TOPICS IN PRIME SUBMODULES AND OTHER ASPECTS OF THE PRIME AVOIDANCE THEOREM

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Let R be a commutative ring with identity and M be a unital R-module. In this paper, we study some properties of prime submodules. Finally, we prove various statements about prime avoidance for modules.

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1. INTRODUCTION

Throughout this paper, let R be a commutative ring (with identity) and M be a unital R-module. A proper submodule N of M with $N :_R M = \mathfrak{p}$ is said to be prime or $\mathfrak{p}$-prime ($\mathfrak{p}$ a prime ideal of R) if $rx \in N$ for $r \in R$ and $x \in M$ implies that either $x \in N$ or $r \in \mathfrak{p}$. Another equivalent notion of prime submodules was first introduced and systematically studied in [4]. Prime submodules have been studied by several authors; see, for example, [1,2,5,7–10,12]. In Section 2, we study the chains of prime submodules and we shall improve the results given in [9]. The prime avoidance theorem states that if an ideal I of a ring is contained in the union of finite number of prime ideals, then I must be contained in one of them. This result's generalization for the non-commutative case has been proved in [6]. In Section 3, we generalize this theorem for modules in different states. Throughout, for any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\mathrm{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$ and we denote $\{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{b}\}$ by $V(\mathfrak{b})$, where $\mathrm{Spec}(R)$ denotes the set of all prime ideals of R. The symbol $\subseteq$ denotes containment and $\subset$ denotes proper containment for sets. If N is a submodule of M, we write $N \leq M$. We denote the annihilator of a factor module M/N of M by $(N :_R M)$. The set of all maximal ideals of R is denoted by $\mathrm{Max}(R)$. For any ideal I of a ring R and for any R-module M, $\Gamma_I(M)$ is defined to be the submodule of M consisting of all elements annihilated by some power of I, *i.e.*, $\bigcup_{n=1}^{\infty}(0 :_M I^n)$. For any unexplained notation and terminology we refer the reader to [3, 11] and [13].

2. CHAINS OF PRIME SUBMODULES

The results of this section are generalizations of some results given in [9] and [2]. First, we need the following definition.

Definition 2.1. Let R be a Noetherian ring and M be a finitely generated R-module. For each $\mathfrak{p} \in \operatorname{Spec}(R)$ we define $\lambda_{\mathfrak{p}}(M)$ as following:

$$\lambda_{\mathfrak{p}}(M) = \dim_{R_{\mathfrak{p}}/\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}}/\mathfrak{p} M_{\mathfrak{p}}).$$

Remark 2.2. Let R be a Noetherian ring and M be a finitely generated R-module. For each $\mathfrak{p} \in \operatorname{Spec}(R)$, $\lambda_{\mathfrak{p}}(M)$ is the number of elements of any minimal generator set of the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ and so $\lambda_{\mathfrak{p}}(M) < \infty$. Also we have $\lambda_{\mathfrak{p}}(M) = 0$ if and only if $\mathfrak{p} \notin \operatorname{Supp}(M)$. Moreover, for any pair $\mathfrak{q} \subseteq \mathfrak{p}$ of prime ideals of R it is easy to see that $\lambda_{\mathfrak{q}}(M) \leq \lambda_{\mathfrak{p}}(M)$.

The following description of prime submodules will be useful in this paper.

LEMMA 2.3. *Let R be a Noetherian ring and $\mathfrak{p} \in \operatorname{Spec}(R)$. Let M be a finitely generated R-module and N be a proper submodule of M. Then the following are equivalent:*
(i) *N is $\mathfrak{p}$-prime submodule of M.*
(ii) $\operatorname{Ass}_R(M/N) = \{\mathfrak{p}\}$ *and* $(N :_R M) = \mathfrak{p}$.
(iii) $(N :_R x) = \mathfrak{p}$, *for each* $x \in M \backslash N$.

Proof. Easily follows from definition. □

The following theorem is the first main result of this paper and a generalization of [9, Lemma 2.6].

THEOREM 2.4. *Let R be a Noetherian ring and $\mathfrak{p} \in \operatorname{Supp}(M)$. Let M be a finitely generated R-module. Then the following statements hold:*
(i) *The length of any chain of $\mathfrak{p}$-prime submodules of M is bounded from above by $\lambda_{\mathfrak{p}}(M) - 1$.*
(ii) *There is a chain of $\mathfrak{p}$-prime submodules of M, which is of length $\lambda_{\mathfrak{p}}(M) - 1$.*
(iii) *Any saturated maximal chain of $\mathfrak{p}$-prime submodules of M is of length $\lambda_{\mathfrak{p}}(M) - 1$.*

Proof. (i) Let $n := \lambda_{\mathfrak{p}}(M)$. Then it follows from the hypothesis $\mathfrak{p} \in \operatorname{Supp}(M)$ that $n > 0$. Suppose the contrary be true. Then there exists a chain of $\mathfrak{p}$-prime submodules of M as:

$$N_0 \subset N_1 \subset \cdots \subset N_n.$$

By Lemma 2.3 we have $\mathfrak{p} \in \operatorname{Supp}(M/N_n)$ and so $l_{R_{\mathfrak{p}}}((M/N_n)_{\mathfrak{p}}) \geq 1$. On the other hand, since by assumption we have $(N_0 :_R M) = \mathfrak{p}$, it follows that there is an exact sequence

$$M/\mathfrak{p} M \to M/N_0 \to 0.$$

Hence we have the following exact sequence:

$$(M/\mathfrak{p}M)_{\mathfrak{p}} \to (M/N_0)_{\mathfrak{p}} \to 0.$$

Therefore, it follows from definition that

$$l_{R_{\mathfrak{p}}}((M/N_0)_{\mathfrak{p}}) = \dim_{R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}}((M/N_0)_{\mathfrak{p}}) \leq \lambda_{\mathfrak{p}}(M) = n.$$

