ELASTIC STRIPS WITH TIMELIKE DIRECTRIX

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We study elastic rectifying strips constructed from timelike curves constituting one of the causal characters of a curve in Minkowski 3-space. Even if elastic surfaces correspond to critical points of Willmore functional, we instead find extremals of the Sadowsky functional, because Willmore functional is proportional to the Sadowsky functional for rectifying strips. We then provide a characterization of timelike critical points of Sadowsky functional with two Euler-Lagrange equations. When we choose different variations, we derive two conservation laws, and by using these rules, we introduce two new kinds of elastic strips with timelike directrix (the base curve). We next establish a relation between elastic strips with timelike directrix and spacelike elastic curves on de Sitter 2-space and pseudohyperbolic 2-space. Finally, we verify that the semi-Riemannian Hopf cylinder associated to the tangent imagine of the timelike curve defining a force-free strip with timelike directrix which is one of the new types elastic strips gives rise to a Willmore surface in anti de Sitter 3-space.

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1. INTRODUCTION

An inextensible strip is a thin shell that deforms in terms of only a pure bending (not by stretching). Its surface is therefore developable, that is, it has vanishing Gaussian curvature [16, 17]. The structure of such strips resembles one dimensional structure of thin rods, and they can be described in terms of the classical equations of thin elastic rods [6]. Surfaces of these thin rods are completely determined (up to Euclidean motions) by curvature and torsion of its centerline. A model for a narrow developable strip has been introduced by Sadowsky (1930), and the extension of this model to a finite width is obtained by Wunderlich (see for details [5, 10, 14, 15, 20, 21]). In particular, Wunderlich proves that the Willmore functional (total squared mean curvature functional) $\int_S H^2 \mathrm{d}A$ of an infinitely narrow inextensible strip is proportional to the

Sadowsky functional

$$S = \int_{\gamma} \kappa^2 \left(1 + \frac{\tau^2}{\kappa^2}\right)^2 \mathrm{d}s. \tag{1}$$

Thus, equilibrium configurations of elastic strips can be obtained in terms of critical points of the Sadowsky functional when it is applied to all variations having fixed end points and the fixed length. In recent years one observes an increasing interest in studying one-dimensional elastic objects like curves, thin rods and ribbons in Euclidean 3-space (see for several applications [4–9]). Papers [15] and [21] which are written in the original language of the authors have recently been translated into English [10] and [20]. In [7], the authors reexamine derivation of the limit energy of an inextensible, isotropic, elastic strip as the width goes to zero. They obtain a limit functional depending on three orthonormal vectors [7]. Hangan [8], Chubelaschwili and Pinkall [4] derive two Euler-Lagrange equations of the functional (1) at different times in Euclidean 3-space. The characterization of the centerline of an elastic strip presented by Chubelaschwili and Pinkall corresponds to the conservation laws generated by the symmetry group of Euclidean motions.

The squared curvature (the bending energy) integrated over a developed surface may be reduced to a single integral over a reference curve (centerline) and the Sadowsky functional is a limit of this quantity for an infinitesimally narrow strip around the centerline. Thus, the Sadowsky functional is a mathematical way to describe the physical model of a thin and narrow elastic strip. This paper is devoted to the study of the Sadowsky strip model in a non-Euclidean space because the knowledge of the balance form of the Euler-Lagrange equations of the functional (1) may be useful for certain problems involving non-Euclidean symmetry groups. One particular example can be given by the description of world lines of relativistic particles in Minkowski space with the Poincare group of isometries as symmetry group [17]. As vectors, curves and surfaces have different causal character with respect to the metric structure of Minkowski space, an elastic strip must separately be investigated in case of the causal character of the centerline. In this paper, we especially study a strip whose directrix is timelike. The reason why we study elastic strips with timelike directrix is to establish a connection between the semi-Riemannian Hopf cylinder and some of such strips. The existence of such a connection implies that the semi-Riemannian Hopf cylinder associated to the tangent of a timelike elastic curve which is directrix of an elastic strip corresponds to a Willmore surface in anti de Sitter 3-space H_1^3.

In Section 2, we obtain two Euler-Lagrange equations for elastic strips with timelike directrix in Minkowski 3-space. By using this differential equation system, we provide a relation between elastic curves and the elastic strips without torsion. We derive two conservation laws in Section 3 in order to find two new classes of integrable elastic strips with timelike directrix. Such results allow us to show that binormal and tangent vectors of the timelike directrix of an elastic strip correspond to a spacelike elastic curve on de Sitter 2-space (pseudosphere with 2-dimension) and pseudohyperbolic 2-space in Section 4. We also introduce momentum strips and force–free strips with timelike directrix. Finally, we verify that semi-Riemannian Hopf cylinder associated to the tangent imagine of the timelike curve defining a force-free strip corresponds to a Willmore surface in anti de Sitter 3-space.

