha\mathcal{QF}_n + q\mathcal{QF}_{n-1} = A\alpha^n. \tag{1.2}$$

Similarly, since $\beta^n = \beta\mathcal{F}_n + q\mathcal{F}_{n-1}$ [5] and letting $B = \sum_{s=0}^{3} \beta^s e_s$, we obtain

$$\beta\mathcal{QF}_n + q\mathcal{QF}_{n-1} = B\beta^n. \tag{1.3}$$

On subtraction of (1.3) from (1.2) gives the Binet formulas for (p,q)-Fibonacci quaternions as

$$\mathcal{QF}_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

where $A = \sum_{s=0}^{3} \alpha^s e_s$ and $B = \sum_{s=0}^{3} \beta^s e_s$. Similarly, adding (1.2) and (1.3), the Binet formula for (p,q)-Lucas quaternions is obtained and it is given by

$$\mathcal{QL}_n = A\alpha^n + B\beta^n.$$

Moreover, it can be seen that $\mathcal{QL}_n = \mathcal{QF}_{n+1} + q\mathcal{QF}_{n-1} = p\mathcal{QF}_n + 2q\mathcal{QF}_{n-1}$ and $\Delta\mathcal{QF}_n = \mathcal{QL}_{n+1} + q\mathcal{QL}_{n-1}$.

2. PRELIMINARIES

In this section, we present some known formulas that are used subsequently.

The following results are given in [5].

LEMMA 2.1. *Let $n \in \mathbb{N}$ and m be a non-zero integer, then*

$$\mathcal{F}_{m+n} = \mathcal{F}_m \mathcal{F}_{n+1} + q\mathcal{F}_{m-1}\mathcal{F}_n$$

and

$$(-q)^{n-1}\mathcal{F}_{m-n} = \mathcal{F}_{m-1}\mathcal{F}_n - \mathcal{F}_m\mathcal{F}_{n-1}.$$

LEMMA 2.2. *Let $n, m \in \mathbb{Z}$, then*

$$\mathcal{L}_{m+n} = \mathcal{F}_n \mathcal{L}_{m+1} + q\mathcal{F}_{n-1}\mathcal{L}_m$$

and

$$(-q)^{n}\mathcal{L}_{m-n} = \mathcal{F}_{m+1}\mathcal{L}_n - \mathcal{L}_{n+1}\mathcal{F}_m.$$

LEMMA 2.3. *Let $m, r \in \mathbb{Z}$ with $m \neq 0$ and $m \neq 1$. Then*

$$\mathcal{F}_{mn+r} = \sum_{j=0}^{n} \binom{n}{j} q^{n-j}\mathcal{F}_m^j \mathcal{F}_{m-1}^{n-j}\mathcal{F}_{j+r}$$

and

$$\mathcal{L}_{mn+r} = \sum_{j=0}^{n} \binom{n}{j} q^{n-j}\mathcal{F}_m^j \mathcal{F}_{m-1}^{n-j}\mathcal{L}_{j+r}.$$

The following results are found in [6].

LEMMA 2.4. *(Binet Formula) Let $\mathcal{QF}_n$ be the n-th (p, q)-Fibonacci quaternion. Then,*

$$\mathcal{QF}_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

where $A = \sum_{s=0}^{3} \alpha^s e_s$ and $B = \sum_{s=0}^{3} \beta^s e_s$.

LEMMA 2.5. *For any non-negative integer n, the ordinary generating function for $\mathcal{QF}_n$ is*

$$G_{\mathcal{F}}(s) = \frac{\mathcal{QF}_0 + (-p\mathcal{QF}_0 + \mathcal{QF}_1)s}{1 - ps - qs^2}.$$

LEMMA 2.6. *For any non-negative integer n, the exponential generating function for $\mathcal{QF}_n$ is*

$$G_{\mathcal{F}}(s) = \frac{Ae^{\alpha s} - Be^{\beta s}}{\alpha - \beta}.$$

LEMMA 2.7. *Let m be a non-negative integer. Then,*

$$\mathcal{QF}_{2n} = \sum_{j=0}^{n} \binom{n}{j} q^{n-j} p^j \mathcal{QF}_j.$$

3. SOME IDENTITIES INVOLVING (p, q)-FIBONACCI AND LUCAS QUATERNIONS

In this section, we derive some new identities concerning (p, q)-Fibonacci and (p, q)-Lucas quaternions.