On the other hand, for each $0 \leq i \leq n-1$ there is an exact sequence

$$0 \to N_{i+1}/N_i \to M/N_i.$$

But, since $N_{i+1}/N_i \neq 0$, it follows from Lemma 2.3 and above exact sequence that

$$\emptyset \neq \operatorname{Ass}_R(N_{i+1}/N_i) \subseteq \operatorname{Ass}_R(M/N_i) = \{\mathfrak{p}\},$$

which implies that $\operatorname{Ass}_R(N_{i+1}/N_i) = \{\mathfrak{p}\}$. In particular $\mathfrak{p} \in \operatorname{Supp}(N_{i+1}/N_i)$, and so $(N_{i+1}/N_i)_{\mathfrak{p}} \neq 0$. Consequently, $l_{R_{\mathfrak{p}}}((N_{i+1}/N_i)_{\mathfrak{p}}) \geq 1$. Whence, we have

$$n = \Sigma_{i=0}^{n-1} 1 \leq \Sigma_{i=0}^{n-1} l_{R_{\mathfrak{p}}}((N_{i+1}/N_i)_{\mathfrak{p}}) = l_{R_{\mathfrak{p}}}((N_n/N_0)_{\mathfrak{p}}) \leq l_{R_{\mathfrak{p}}}((M/N_0)_{\mathfrak{p}}) - 1 \leq n-1,$$

which is a contradiction.

(ii) Let $\lambda_{\mathfrak{p}}(M) = n$. Then $n > 0$. As $\mathfrak{p} \in \operatorname{Supp}(M)$ it follows that $(\mathfrak{p}M :_R M) = \mathfrak{p}$. Therefore, $\mathfrak{p} \in \operatorname{Ass}_R(M/\mathfrak{p}M)$. Let $N_0 = \mathfrak{p}M$, whenever $\operatorname{Ass}_R(M/\mathfrak{p}M) = \{\mathfrak{p}\}$. In other case, suppose

$$\operatorname{Ass}_R(M/\mathfrak{p}M)\backslash\{\mathfrak{p}\} := \{\mathfrak{q}_1, ..., q_k\}.$$

Let $I = \cap_{j=1}^{k} \mathfrak{q}_j$ and $N_0/\mathfrak{p}M := \Gamma_I(M/\mathfrak{p}M)$. Then we have

$$\operatorname{Ass}_R(M/N_0) = \operatorname{Ass}_R((M/\mathfrak{p}M)/\Gamma_I(M/\mathfrak{p}M)) = \operatorname{Ass}_R(M/\mathfrak{p}M)\backslash V(I).$$

But, since for each $1 \leq j \leq k$ we have $\operatorname{Ann}_R(M/\mathfrak{p}M) = \mathfrak{p} \subseteq \mathfrak{q}_j$ and $\mathfrak{q}_j \neq \mathfrak{p}$, it follows that $\mathfrak{p} \notin V(\mathfrak{q}_j)$. Therefore

$$\mathfrak{p} \notin \bigcup_{j=1}^{k} V(\mathfrak{q}_j) = V(\cap_{j=1}^{k} \mathfrak{q}_j) = V(I).$$

Therefore,

$$\operatorname{Ass}_R(M/N_0) = \operatorname{Ass}_R(M/\mathfrak{p}M)\backslash V(I) = \{\mathfrak{p}\},$$

which results $\operatorname{Ann}_R(M/N_0) \subseteq \mathfrak{p}$. Therefore, we have $\mathfrak{p} = (\mathfrak{p}M :_R M) \subseteq (N_0 :_R M) \subseteq \mathfrak{p}$ and so $(N_0 :_R M) = \mathfrak{p}$. Also as

$$\operatorname{Ass}_R(N_0/\mathfrak{p}M) = \operatorname{Ass}_R(\Gamma_I(M/\mathfrak{p}M)) = \operatorname{Ass}_R(M/\mathfrak{p}M) \cap V(I),$$

it follows that $\mathfrak{p} \notin \operatorname{Supp}(N_0/\mathfrak{p}M)$ and hence $(N_0/\mathfrak{p}M)_{\mathfrak{p}} = 0$. Now in both cases it follows from Lemma 2.3 that N_0 is a $\mathfrak{p}$-prime submodule of M. We shall construct the chain $N_0 \subset \cdots \subset N_{n-1}$ of $\mathfrak{p}$-prime submodules of M such that $l_{R_\mathfrak{p}}((N_{i+1}/N_i)_\mathfrak{p}) = 1$, for each $0 \leq i \leq n-2$, by an inductive process. To do this, assume that $0 \leq j < n-1$, and that we have already constructed $N_0 \subset N_1 \subset \cdots \subset N_j$. We show how to construct N_{j+1}. To do this, since by definition $M \neq N_j$ it follows that there is an element $x \in M \backslash N_j$. Let $L := Rx + N_j$. In view of Lemma 2.3 we have $L/N_j \cong R/\mathfrak{p}$. In particular, we have $l_{R_\mathfrak{p}}((L/N_j)_\mathfrak{p}) = 1$. By inductive hypothesis we have

$$l_{R_\mathfrak{p}}((M/L)_\mathfrak{p}) = l_{R_\mathfrak{p}}((M/N_0)_\mathfrak{p}) - l_{R_\mathfrak{p}}((L/N_0)_\mathfrak{p}) =$$

$$l_{R_\mathfrak{p}}((M/\mathfrak{p}M)_\mathfrak{p}) - [l_{R_\mathfrak{p}}((L/N_j)_\mathfrak{p}) + \Sigma_{i=0}^{j-1} l_{R_\mathfrak{p}}((N_{i+1}/N_i)_\mathfrak{p})] = n-(1+j) = n-j-1 > 0.$$