2. ELASTIC STRIPS WITH TIMELIKE DIRECTRIX

In Euclidean or Minkowski 3-space, the Euler-Lagrange equation is given by the fourth order equation

$$\Delta H + 2H\left(H^2 - K\right) = 0$$

where Δ denotes the Laplace-Beltrami operator and K is the Gaussian curvature [3]. Thus, as Euclidean 3-space, Willmore functional of a developable ruled surface with directrix choosing with regard to the Darboux vector is proportional to the Sadowsky functional in Minkowski 3-space $\mathbb{R}^3_1$. In this section, we introduce timelike rectifying strips, build the variational problem for elastic strips with timelike directrix, and then find two Euler-Lagrange equations for the critical points of modified Sadowsky functional. In particular, we provide an example solving this system of equations.

We recall that Minkowski 3-space $\mathbb{R}^3_1$ is a three-dimensional real vector space equipped with the metric

$$< x, y >= x_1y_1 + x_2y_2 - x_3y_3, \quad x = (x_1, x_2, x_3)\,,\; y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$$

which is a non-degenerate, symmetric and bilinear form. A smooth curve in $\mathbb{R}^3_1$ is a timelike (resp., a spacelike and a lightlike), if its tangent vector is a timelike (resp., a spacelike and a lightlike) [11].

An elastic strip is a surface with the minimum bending energy. For a detailed description of these strips, we first introduce a developable ruled surface whose directrix is a timelike curve. For that reason, assume that $\gamma(s)$ is a timelike curve in Minkowski 3-space and consider the ruled surface with $\gamma(s)$:

$$\begin{array}{llll} F_\gamma : & [0,\ell] \times [-\varepsilon, \varepsilon] & \rightarrow & \mathbb{R}^3_1 \\ & (t,\delta) & \rightarrow & F_\gamma(t,\delta) = \gamma(t) + \delta\left(B(t) + \lambda T(t)\right), \end{array} \tag{2}$$

where T is the unit timelike tangent vector, B is the unit spacelike binormal and $\lambda = \dfrac{\tau}{\kappa}$ is the modified torsion in which κ and τ are the curvature and the torsion of γ respectively, which are defined by

$$\kappa = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3} \text{ and } \tau = \frac{\det(\gamma', \gamma'', \gamma''')}{\|\gamma' \times \gamma''\|^2}. \tag{3}$$

Definition 1. Any developable ruled surface defined as in (2) is called a timelike rectifying strip.

We next study infinitely narrow timelike rectifying strips constructed by using critical points of the Sadowsky functional (1) within all space curves with fixed end points and

$$\dot{\ell} := \frac{\partial}{\partial \delta} \ell(\gamma_t) \bigg|_{\delta=0} = 0, \tag{4}$$

where $\gamma(t)$ is the timelike directrix curve of the timelike rectifying surface. The results obtained by Wunderlich imply that we can work on a variational problem finding the critical base curve constituting the surface, instead of the variational problem finding the critical surface. We therefore look for elastic strips with timelike directrix defined as the following.

Definition 2. Let F_γ be a timelike rectifying strip in Minkowski 3-space. If the timelike directrix γ of F_γ is an extremal for the modified Sadowsky functional

$$S_\mu(\gamma) = \int_0^\ell \left(\kappa^2 \left(1+\lambda^2\right)^2 - \mu\right) \upsilon \, ds, \tag{5}$$

then F_γ is an elastic strip with timelike directrix. Here μ is a Lagrange multiplier, standing for the length constraint.

In Definition 2, observe that timelike curve $\gamma : (0, \ell) \to \mathbb{R}^3_1$ defines an elastic strip, so does $\gamma : [0, \ell] \to \mathbb{R}^3_1$.

In order to determine the timelike directrix γ for which F_γ is an elastic strip, we denote the Frenet frame $\{T, N, B\}$ of γ at the point $\gamma(s)$. Note that for the curve γ with velocity $\upsilon = \|\gamma'\|$, the Frenet formulas are given by

$$\begin{aligned} T' &= \upsilon\kappa N \\ N' &= \upsilon\kappa T + \upsilon\lambda\kappa B \\ B' &= -\upsilon\lambda\kappa N. \end{aligned} \tag{6}$$

Before finding the critical points of the Sadowsky functional, we need to determine a variation first, and then, calculate some derivatives related to the variation in the following lemma.