THEOREM 3.1. *If m, k and n are positive integers with $m \geq n$, then*

(i) $\mathcal{QF}_{kn+m} = \mathcal{F}_m \mathcal{QF}_{kn+1} + q\mathcal{F}_{m-1} \mathcal{QF}_{kn}$
(ii) $(-q)^{n-1} \mathcal{QF}_{m-kn} = \mathcal{F}_{kn} \mathcal{QF}_{m-1} - \mathcal{F}_{kn-1} \mathcal{QF}_m$
(iii) $\mathcal{QL}_{kn+m} = \mathcal{F}_{kn} \mathcal{QL}_{m+1} + q\mathcal{F}_{kn-1} \mathcal{QL}_m$
(iv) $(-q)^n \mathcal{QL}_{m-kn} = \mathcal{L}_{kn} \mathcal{QF}_{m+1} - \mathcal{L}_{kn+1} \mathcal{QF}_m$

Proof. In order to prove the identity (i), we use Lemma 2.1 to get

$$\begin{aligned} \mathcal{QF}_{kn+m} &= \sum_{s=0}^{3} \mathcal{F}_{kn+m+s} e_s \\ &= \mathcal{F}_m \sum_{s=0}^{3} \mathcal{F}_{kn+1+s} e_s + q\mathcal{F}_{m-1} \sum_{s=0}^{3} \mathcal{F}_{kn+s} e_s, \end{aligned}$$

and the result follows. The proof of (ii) is similar as (i). The identities (iii) and (iv) can be done similarly using Lemma 2.2. □

THEOREM 3.2. *Any natural numbers k, m and n, the generating functions of the quaternions $\mathcal{QF}_{kn+m}$ and $\mathcal{QL}_{kn+m}$ are given by*

$$\sum_{n=0}^{\infty} \mathcal{QF}_{kn+m} s^n = \frac{\mathcal{QF}_m - (-q)^k \mathcal{QF}_{m-k} s}{1 - \mathcal{L}_k s + (-q)^k s^2}$$

and

$$\sum_{n=0}^{\infty} \mathcal{QL}_{kn+m} s^n = \frac{\mathcal{QL}_m - (-q)^k \mathcal{QL}_{m-k} s}{1 - \mathcal{L}_k s + (-q)^k s^2}.$$

Proof. Using the Binet formula for $\mathcal{QF}_n$, we have

$$\sum_{n=0}^{\infty} \mathcal{QF}_{kn+m} s^n = \sum_{n=0}^{\infty} \left(\frac{A\alpha^{kn+m} - B\beta^{kn+m}}{\alpha - \beta} \right) s^n$$

$$= \frac{1}{\alpha-\beta}\left(\frac{A\alpha^m}{1-\alpha^k s} - \frac{B\beta^m}{1-\beta^k s}\right)$$
$$= \frac{1}{\alpha-\beta}\left[\frac{(A\alpha^m - B\beta^m) - (-q)^k(A\alpha^{m-k} - B\beta^{m-k})s}{1-(\alpha^k+\beta^k)s + (\alpha\beta)^k s^2}\right]$$
$$= \frac{\mathcal{QF}_m - (-q)^k \mathcal{QF}_{m-k} s}{1-\mathcal{L}_k s + (-q)^k s^2},$$

which is the desired result.

For $\sum_{n=0}^{\infty} \mathcal{QL}_{kn+m} s^n$, the proof is similar as $\sum_{n=0}^{\infty} \mathcal{QF}_{kn+m} s^n$. □

Observation 3.3. It can be seen that for $k = 1$ and $m = 0$ in the first result of Theorem 3.2, we get the generating functions for $\mathcal{QF}_n$ as

$$\sum_{n=0}^{\infty} \mathcal{QF}_n s^n = \frac{\mathcal{QF}_0 + q\mathcal{QF}_{-1}s}{1-ps-qs^2} = \frac{\mathcal{QF}_0 + (-p\mathcal{QF}_0 + \mathcal{QF}_1)s}{1-ps-qs^2}.$$

This result which is given in Lemma 2.5 is already shown in [6].