Therefore, $(M/L)_\mathfrak{p} \neq 0$. Now it is easy to see that $(L :_R M) = \mathfrak{p}$, and so $\mathfrak{p} \in \mathrm{Ass}_R(M/L)$. Let $N_{j+1} = L$, whenever $\mathrm{Ass}_R(M/L) = \{\mathfrak{p}\}$. In other case suppose

$$\mathrm{Ass}_R(M/L) \backslash \{\mathfrak{p}\} := \{\mathfrak{q}'_1, ..., q'_t\}.$$

Let $J = \cap_{i=1}^{t} \mathfrak{q}'_i$ and $N_{j+1}/L := \Gamma_J(M/L)$. Then we have

$$\mathrm{Ass}_R(M/N_{j+1}) = \mathrm{Ass}_R((M/L)/\Gamma_J(M/L)) = \mathrm{Ass}_R(M/L) \backslash V(J).$$

But, since for each $1 \leq i \leq t$ we have $\mathrm{Ann}_R(M/L) = \mathfrak{p} \subseteq \mathfrak{q}'_i$ and $\mathfrak{q}'_i \neq \mathfrak{p}$, it follows that $\mathfrak{p} \notin V(\mathfrak{q}'_i)$. Therefore,

$$\mathrm{Ass}_R(M/N_{j+1}) = \mathrm{Ass}_R(M/L) \backslash V(J) = \{\mathfrak{p}\},$$

which results $\mathrm{Ann}_R(M/N_{j+1}) \subseteq \mathfrak{p}$. Therefore, we have $\mathfrak{p} = (L :_R M) \subseteq (N_{j+1} :_R M) \subseteq \mathfrak{p}$ and so $(N_{j+1} :_R M) = \mathfrak{p}$. Also as

$$\mathrm{Ass}_R(N_{j+1}/L) = \mathrm{Ass}_R(\Gamma_J(M/L)) = \mathrm{Ass}_R(M/L) \cap V(J),$$

it follows that $\mathfrak{p} \notin \mathrm{Supp}(N_{j+1}/L)$ and hence $(N_{j+1}/L)_\mathfrak{p} = 0$. Whence,

$$l_{R_\mathfrak{p}}((N_{j+1}/N_j)_\mathfrak{p}) = l_{R_\mathfrak{p}}((N_{j+1}/L)_\mathfrak{p}) + l_{R_\mathfrak{p}}((L/N_j)_\mathfrak{p}) = 1 + 0 = 1.$$

Now in both cases it follows from Lemma 2.3 that N_{j+1} is a $\mathfrak{p}$-prime submodule of M such that $l_{R_\mathfrak{p}}((N_{j+1}/N_j)_\mathfrak{p}) = 1$. This completes the inductive step in the construction.

(iii) Let $\lambda_\mathfrak{p}(M) = n$ and $N_0 \subset \cdots \subset N_k$ be a saturated maximal chain of $\mathfrak{p}$-prime submodules of M. We show that $k = n-1$. By (i) we have $k \leq n-1$. Since by assumption this chain is maximal it follows from the proof of (ii) that $l_{R_\mathfrak{p}}((M/N_k)_\mathfrak{p}) = 1$. Now suppose the contrary be true. Then the set

$$E := \{N \ : \ N \text{ is a } \mathfrak{p}\text{-prime submodule of } M\},$$

has a unique minimal element $N' := \cap_{N \in E} N$ with respect to " $\subseteq$ ". So it follows from hypothesis that $N_0 = N'$. Also using (i) it follows from the proof of (ii) that $(N_0/\mathfrak{p}M)_\mathfrak{p} = 0$. Therefore,

$$l_{R_\mathfrak{p}}((N_k/N_0)_\mathfrak{p}) = n-1.$$

Now suppose the contrary be true and $k < n-1$. Then we deduce that there is $0 \leq j \leq k-1$, such that $l_{R_{\mathfrak{p}}}((N_{j+1}/N_j)_{\mathfrak{p}}) \geq 2$. Then there is $x \in N_{j+1} \backslash N_j$. By Lemma 2.3 we have $(N_j + Rx)/N_j \cong R/\mathfrak{p}$ and so $l_{R_{\mathfrak{p}}}(((N_j + Rx)/N_j)_{\mathfrak{p}}) = 1$. Let $L := N_j + Rx$. Since N_{j+1}/L is the unique minimal element of the set

$$\{N/L \ : \ N/L \text{ is a } \mathfrak{p}\text{-prime submodule of } M/L\},$$

again using (i) it follows from the proof of (ii) that $(N_{j+1}/L)_{\mathfrak{p}} = 0$. Thus we have

$$2 \leq l_{R_{\mathfrak{p}}}((N_{j+1}/N_j)_{\mathfrak{p}}) = l_{R_{\mathfrak{p}}}((N_{j+1}/L)_{\mathfrak{p}}) + l_{R_{\mathfrak{p}}}((L/N_j)_{\mathfrak{p}}) = 0 + 1 = 1,$$

which is a contradiction. This completes the proof. $\square$

Now we need the following definitions.