Lemma 1. *If* $\gamma_0 : [0,\ell] \to \mathbb{R}^3_1$ *is an arc length parametrized timelike curve and*

$$\begin{array}{rcll} \gamma: & [0,\ell]\times[-\varepsilon,\varepsilon] & \to & \mathbb{R}^3_1 \\ & (s,\delta) & \to & \gamma(s,\delta)=\gamma_\delta(s)=\gamma_0(s)+\delta\dot{\gamma}(s) \end{array}$$

is a variation of γ_0 *with variational vector*

$$\begin{aligned} \dot{\gamma}(s) &= \frac{\partial}{\partial\delta}\Big|_{\delta=0}\gamma_\delta(s) \\ &= u_1(s)\,T(s)+u_2(s)\,N(s)+u_3(s)\,B(s), \end{aligned} \tag{7}$$

where $u_1(s)=-<\dot{\gamma}(s),T(s)>$, $u_2=<\dot{\gamma}(s),N(s)>$ *and* $u_3=<\dot{\gamma}(s),B(s)>$, *then we have*

$$\dot{\upsilon}=u_1'+u_2\kappa, \tag{8}$$

$$\dot{\kappa}=u_1\kappa'-u_2\kappa^2\left(1+\lambda^2\right)-2u_3'\lambda\kappa-u_3(\lambda\kappa)'+u_2'', \tag{9}$$

$$\begin{aligned} \dot{\lambda} &= u_1\lambda'+u_2\left(\frac{(\lambda\kappa)''}{\kappa^2}-\frac{(\lambda\kappa)'\kappa'}{\kappa^3}+\lambda^3\kappa-\lambda\kappa\right)+u_2'\left(2\frac{\lambda'}{\kappa}+\frac{(\lambda\kappa)'}{\kappa^2}\right) \\ &\quad +u_2''\frac{\lambda}{\kappa}-u_3\lambda\lambda'-u_3'\left(1-\lambda^2\right)-u_3''\frac{\kappa'}{\kappa^3}+u_3'''\frac{1}{\kappa^2}. \end{aligned} \tag{10}$$

Proof. By using the fact $\left(\dot{\gamma}\right)'=(\dot{\gamma'})$, we obtain the equality

$$\dot{\upsilon}T+\dot{T}=\left(u_1'+u_2\kappa\right)T+\left(u_1\kappa+u_2'-u_3\lambda\kappa\right)N+\left(u_2\lambda\kappa+u_3'\right)B,$$

from which we derive (8) and

$$\dot{T}=\left(u_1\kappa+u_2'-u_3\lambda\kappa\right)N+\left(u_2\lambda\kappa+u_3'\right)B. \tag{11}$$

Employing (6) and (11), we conclude the following equalities

$$\begin{aligned} (\dot{T'}) &= \dot{\upsilon}\kappa N+\dot{\kappa}N+\kappa\dot{N}, \\ \left(\dot{T}\right)' &= \left(u_1\kappa+u_2'-u_3\lambda\kappa\right)'N+\left(u_1\kappa+u_2'-u_3\lambda\kappa\right)(\kappa T+\lambda\kappa B) \\ &\quad +\left(u_2\lambda\kappa+u_3'\right)'B-\left(u_2\lambda\kappa+u_3'\right)\lambda\kappa N. \end{aligned}$$

The equation (9) directly follows from the equation (8) and the fact $\left(\dot{T}\right)'=(\dot{T'})$, both of which imply that

$$\begin{aligned} \dot{N} &= \left(u_1\kappa+u_2'-u_3\lambda\kappa\right)T \\ &\quad +\tfrac{1}{\kappa}\left(\left(u_2\lambda\kappa+u_3'\right)'+\lambda\kappa\left(u_1\kappa+u_2'-u_3\lambda\kappa\right)\right)B. \end{aligned} \tag{12}$$

Similarly, by the equality $\left(\dot{N}\right)' = (\dot{N'})$ and equations (8) and (9), we obtain

$$\begin{aligned}\dot{\lambda}\kappa &= u_1\lambda'\kappa + u_2\left(\tfrac{(\lambda\kappa)''}{\kappa} - \tfrac{(\lambda\kappa)'\kappa'}{\kappa^2} + \lambda^3\kappa^2 - \lambda\kappa^2\right) \\ &\quad + u_2'\left(2\lambda' + \tfrac{(\lambda\kappa)'}{\kappa}\right) + u_2''\lambda - u_3\left(\lambda\lambda'\kappa\right) + u_3'\left(-1+\lambda^2\right)\kappa \\ &\quad - u_3''\tfrac{\kappa'}{\kappa^2} + u_3'''\tfrac{1}{\kappa},\end{aligned}$$

from which the equality (10) follows.