Again putting for $k = 1$ and $m = 0$ in the second result of Theorem 3.2, we obtain the generating functions for (p, q)-Lucas quaternions $\mathcal{QL}_n$ as

$$\sum_{n=0}^{\infty} \mathcal{QL}_n s^n = \frac{\mathcal{QL}_0 + q\mathcal{QL}_{-1}s}{1-ps-qs^2} = \frac{\mathcal{QL}_0 + (-p\mathcal{QL}_0 + \mathcal{QL}_1)s}{1-ps-qs^2}.$$

Theorem 3.4. *For $k, m, n \in \mathbb{N}$, the exponential generating functions for $\mathcal{QF}_{kn+m}$ and $\mathcal{QL}_{kn+m}$ are*

$$\sum_{n=0}^{\infty} \frac{\mathcal{QF}_{kn+m}}{n!} s^n = \frac{A\alpha^m e^{\alpha^k s} - B\beta^m e^{\beta^k s}}{\alpha-\beta}$$

and

$$\sum_{n=0}^{\infty} \frac{\mathcal{QL}_{kn+m}}{n!} s^n = A\alpha^m e^{\alpha^k s} + B\beta^m e^{\beta^k s},$$

where $A = \sum_{s=0}^{3} \alpha^s e_s$ and $B = \sum_{s=0}^{3} \beta^s e_s$.

Proof. Using Binet formula for $\mathcal{QL}_n$, we obtain

$$\sum_{n=0}^{\infty} \frac{\mathcal{QL}_{kn+m}}{n!} s^n = \sum_{n=0}^{\infty} \left(A\alpha^{kn+m} + B\beta^{kn+m}\right)\frac{s^n}{n!}$$
$$= A\alpha^m \sum_{n=0}^{\infty} \frac{(\alpha^k s)^n}{n!} + B\beta^m \sum_{n=0}^{\infty} \frac{(\beta^k s)^n}{n!}$$
$$= A\alpha^m e^{\alpha^k s} + B\beta^m e^{\beta^k s},$$

which ends the proof. The proof is similar for (p,q)-Fibonacci quaternions $\mathcal{QF}_{kn+m}$. $\square$

THEOREM 3.5 (Catalan's identity). *Let $n, r \in \mathbb{N}$ with $n \geq r$, then*

$$\mathcal{QF}_{n-r}\mathcal{QF}_{n+r} - \mathcal{QF}_n^2 = \frac{(-q)^{n-r}(\alpha^r - \beta^r)\left[\beta^r AB - \alpha^r BA\right]}{\Delta}$$

and

$$\mathcal{QL}_{n-r}\mathcal{QL}_{n+r} - \mathcal{QL}_n^2 = (-q)^{n-r}(\alpha^r - \beta^r)\left[\alpha^r BA - \beta^r AB\right],$$

where $\Delta = p^2 + 4q$.

Proof. Using the Binet formula for $\mathcal{QF}_n$ and the fact $\alpha\beta = -q$, we have

$$\begin{aligned}
&\mathcal{QF}_{n-r}\mathcal{QF}_{n+r} - \mathcal{QF}_n^2 \\
&= \left(\frac{A\alpha^{n-r} - B\beta^{n-r}}{\alpha - \beta}\right)\left(\frac{A\alpha^{n+r} - B\beta^{n+r}}{\alpha - \beta}\right) - \left(\frac{A\alpha^n - B\beta^n}{\alpha - \beta}\right)^2 \\
&= \frac{AB(-q)^n\left(1 - \left(\frac{\beta}{\alpha}\right)^r\right) + BA(-q)^n\left(1 - \left(\frac{\alpha}{\beta}\right)^r\right)}{(\alpha - \beta)^2} \\
&= \frac{AB(-q)^{n-r}\beta^r(\alpha^r - \beta^r) - BA(-q)^{n-r}\alpha^r(\alpha^r - \beta^r)}{\Delta},
\end{aligned}$$

which completes the proof. The proof for the (p,q)-Lucas quaternions is similar. $\square$

The next result directly follows from Theorem 3.5 for $r = 1$.

COROLLARY 3.6 (Cassini's identity). *Let $n \in \mathbb{N}$, then*

$$\mathcal{QF}_{n-1}\mathcal{QF}_{n+1} - \mathcal{QF}_n^2 = \frac{(-q)^{n-1}(\beta AB - \alpha BA)}{\sqrt{\Delta}}$$

and

$$\mathcal{QL}_{n-1}\mathcal{QL}_{n+1} - \mathcal{QL}_n^2 = (-q)^{n-1}\sqrt{\Delta}\left[\alpha BA - \beta AB\right].$$

THEOREM 3.7 (d'Ocagne's identity). *Let $m, n \in \mathbb{N}$ with $n \geq m$, then*

$$\mathcal{QF}_{m+1}\mathcal{QF}_n - \mathcal{QF}_m\mathcal{QF}_{n+1} = \frac{(-q)^m\left[BA\alpha^{n-m} - AB\beta^{n-m}\right]}{\sqrt{\Delta}}$$

and

$$\mathcal{QL}_{m+1}\mathcal{QL}_n - \mathcal{QL}_m\mathcal{QL}_{n+1} = (-q)^m\sqrt{\Delta}\left[AB\beta^{n-m} - BA\alpha^{n-m}\right].$$