Definition 2.5. Let R be a Noetherian ring and M be a finitely generated R-module. For each $\mathfrak{p}$-prime submodule N of M we define $\mathfrak{p}$-*height of* N as:

$$\mathfrak{p}\text{-ht}(N) := \sup\{k \in \mathbb{N}_0 \ : \ \exists\ N_0 \subset \cdots \subset N_k = N, \text{ with } N_i \in \mathrm{Spec}_R^{\mathfrak{p}}(M),\ \forall\, i\},$$

where $\mathrm{Spec}_R^{\mathfrak{p}}(M)$ denotes to the set of all $\mathfrak{p}$-prime submodules of M as an R-module.

Definition 2.6. Let R be a Noetherian ring and M be a finitely generated R-module. For each $\mathfrak{p}$-prime submodule N of M we define *height of* N as:

$$\mathrm{ht}(N) := \sup\{k \in \mathbb{N}_0 \ : \ \exists\ N_0 \subset \cdots \subset N_k = N, \text{ with } N_i \in \mathrm{Spec}_R(M),\ \forall\, i\},$$

where $\mathrm{Spec}_R(M)$ denotes to the set of all prime submodules of M as an R-module.

Definition 2.7. Let R be a Noetherian ring and M be a finitely generated R-module. Then we define $dimSpec_R(M)$ as:

$$\mathrm{dimSpec}_R(M) := \sup\{\mathrm{ht}(N) \ : \ N \in \mathrm{Spec}_R(M)\}.$$

The following result is an immediate consequence of Theorem 2.4.

COROLLARY 2.8. *Let R be a Noetherian ring and M be a finitely generated R-module and N be a $\mathfrak{p}$-prime submodule of M. Then*

$$\mathfrak{p}\text{-ht}(N) = l_{R_{\mathfrak{p}}}((N/\mathfrak{p}M)_{\mathfrak{p}}) = \dim_{R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}}(N_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}).$$

Proof. Let $k := \mathfrak{p}\text{-ht}(N)$. Then there is saturated chain of $\mathfrak{p}$-prime submodules of M as $N_0 \subset \cdots \subset N_k = N$. By the proof of Theorem 2.4 this chain can be extended to a maximal saturated chain of $\mathfrak{p}$-prime submodules of M as

$$N_0 \subset \cdots \subset N_k = N \subset \cdots \subset N_{n-1},$$

Where $n = \lambda_{\mathfrak{p}}(M)$. Then by the proof of Theorem 2.4 we have $(N_0/\,\mathfrak{p}\,M)_{\mathfrak{p}} = 0$ and $l_{R_{\mathfrak{p}}}((N_{i+1}/N_i)_{\mathfrak{p}}) = 1$, for each $0 \leq i \leq n-2$. Now clearly the assertion holds. $\square$

As an application of Theorem 2.4 we prove the following.

THEOREM 2.9. *Let R be a Noetherian ring and M be a finitely generated R-module and N be a $\mathfrak{p}$-prime submodule of M. Then*

$$\mathrm{ht}(N) \leq (\lambda_{\mathfrak{p}}(M))(\dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})) < \infty.$$

Proof. Let $N_k \subset \cdots \subset N_0 = N$ be a chain of prime submodules of M, such that for each $0 \leq i \leq k$, N_i is $\mathfrak{p}_i$-prime, where $\mathfrak{p}_0 = \mathfrak{p}$. Then it easily follows from definition that

$$\mathfrak{p}_k \subseteq \cdots \subseteq \mathfrak{p}_0 = \mathfrak{p}\,.$$

Therefore, the set $\{\mathfrak{p}_i\}_{i=0}^{k}$ has at most $\dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ elements. (Note that $\mathfrak{p}_i \in \mathrm{Supp}(M)$, for all $0 \leq i \leq k$). Let

$$\{\mathfrak{p}_i\}_{i=0}^{k} = \{\mathfrak{q}_0 = \mathfrak{p}, ..., \mathfrak{q}_t\},$$

where $t \leq \dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ and $\mathfrak{p} = \mathfrak{q}_0 \supset \cdots \supset \mathfrak{q}_t$. Let $A_j := \mathrm{Spec}_R^{\mathfrak{q}_j}(M) \cap \{N_i\}_{i=0}^{k}$, for each $0 \leq j \leq t$. Then by Theorem 2.4 the set A_j has at most $\lambda_{\mathfrak{q}_j}(M)$ elements. But $\lambda_{\mathfrak{q}_j}(M) \leq \lambda_{\mathfrak{p}}(M)$, because $\mathfrak{q}_j \subseteq \mathfrak{p}$. Therefore as

$$\bigcup_{j=1}^{t} A_j = \{N_i\}_{i=0}^{k},$$

it follows that $k \leq t\lambda_{\mathfrak{p}}(M) \leq (\dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}))\lambda_{\mathfrak{p}}(M)$. Which implies that

$$\mathrm{ht}(N) \leq (\lambda_{\mathfrak{p}}(M))(\dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})) < \infty,$$

as required. $\square$

3. PRIME AVOIDANCE THEOREM

The results of this section improve some well known results given in [7].