Assume that an arc length parametrized timelike curve $\gamma : [0,\ell] \to \mathbb{R}^3_1$ becomes the directrix of an elastic strip with timelike directrix. We next consider a variation of γ having the variational vector field (7). We may calculate the first variation of the Sadowsky functional

$$S_\mu\left(\gamma_\delta\right) = \int_0^\ell \left(\kappa_\delta^2\left(1+\lambda_\delta^2\right)^2 - \mu\right) \upsilon_\delta \mathrm{d}t$$

as

$$\begin{aligned}\left.\frac{\partial}{\partial\delta} S_\mu\left(\gamma_\delta\right)\right|_{\delta=0} &= \left.\frac{\partial}{\partial\delta}\ell\left(\gamma_t\right)\right|_{\delta=0}\left.\left(\left(\kappa_\delta^2\left(1+\lambda_\delta^2\right)^2 - \mu\right)\upsilon_\delta\right)\right|_{\delta=\ell(\gamma_t)} \\ &\quad + \int_0^\ell \frac{\partial}{\partial\delta}\left(\kappa_\delta^2\left(1+\lambda_\delta^2\right)^2 - \mu\right)\left.\upsilon_\delta\right|_{\delta=0}\mathrm{d}t,\end{aligned}$$

and then by taking the condition (4) into consideration, we obtain the following equality

$$\left.\frac{\partial}{\partial\delta} S_\mu\left(\gamma_\delta\right)\right|_{\delta=0} = \int_0^\ell \frac{\partial}{\partial\delta}\left(\kappa_\delta^2\left(1+\lambda_\delta^2\right)^2 - \mu\right)\left.\upsilon_\delta\right|_{\delta=0}\mathrm{d}t.$$

Now from the equalities (8), (9) and (10), we conclude that

$$\frac{1}{2}\int_0^\ell \frac{\partial}{\partial\delta}\left(\kappa_\delta^2\left(1+\lambda_\delta^2\right)^2 - \mu\right)\left.\upsilon_\delta\right|_{\delta=0}\mathrm{d}t = \int_0^\ell \left(u_2 f_1 + u_3 f_2 + b'\right)\mathrm{d}t, \tag{13}$$

where

$$\begin{aligned} f_1 &= \left(\kappa'\left(1+\lambda^2\right)^2 + 2\kappa\left(1+\lambda^2\right)\lambda\lambda'\right)' \\ &\quad - \tfrac{\kappa}{2}\left(\kappa^2\left(1+\lambda^2\right)\left(1+9\lambda^2\right)+\mu\right) \\ &\quad + \lambda\kappa\left(\kappa^2\left(1+\lambda^2\right)^2\lambda + \left(\tfrac{\kappa'}{\kappa}\left(1+\lambda^2\right)2\lambda\right)' + \left(\left(1+\lambda^2\right)2\lambda\right)''\right),\end{aligned} \tag{14}$$

$$\begin{aligned} f_2 &= -(\kappa^2\left(1+\lambda^2\right)^2\lambda + \left(\tfrac{\kappa'}{\kappa}\left(1+\lambda^2\right)2\lambda\right)' \\ &\quad - 4\kappa^2\lambda\left(1+\lambda^2\right) + (\left(\left(1+\lambda^2\right)2\lambda\right)'')' \\ &\quad + \kappa\lambda\left(\kappa'\left(1+\lambda^2\right)^2 + 2\kappa\left(1+\lambda^2\right)\lambda\lambda'\right)\end{aligned} \tag{15}$$

and

$$
\begin{aligned}
b = {} & u_1\left(\tfrac{1}{2}\left(\kappa^2\left(1+\lambda^2\right)^2-\mu\right)\right)\\
& + u_2\left(\left(6\lambda\lambda'\kappa+2\lambda^2\kappa'\right)\left(1+\lambda^2\right)-\left(\kappa\left(3\lambda^2+1\right)\left(1+\lambda^2\right)\right)'\right)\\
& + u_2'\left(\kappa\left(3\lambda^2+1\right)\left(1+\lambda^2\right)\right)\\
& + u_3\left(\left(-4\kappa^2\lambda\left(1+\lambda^2\right)\right)+\left(2\lambda\tfrac{\kappa'}{\kappa}\left(1+\lambda^2\right)\right)'+\left(2\lambda\left(1+\lambda^2\right)\right)''\right)\\
& - u_3'\left(2\lambda\tfrac{\kappa'}{\kappa}\left(1+\lambda^2\right)+\left(2\lambda\left(1+\lambda^2\right)\right)'\right)+u_3''\left(2\lambda\left(1+\lambda^2\right)\right). \quad \square
\end{aligned} \tag{16}
$$