Proof. Using the Binet formula for $\mathcal{QL}_n$, we have

$$\begin{aligned}
&\mathcal{QL}_{m+1}\mathcal{QL}_n - \mathcal{QL}_m\mathcal{QL}_{n+1} \\
&= \left(A\alpha^{m+1} + B\beta^{m+1}\right)(A\alpha^n + B\beta^n) - (A\alpha^m + B\beta^m)\left(A\alpha^{n+1} + B\beta^{n+1}\right)
\end{aligned}$$

$$= (-q)^m \left[AB\beta^{n-m}(\alpha-\beta) + BA\alpha^{n-m}(\beta-\alpha)\right]$$
$$= (-q)^m\sqrt{\Delta}\left[AB\beta^{n-m} - BA\alpha^{n-m}\right],$$

and the result follows. Similarly, using Binet formula for (p,q)-Fibonacci quaternions, the second part can be easily shown. □

THEOREM 3.8. *For any natural numbers m and k with $m > k \geq 0$, we have*

$$\sum_{r=0}^{n} \mathcal{QF}_{mr+k} = \frac{(-q)^m\mathcal{QF}_{mn+k} - \mathcal{QF}_{mn+m+k} - (-q)^m\mathcal{QF}_{k-m} + \mathcal{QF}_k}{1+(-q)^m - \mathcal{L}_m}$$

and

$$\sum_{r=0}^{n} \mathcal{QL}_{mr+k} = \frac{(-q)^m\mathcal{QL}_{mn+k} - \mathcal{QL}_{mn+m+k} - (-q)^m\mathcal{QL}_{k-m} + \mathcal{QL}_k}{1+(-q)^m - \mathcal{L}_m}.$$

Proof. Using the Binet formula of $\mathcal{QF}_k$, we have

$$\begin{aligned}\sum_{r=0}^{n} \mathcal{QF}_{mr+k} &= \sum_{r=0}^{n} \frac{A\alpha^{mr+k} - B\beta^{mr+k}}{\alpha-\beta}\\ &= \frac{1}{\alpha-\beta}\left[A\alpha^k\left(\frac{\alpha^{mn+m}-1}{\alpha^m-1}\right) - B\beta^k\left(\frac{\beta^{mn+m}-1}{\beta^m-1}\right)\right]\\ &= \frac{1}{\alpha-\beta}\left[\frac{(-q)^m(A\alpha^{mn+k}-B\beta^{mn+k})-(A\alpha^{mn+m+k}-B\beta^{mn+m+k})}{1+(-q)^m-(\alpha^m+\beta^m)}\right.\\ &\quad\left.+\frac{(A\alpha^k-B\beta^k)-(-q)^m(A\alpha^{k-m}-B\beta^{k-m})}{1+(-q)^m-(\alpha^m+\beta^m)}\right],\end{aligned}$$

which follows the result. The proof for the (p,q)-Lucas quaternions is similar. □

THEOREM 3.9. *For $m, n \geq 0$,*

$$\mathcal{QF}_{mn} = \sum_{j=0}^{n}\binom{n}{j} q^{n-j}\mathcal{F}_m^j\mathcal{F}_{m-1}^j\mathcal{QF}_j$$

and

$$\mathcal{QL}_{mn} = \sum_{j=0}^{n}\binom{n}{j} q^{n-j}\mathcal{F}_m^j\mathcal{F}_{m-1}^{n-j}\mathcal{QL}_j.$$

Proof. By virtue of Lemma 2.3, we have

$$\mathcal{QL}_{mn} = \sum_{s=0}^{3}\mathcal{L}_{mn+s}e_s$$

$$= \sum_{s=0}^{3}\left(\sum_{j=0}^{n}\binom{n}{j} q^{n-j}\mathcal{F}_m^j\mathcal{F}_{m-1}^j\mathcal{L}_{j+s}\right) e_s$$
$$= \sum_{j=0}^{n}\binom{n}{j} q^{n-j}\mathcal{F}_m^j\mathcal{F}_{m-1}^j\left(\sum_{s=0}^{3}\mathcal{L}_{j+s}e_s\right),$$

which completes the proof. The first part can be proved similarly using Lemma 2.3. □