PROPOSITION 3.1. *Let R be any ring and M be a non-zero R-module and N be a submodule of M. Let $\mathfrak{p}_1, ..., \mathfrak{p}_n$ be distinct prime ideals of R. Let for each $1 \leq i \leq n$, N_i be a $\mathfrak{p}_i$-prime submodule of M. If $N \subseteq \cup_{i=1}^{n} N_i$, then $N \subseteq N_j$ for some $1 \leq j \leq n$.*

Proof. We use induction on n. The case $n = 2$ is easy. Now let $n \geq 3$ and the case $n-1$ is settled. By definition for each $1 \leq i \leq n$ we have $\mathfrak{p}_i = (N_i :_R M)$. From the hypothesis $N \subseteq \cup_{i=1}^{n} N_i$ it follows that $N =$

$\cup_{i=1}^{n}(N_i \cap N)$. Now let the contrary be true. Then $N \not\subseteq N_i$ and hence $(N_i \cap N) \neq N$, for any $1 \leq i \leq n$. Also from the inductive hypothesis it follows that $N \neq \cup_{i\in(\{1,...,n\}\backslash\{k\})}(N_i \cap N)$ for each $1 \leq k \leq n$ and so $(N_k \cap N) \not\subseteq \cup_{i\in(\{1,...,n\}\backslash\{k\})}(N_i \cap N)$. Let $\mathfrak{q}$ be a minimal element of the set $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ with respect to " $\subseteq$ ". Then $\mathfrak{p}_i \not\subseteq \mathfrak{q}$ for each $\mathfrak{p}_i \in (\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}\backslash\{q\})$. Without loss of generality we may assume that $\mathfrak{q} = \mathfrak{p}_n$. Let $J_i := (N_i :_R N)$, for all $i = 1, ..., n$. Then from the definition it follows that $\mathfrak{p}_i \subseteq J_i$, for all $i = 1, ..., n$. On the other hand, for each $x \in N$ and $r \in R$, if $rx \in (N_i \cap N)$ and $x \notin (N_i \cap N)$, then $rx \in N_i$ and $x \notin N_i$. Therefore it follows from the definition that $r \in \mathfrak{p}_i$. So $rM \subseteq N_i$, and consequently, $rN \subseteq (N_i \cap N)$. As $(N_i \cap N) \neq N$ it follows that there exists an element $y \in (N\backslash(N_i \cap N))$. Now for each $s \in J_i$ we have $sy \in (N_i \cap N) \subseteq N_i$ and $y \notin N_i$. So it follows from the definition that $s \in \mathfrak{p}_i$. Therefore, $(N_i :_R N) = J_i = \mathfrak{p}_i = (N_i :_R M)$. But it is easy to see that $(N_i :_R N) = ((N_i \cap N) :_R N)$. Thus for each $1 \leq i \leq n$, $N_i \cap N$ is $\mathfrak{p}_i$-prime submodule of N. Therefore without loss of generality we may assume that $N = M = \cup_{i=1}^{n}N_i$ and $N_n \not\subseteq \cup_{i=1}^{n-1}N_i$. Next let $T := \cap_{i=1}^{n}N_i$. Then it is not difficult to see that for each $1 \leq i \leq n$, N_i/T is $\mathfrak{p}_i$-prime submodule of M/T and $M/T = \cup_{i=1}^{n}N_i/T$. Therefore, without loss of generality we may assume $M = \cup_{i=1}^{n}N_i$ and $\cap_{i=1}^{n}N_i = 0$ and $N_n \not\subseteq \cup_{i=1}^{n-1}N_i$. Then there is an exact sequence $0 \to M \to \oplus_{i=1}^{n}M/N_i$, which implies that $\cap_{i=1}^{n}\mathfrak{p}_i = \mathrm{Ann}_R(\oplus_{i=1}^{n}M/N_i) \subseteq \mathrm{Ann}_R(M)$. On the other hand for each $1 \leq i \leq n$ we have $\mathrm{Ann}_R(M) \subseteq (N_i :_R M) = \mathfrak{p}_i$. So $\mathrm{Ann}_R(M) \subseteq \cap_{i=1}^{n}\mathfrak{p}_i$. Hence $\mathrm{Ann}_R(M) = \cap_{i=1}^{n}\mathfrak{p}_i$. Now if we have $\cap_{i=1}^{n-1}N_i = 0$, then there is an exact sequence $0 \to M \to \oplus_{i=1}^{n-1}M/N_i$, which implies that $\cap_{i=1}^{n-1}\mathfrak{p}_i = \mathrm{Ann}_R(\oplus_{i=1}^{n-1}M/N_i) \subseteq \mathrm{Ann}_R(M) = \cap_{i=1}^{n}\mathfrak{p}_i \subseteq \mathfrak{p}_n$. So $\mathfrak{p}_t \subseteq \mathfrak{p}_n$, for some $1 \leq t \leq n-1$, which is a contradiction. So $\cap_{i=1}^{n-1}N_i \neq 0$. Then there is an element $0 \neq a \in \cap_{i=1}^{n-1}N_i$. As $\cap_{i=1}^{n}N_i = 0$, it follows that $a \notin N_n$. On the other hand since $N_n \not\subseteq \cup_{i=1}^{n-1}N_i$, it follows that there is an element $b \in N_n$ such that $b \notin \cup_{i=1}^{n-1}N_i$. Now as $a + b \in \cup_{i=1}^{n}N_i$, it follows that $a + b \in N_k$ for some $1 \leq k \leq n$, which is a contradiction. This completes the inductive step. □

