A PROBABILISTIC PROOF OF EULER'S FORMULA FOR $\zeta(2n)$

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In this paper, we utilize the probabilistic approach to Bernoulli polynomials to give a new proof of Euler's formula for $\zeta(2n)$, where n is a positive integer.

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1. INTRODUCTION

Let $\zeta(z) := \sum_{k=1}^{\infty} \frac{1}{k^z}$ be the Riemann zeta function (see [7]). The classical formula

$$\zeta(2n) = (-1)^{n+1} \frac{2^{2n-1}\pi^{2n} B_{2n}}{(2n)!}, \qquad n = 1, 2, \ldots \tag{1.1}$$

was discovered and proved by Euler in his pioneering work [5] published in 1740. The numbers B_n are Bernoulli numbers (see [1, p. 804]). They are the coefficients of the Bernoulli polynomials which are defined by the series expansion

$$\frac{ue^{ux}}{e^u - 1} = \sum_{n=0}^{\infty} \frac{u^n}{n!} B_n(x), \tag{1.2}$$

and, hence,

$$\frac{u}{e^u - 1} = \sum_{n=0}^{\infty} \frac{u^n}{n!} B_n.$$

Now, for $z = u \in \mathbb{R}$ such that $|u| < \rho$ we obtain

$$\frac{1}{\mathbb{E}(e^{u\xi})} = \sum_{n=0}^{\infty} c_n u^n = \sum_{n=0}^{\infty} \hat{c}_n \frac{u^n}{n!},$$

where $\hat{c}_n = c_n n!$

Note that the Bernoulli polynomials $B_n(x)$ are, in fact, Appell polynomials associated with the uniformly distributed random variable ξ on $[0,1]$, *i.e.*, expansion (1.2) can be written as (see, [9, Section 3.1])

$$\frac{e^{ux}}{\mathbb{E}(e^{u\xi})} = \sum_{n=0}^{\infty} \frac{u^n}{n!} B_n(x). \tag{1.3}$$

Bernoulli polynomials associated to ξ have an interesting property, the so-called mean value property

$$\mathbb{E}(B_n(x+\xi)) = x^n.$$

This property can be used to derive many other interesting properties (see also [9, p. 282] for simple proofs), *e.g.*, the Bernoulli numbers can be computed by the recursion formula

$$B_n = -\sum_{k=0}^{n-1} \binom{n}{k} B_k \mathbb{E}(\xi^{n-k}) = -\sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{n-k+1}, \tag{1.4}$$

and the relationship between the cumulants κ_n of ξ and the Bernoulli numbers

$$\kappa_{n+1} = \frac{B_{n+1}}{n+1}, \quad n = 1, 2, \ldots, \tag{1.5}$$

where the cumulants κ_n are the coefficients of the expansion

$$\log \mathbb{E}(e^{u\xi}) = \sum_{n=0}^{\infty} \frac{u^n}{n!} \kappa_n.$$

Straightforward calculations from recursion equation (1.4) give

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -1/30, \ldots.$$

Using the probabilistic approach (1.3), the Bernoulli polynomials have the moment representation (see [9, p. 284])

$$B_n(x) = \mathbb{E}(x - \frac{1}{2} + i\eta)^n, \tag{1.6}$$

where the random variable η has the logistic distribution with the density (see [6, p. 60])

$$f^{(\eta)}(x) = \frac{\pi}{2} \operatorname{sech}^2(\pi x), \quad x \in \mathbb{R}.$$

From (1.6) Bernoulli numbers can be represented by moments of the random variable η (see [9, formula (38), p. 285])

$$B_{2n} = (-1)^{n-1} \frac{\mathbb{E}(\eta^{2n})}{(1 - 2^{1-2n})}. \tag{1.7}$$

The logistic distribution η can be characterized as an infinite sum of independent, identically distributed Laplace random variables $L_j, j = 1, 2, \dots$ with density $g(x) = \frac{1}{2}e^{-|x|}$, we refer to Sun [8], Talacko [10] for the characterization of η and Kotz *et al.* [6] for the definition and basic properties of Laplace distribution,

$$\eta \stackrel{(d)}{=} \sum_{j=1}^{\infty} \frac{L_j}{2\pi j}, \tag{1.8}$$

where $\stackrel{(d)}{=}$ denotes equality in distribution.

Euler's identity (1.1) has been much studied by a number of authors (see, *e.g.*, [2, 3, 4, 11]). In this paper, we will prove identity (1.1) by using the probabilistic approach to Bernoulli polynomials. More precisely, the moment representation of Bernoulli polynomials (1.6), the identity (1.5) and the characterization of the logistic distribution (1.8) are utilized as the key in our proof.

2. PROOF EULER'S FORMULA

Let us now prove formula (1.1). From (1.3) and (1.6), putting $x = 0$ we get

$$\mathbb{E}(e^{(i\eta - \frac{1}{2})u}) = \frac{1}{\mathbb{E}(e^{u\xi})},$$

and, hence,

$$\log \mathbb{E}(e^{(i\eta - \frac{1}{2})u}) = -\log \mathbb{E}(e^{u\xi}).$$

Denote $\kappa_n(\eta)$ and $\kappa_n(\xi)$ the n-th cumulants of random variables η and ξ, respectively. We have

$$\kappa_n(i\eta - \frac{1}{2}) = -\kappa_n(\xi).$$

Consequently,

$$(-1)^n \kappa_{2n}(\eta) = -\kappa_{2n}(\xi), \quad \text{for all} \quad n \geq 1. \tag{2.1}$$

By (1.8) we obtain

$$\kappa_{2n}(\eta) = \kappa_{2n}\Big(\sum_{j=1}^{\infty} \frac{L_j}{2\pi j}\Big) = \frac{1}{(2\pi)^{2n}} \sum_{j=1}^{\infty} \frac{\kappa_{2n}(L_j)}{j^{2n}}.$$

Notice that $L_j, j = 1, 2, \dots$ are i.i.d. Laplace random variables, hence (see [6, p. 19–20])

$$\kappa_{2n}(L_j) = \kappa_{2n} = 2(2n-1)!.$$

So we get

$$\kappa_{2n}(\eta) = \frac{2(2n-1)!}{(2\pi)^{2n}} \sum_{j=1}^{\infty} \frac{1}{j^{2n}} = \frac{2(2n-1)!}{(2\pi)^{2n}} \zeta(2n). \tag{2.2}$$

From (1.5), (2.1) and (2.2) we now obtain

$$(-1)^n \frac{2(2n-1)!}{(2\pi)^{2n}} \zeta(2n) = -\frac{B_{2n}}{2n},$$

which gives (1.1).

Remark 2.1. From formula (1.7) we have the representation of $\zeta(2n)$ *via* the moment of the logistic distribution

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!(1-2^{1-2n})} \mathbb{E}(\eta^{2n}).$$

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