Observation 3.10. Setting $m = 2$ in the first result of Theorem 3.9, we obtain $\mathcal{QF}_{2n} = \sum_{j=0}^{n}\binom{n}{j} q^{n-j}p^j\mathcal{QF}_j$. This result is given in Lemma 2.7 which is already shown in [6]. Putting $m = 2$ in the second result of Theorem 3.9, we have

$$\mathcal{QL}_{2n} = \sum_{j=0}^{n}\binom{n}{j} q^{n-j}p^j\mathcal{QL}_j.$$

THEOREM 3.11. *For every $n \in \mathbb{N}$, $\mathcal{QL}_n^2 - \Delta\mathcal{QF}_n^2 = 2(-q)^n(AB + BA)$.*

Proof. Using the Binet formula of $\mathcal{QF}_k$ and $\mathcal{QL}_k$, we have

$$\begin{aligned}\mathcal{QL}_n^2 - \Delta\mathcal{QF}_n^2 &= (A\alpha^n + B\beta^n)^2 - \Delta\left(\frac{A\alpha^n - B\beta^n}{\alpha-\beta}\right)^2\\ &= \left[A^2\alpha^{2n} + AB(\alpha\beta)^n + BA(\alpha\beta)^n + B^2\beta^{2n}\right]\\ &- \left[A^2\alpha^{2n} - AB(\alpha\beta)^n - BA(\alpha\beta)^n + B^2\beta^{2n}\right]\\ &= 2(-q)^n(AB+BA),\end{aligned}$$

which ends the proof. □

Matrix methods are useful tools to obtain results for different identities and algebraic representations in the study of recurrence relations. (p,q)-Fibonacci numbers are also generated through matrices. That is, $Q_{\mathcal{F}} = \begin{pmatrix} p & q\\ 1 & 0\end{pmatrix}$ and shown that $Q_{\mathcal{F}}^n = \begin{pmatrix}\mathcal{F}_{n+1} & q\mathcal{F}_n\\ \mathcal{F}_n & q\mathcal{F}_{n-1}\end{pmatrix}$.

We define (p,q)-Fibonacci quaternion matrix as a second order matrix whose entries being (p,q)-Fibonacci quaternions as follows.

$$M_{Q_{\mathcal{F}}^n} = \begin{pmatrix}\mathcal{QF}_{n+1} & q\mathcal{QF}_n\\ \mathcal{QF}_n & q\mathcal{F}_{n-1}\end{pmatrix}.$$

THEOREM 3.12. *For an integer $n \geq 1$,*

$$\begin{pmatrix}\mathcal{QF}_{n+1} & q\mathcal{QF}_n\\ \mathcal{QF}_n & q\mathcal{QF}_{n-1}\end{pmatrix} = \begin{pmatrix}\mathcal{QF}_2 & q\mathcal{QF}_1\\ \mathcal{QF}_1 & q\mathcal{QF}_0\end{pmatrix}\begin{pmatrix} p & q\\ 1 & 0\end{pmatrix}^{n-1}.$$

Proof. We will use the method of induction to prove this result. Basis step is clear for $n = 2$. As an inductive hypothesis, assume that the relation holds for all positive integers $n < k$ ($k \in \mathbb{Z}_{\geq 1}$). Finally, in the inductive step,

$$\begin{pmatrix} \mathcal{QF}_2 & q\mathcal{QF}_1 \\ \mathcal{QF}_1 & q\mathcal{QF}_0 \end{pmatrix} \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \mathcal{QF}_{n+1} & q\mathcal{QF}_n \\ \mathcal{QF}_n & q\mathcal{QF}_{n-1} \end{pmatrix} \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} p\mathcal{QF}_{n+1} + q\mathcal{QF}_n & q\mathcal{QF}_{n+1} \\ p\mathcal{QF}_n + q\mathcal{QF}_{n-1} & q\mathcal{QF}_n \end{pmatrix}.$$

Hence, by definition (1.1), the result holds for $k = n$. □

Cassini formula for (p, q)-Fibonacci quaternion can also be obtained from (p, q)-Fibonacci quaternion matrix. The following result demonstrates this fact.

COROLLARY 3.13. *For an integer $n \geq 1$,*

$$\mathcal{QF}_{n+1}\mathcal{QF}_{n-1} - \mathcal{QF}_n^2 = \frac{(-q)^{n-1}(\beta AB - \alpha BA)}{(\alpha - \beta)}.$$

Proof. Evaluating the determinants from both sides of the above result and after some algebraic manipulation, the desired result is obtained. □

4. CONCLUSION

In this article, we have introduced (p, q)-Lucas quaternions which are based on generalized Lucas numbers. Some new identities of (p, q)-Fibonacci and Lucas quaternions are also established. Some results of İpek [6] are the particular cases of the present work.

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