Remark. Proposition 3.1 does not hold in general. For example, let $p \geq 2$ be a prime number and $2 \leq n \in \mathbb{N}$. Let $R = \mathbb{Z}_p = \{\overline{0}, \overline{1}, ..., \overline{p-1}\}$ and $M = \oplus_{i=1}^{n}\mathbb{Z}_p$. Let

$$\mathfrak{A} = \{N \;:\; N = Rx, \text{ for some } 0 \neq x \in M\}.$$

Then $\mathfrak{A}$ is a finite set that has at most 2^{p^n} elements and for each $N \in \mathfrak{A}$, N is a $\{\overline{0}\}$-prime submodule of M such that $M \subseteq \cup_{N\in\mathfrak{A}}N$. But $M \not\subseteq N$ for any $N \in \mathfrak{A}$. □

The following proposition is a generalization of [11, Ex. 16.8].

PROPOSITION 3.2. *Let R be a ring, M a non-zero R-module, N a submodule of M and $x \in M$. Let $\mathfrak{p}_1, ..., \mathfrak{p}_n$ be distinct prime ideals of R. Let for* *each $1 \leq i \leq n$, N_i be a $\mathfrak{p}_i$-prime submodule of M. If $N + Rx \not\subseteq \cup_{i=1}^{n} N_i$, then there exists $a \in N$ such that $a + x \notin \cup_{i=1}^{n} N_i$.*

Proof. We use induction on n. Let $n = 1$. If $x \in N_1$ then $N \not\subseteq N_1$. So there is $a \in N \backslash N_1$ and it is easy to see that $a + x \notin N_1$. But if $x \notin N_1$, then by choosing $a = 0 \in N$ the assertion holds. Now suppose $n \geq 2$ and the case $n-1$ is settled. Let $\mathfrak{q}$ be a minimal element of the set $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ with respect to "$\subseteq$". Then $\mathfrak{p}_i \not\subseteq \mathfrak{q}$ for each $\mathfrak{p}_i \in (\{\mathfrak{p}_1, ..., \mathfrak{p}_n\} \backslash \{q\})$. Without loss of generality we may assume that $\mathfrak{q} = \mathfrak{p}_n$. Then it is easy to see that $\cap_{i=1}^{n-1} \mathfrak{p}_i \not\subseteq \mathfrak{p}_n$. By inductive hypothesis there is an element $b \in N$ such that $b + x \notin \cup_{i=1}^{n-1} N_i$. So the assertion holds for $a = b$, whenever $b + x \notin N_n$. So we may assume $b + x \in N_n$. Then we claim that $N \not\subseteq N_n$. Because, if $N \subseteq N_n$ then $x \in N_n$ and so $N + Rx \subseteq N_n \subseteq \cup_{i=1}^{n} N_i$, which is a contradiction. Therefore, there exists an element $c \in N \backslash N_n$. As $\cap_{i=1}^{n-1} \mathfrak{p}_i \not\subseteq \mathfrak{p}_n$ it follows that there exists an element $r \in (\cap_{i=1}^{n-1} \mathfrak{p}_i) \backslash \mathfrak{p}_n$. Then it easily follows from the definition of the $\mathfrak{p}_n$-prime submodule that $rc \notin N_n$. Moreover, since $r \in \cap_{i=1}^{n-1} \mathfrak{p}_i$ it follows from the definition that $rc \in \cap_{i=1}^{n-1} N_i$. Now it is easy to see that $rc + b + x \notin \cup_{i=1}^{n} N_i$. Therefore, the assertion holds for $a := rc + b \in N$. This completes the induction step. □

Remark. Proposition 3.2 does not hold in general. For example, let $p \geq 2$ be a prime number and $R = \mathbb{Z}_p = \{\overline{0}, \overline{1}, ..., \overline{p-1}\}$ and $M = \mathbb{Z}_p \oplus \mathbb{Z}_p$. Let $N = (\overline{1}, \overline{0})\mathbb{Z}_p$, $x = (\overline{0}, \overline{1})$ and $N_i = (\overline{i}, \overline{1})\mathbb{Z}_p$, for $i = 0, ..., p-1$. Then N_i is $\{\overline{0}\}$-prime submodule of the R-module M, for all $i = 0, ..., p-1$. Also as $(\overline{1}, \overline{0}) \in N + Rx$ and $(\overline{1}, \overline{0}) \notin \cup_{i=0}^{p-1} N_i$, it follows that $N + Rx \not\subseteq \cup_{i=0}^{p-1} N_i$. But for any $a \in N$ we have $a + x \in \cup_{i=0}^{p-1} N_i$.

Now we give other aspects of prime avoidance Theorem in different states.

Proposition 3.3. *Let R be a ring, M a non-zero R-module, N a submodule of M and $k \in \mathbb{N}$. Let for each $1 \leq i \leq k$, $n_i \in \mathbb{N}$ and for $1 \leq i \leq k$ and $1 \leq j \leq n_i$, the ideals $\mathfrak{p}_{i,j}$ be distinct elements of* $\mathrm{Spec}(R)$. *Let for each $1 \leq i \leq k$ and $1 \leq j \leq n_i$, $N_{i,j}$ be a $\mathfrak{p}_{i,j}$-prime submodule of M. Let for each $1 \leq i \leq k$, $N_i = \cap_{j=1}^{n_i} N_{i,j}$. If $N \subseteq \cup_{i=1}^{k} N_i$, then $N \subseteq N_t$ for some $1 \leq t \leq k$.*

Proof. Let the contrary be true. Then for each $1 \leq i \leq k$ we have $N \not\subseteq N_i$. Therefore there exists $1 \leq s_i \leq n_i$ such that $N \not\subseteq N_{i,s_i}$. But in this situation we have

$$N \subseteq \cup_{i=1}^{k} N_i \subseteq \cup_{i=1}^{k} N_{i,s_i}.$$

Consequently, it follows from Proposition 3.1 that there is $1 \leq l \leq k$, such that $N \subseteq N_{l,s_l}$, which is a contradiction. □

Proposition 3.4. *Let R be a ring, M a non-zero R-module, N a submodule of M, $x \in M$ and $k \in \mathbb{N}$. Let for each $1 \leq i \leq k$, $n_i \in \mathbb{N}$ and for $1 \leq i \leq k$ and $1 \leq j \leq n_i$, the ideals $\mathfrak{p}_{i,j}$ be distinct elements of* $\mathrm{Spec}(R)$. *Let for each $1 \leq i \leq k$ and $1 \leq j \leq n_i$, $N_{i,j}$ be a $\mathfrak{p}_{i,j}$-prime submodule of M. Let for each $1 \leq i \leq k$, $N_i = \cap_{j=1}^{n_i} N_{i,j}$. If $N + Rx \not\subseteq \cup_{i=1}^{k} N_i$, then there exists $a \in N$ such that $a + x \notin \cup_{i=1}^{k} N_i$.*

Proof. For each $1 \leq i \leq k$ we have $N + Rx \not\subseteq N_i$. Therefore there exists $1 \leq s_i \leq n_i$ such that $N + Rx \not\subseteq N_{i,s_i}$. But in this situation using Proposition 3.1 we have

$$N + Rx \not\subseteq \cup_{i=1}^{k} N_{i,s_i}.$$

Consequently, it follows from Proposition 3.2 that there is $a \in N$, such that $a+x \notin \cup_{i=1}^{k} N_{i,s_i}$. But since $\cup_{i=1}^{k} N_i \subseteq \cup_{i=1}^{k} N_{i,s_i}$, it follows that $a+x \notin \cup_{i=1}^{k} N_i$, as required. □

Proposition 3.5. *Let R be a ring, I an ideal of R and $x \in R$. Let $J_1, ..., J_n$, $(n \geq 1)$ be ideals of R such that for each $1 \leq i \leq n$ we have $\mathrm{Rad}(J_i) = J_i$. If $I + Rx \not\subseteq \cup_{i=1}^{n} J_i$, then there exists an element $a \in I$ such that $a + x \notin \cup_{i=1}^{n} J_i$.*

Proof. For each $1 \leq i \leq n$ we have $I + Rx \not\subseteq J_i$. Therefore for each $1 \leq i \leq n$, since $J_i = \cap_{\mathfrak{q} \in V(J_i)} \mathfrak{q}$ it follows that there exists $\mathfrak{p}_i \in V(J_i)$ such that $I + Rx \not\subseteq \mathfrak{p}_i$. But in this situation we have $I + Rx \not\subseteq \cup_{i=1}^{n} \mathfrak{p}_i$. Consequently, it follows from [11, Ex. 16.8] that there is $a \in I$, such that $a + x \notin \cup_{i=1}^{n} \mathfrak{p}_i$. But since $\cup_{i=1}^{n} J_i \subseteq \cup_{i=1}^{n} \mathfrak{p}_i$, it follows that $a + x \notin \cup_{i=1}^{n} J_i$, as required. □

Before bringing the next result we need the following well known lemma.

Lemma 3.6. *Let $(R, \mathfrak{m})$ be a commutative local ring such that $R/\mathfrak{m}$ is infinite. Let M be an R-module and $N_1, ..., N_t$ be submodules of M such that $M = \bigcup_{i=1}^{t} N_i$. Then there exists $1 \leq j \leq t$, $M = N_i$*

Proof. The assertion follows using NAK Lemma.

Proposition 3.7. *Let R be a commutative ring, M be an R-module and $N_1, ..., N_t$ be submodules of M such that $M = \bigcup_{i=1}^{t} N_i$. Then $\bigcap_{i=1}^{t} SuppM/N_i \subseteq Max(R)$.*

Proof. Suppose the contrary be true. Then there exists $\mathfrak{p} \in (\bigcap_{i=1}^{t} SuppM/N_i) \setminus \mathrm{Max}(R)$. So $R/\mathfrak{p}$ is an integral domain but not a field and therefore $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is infinite. By hypothesis and Proposition 3.6 there exists $1 \leq j \leq t$ such that $(M/N_j)_{\mathfrak{p}} = 0$ and so $\mathfrak{p} \notin \mathrm{Supp}\, M/N_j$ which is a contradiction.

Corollary 3.8. *Let R be a commutative ring and $\mathfrak{p} \in Spec(R) \setminus \mathrm{Max}(R)$. Let M be an R-module and $N_1, ..., N_t$ be $\mathfrak{p}$-prime submodules of M and N a* *submodule of M such that $N \subseteq \bigcup_{i=1}^{t} N_i$. Then there exists $1 \leq j \leq t$ such that $N \subseteq N_j$.*

Proof. Let for any $1 \leq j \leq t$, $N \nsubseteq N_j$. Then for all $1 \leq j \leq t$, we have $N \cap N_j \neq N$. Since $\mathfrak{p}M \subseteq N_j$, it follows that $\mathfrak{p}N \subseteq N_j$ and so $\mathfrak{p}N \subseteq N \cap N_j$. Hence $\mathfrak{p} \subseteq (N \cap N_j : N)$. On the other hand there exists $x \in N \backslash N \cap N_j$ and so $x \notin N_i$. Let $r \in (N_i \cap N : N)$. Then $rx \in N_i \cap N \subseteq N_i$ and $x \notin N_i$, so $r \in (N_i : M) = \mathfrak{p}$. Consequently $(N_i \cap N : N) \subseteq \mathfrak{p}$ and so $(N_i \cap N : N) = \mathfrak{p}$. Now it is easy to show that $N_i \cap N$ is a $\mathfrak{p}$-prime submodule of N. Since $N \subseteq \bigcup_{i=1}^{t} N_i$ it follows that $N = \bigcup_{i=1}^{t} (N \cap N_i)$. But in this case $\mathfrak{p} \in \bigcap_{i=1}^{t} \mathrm{Supp}(N/N_i \cap N)$. Since $\mathfrak{p} \in Spec(R) \backslash \mathrm{Max}(R)$ this is impossible by Proposition 3.7.

Proposition 3.9. *Let R be a commutative ring and $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $R/\mathfrak{p}$ infinite. Let M be an R-module and $N_1, ..., N_t$ be $\mathfrak{p}$-prime submodules of M and N a submodule of M such that $N \subseteq \bigcup_{i=1}^{t} N_i$. Then there exists $1 \leq j \leq t$ such that $N \subseteq N_j$.*

Proof. If $\mathfrak{p} \notin \mathrm{Max}(R)$, the assertion follows from Corollary 3.8. So let $\mathfrak{p} \in \mathrm{Max}(R)$ and for all $1 \leq i \leq t$, we have $N \nsubseteq N_i$. Hence for any $1 \leq j \leq t$, there exists $x_j \in N \backslash N_j$. Set $N' = (x_1, ..., x_t) \subseteq N$ and so we have $N'/\mathfrak{p}N' = \bigcup_{i=1}^{t} ((N' \cap N_i) + \mathfrak{p}N')/\mathfrak{p}N'$. Since $R/\mathfrak{p}$ is infinite, there exists $1 \leq j \leq t$ such that $N'/\mathfrak{p}N' = ((N' \cap N_j) + \mathfrak{p}N')/\mathfrak{p}N'$. This implies that $N' = (N' \cap N_j) + \mathfrak{p}N') \subseteq \mathfrak{p}M + N_j = N_j$. Hence $N' \subseteq N_j$ which is a contradiction.

Proposition 3.10. *Let R be a commutative ring and $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $R/\mathfrak{p}$ infinite. Let M be an R-module and $N_1, ..., N_t$ be $\mathfrak{p}$-prime submodules of M and N a submodule of M. Let $x \in M$ such that $N + Rx \nsubseteq \bigcup_{i=1}^{t} N_i$. Then there exists $a \in N$ such that $a + x \notin \bigcup_{i=1}^{t} N_i$.*

Proof. It is certainly true for $t = 1$. Let $t > 1$ and the result has been proved for $t - 1$. If $N \subseteq \bigcup_{i=1}^{t} N_i$ then by Proposition 3.9 there exists $1 \leq j \leq t$, such that $N \subseteq N_j$. Without loss of generality we may assume that $j = t$. By induction hypothesis there exists $b \in N$ such that $b + x \notin \bigcup_{i=1}^{t-1} N_i$. Since $b + x \notin N_t$ it follows that $b + x \notin \bigcup_{i=1}^{t} N_i$ and so the assertion follows. Now suppose that $N \nsubseteq \bigcup_{i=1}^{t} N_i$, then there exists $c \in N \backslash \bigcup_{i=1}^{t} N_i$. In this case if $x \notin \bigcup_{i=1}^{t} N_i$ we set $a = 0$ and if $x \in \cap_{i=1}^{t} N_i$ then we set $a = c$. Now suppose that the above conditions are not true. We may assume that there exists $1 \leq k \leq t - 1$ such that $x \in \cap_{i=1}^{k} N_i$ and $x \notin \bigcup_{i=k+1}^{t} N_i$. Since $R/\mathfrak{p}$ is infinite, so there exist $t - k + 1$ non-zero distinct elements in $R/\mathfrak{p}$ such as $s_1 + \mathfrak{p}, ..., s_{t-k+1} + \mathfrak{p}$. Set $A = \{s_i c + x | i = 1, ..., t - k + 1\}$. If there exists an element $s_j c + x$ in A such that $s_j c + x \notin \bigcup_{i=1}^{t} N_i$ then the proof is complete. Otherwise, for each $1 \leq l \leq t - k + 1$, there is $1 \leq j \leq t$ such that $s_l c + x \in N_j$. If $1 \leq j \leq k$ then $s_l \in \mathfrak{p}$ and so $s_l + \mathfrak{p} = \mathfrak{p}$ which is a contradiction. So

$k+1 \leq j \leq t$ and hence $A \subseteq \bigcup_{i=k+1}^{t} N_i$. Whence, according to the Dirichlet drawer principle, there exists $k+1 \leq j \leq t$ and $1 \leq l_1 < l_2 \leq t-k+1$ such that $s_{l_1}c + x$ and $s_{l_2}c + x$ belong to N_j. Therefore $s_{l_1} + \mathfrak{p} = s_{l_2} + \mathfrak{p}$ which is a contradiction. □

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