THEOREM 1. *If F_γ is an elastic strip with timelike directrix γ, then γ satisfies Euler-Lagrange equations*

$$f_1 = f_2 = 0 \text{ and } b' = 0. \tag{17}$$

Proof. Suppose that a timelike curve γ is an extremal for S_μ in which γ is parametrized with respect to the arclength. From (13), we obtain

$$\frac{\partial}{\partial\delta}S_\mu\left(\gamma_\delta\right)\bigg|_{\delta=0} = \int_0^\ell \left(u_2\left(s\right)f_1\left(s\right)+u_3\left(s\right)f_2\left(s\right)\right)\mathrm{d}s + b\left(\ell\right)-b\left(0\right)=0.$$

On the other hand, since $b\left(\ell\right) = b\left(0\right) = 0$ for a suitable variation, claimed Euler-Lagrange equations hold. Furthermore, when γ is a critical for S_μ it satisfies Euler-Lagrange equations stated in the first part of (17) so that

$$\frac{1}{2}\frac{\partial}{\partial\delta}\bigg|_{\delta=0}\left(\kappa_\delta^2\left(1+\lambda_\delta^2\right)^2-\mu\right)\upsilon_\delta = u_2f_1+u_3f_2+b' = b' = 0 \quad \square.$$

One of the consequences of Theorem 1 is that the critical points with no torsion of the functional (5) in Minkowski 3-space correspond to timelike elastic curves on pseudo-plane [1].

Recall that if a timelike curve in Minkowski 3-space is a helix, then $\lambda = \frac{\tau}{\kappa}$ is a constant function [10]. We next provide two examples for elastic strips with directrix which is a timelike helix.

Example 1. Let $\psi\left(t\right) = \left(t, \sqrt{2}\cosh t, \sqrt{2}\sinh t\right)$ be a unit speed timelike helix [18] with the curvature, the torsion and the modified torsion given by

$$\kappa_\psi = \sqrt{2},\ \tau_\psi = 1 \text{ and } \lambda_\psi = \frac{1}{\sqrt{2}}. \tag{18}$$

From these values (18) together with Euler-Lagrange equations (17), it easily follows that the timelike rectifying strip with directrix ψ

$$F_\psi\left(t,\delta\right) = \left(t-\frac{\sqrt{2}}{2}\delta, \sqrt{2}\cosh t, \sqrt{2}\sinh t\right)$$

is an elastic strip with timelike directrix if we choose $\mu = -12$.

3. CONSERVATION LAWS OF ELASTIC STRIPS WITH TIMELIKE DIRECTRIX

In this section, we set up two different variations including only Lorentzian translations or rotations. We then state the first and second conservation laws of elastic strips with timelike directrix in $\mathbb{R}_1^3$.

We recall that $\left(\kappa^2\left(1+\lambda^2\right)^2-\mu\right)$ is invariant under Lorentzian motions. We only consider translations in the following variation

$$\gamma_\delta(s)=\gamma(s)+\delta\Gamma,$$

with the variation vector field

$$\Rightarrow \dot{\gamma}_\delta(s)=\Gamma=\underbrace{-<\Gamma,T>}_{u_1}T+\underbrace{<\Gamma,N>}_{u_2}N+\underbrace{<\Gamma,B>}_{u_3}B,$$

where Γ is an arbitrary point in Minkowski 3-space. The derivatives u_2', u_3', u_3'' are calculated as

$$\begin{aligned} u_2' &= \kappa<\Gamma,T>+\lambda\kappa<\Gamma,B>,\\ u_3' &= -\lambda\kappa<\Gamma,N>,\\ u_3'' &= -(\lambda\kappa)'<\Gamma,N>-\lambda\kappa^2<\Gamma,T>-\lambda^2\kappa^2<\Gamma,B>. \end{aligned} \tag{19}$$

Combined the equation (16) with the derivatives (19), we obtain

$$b=<\Gamma,W_0>, \tag{20}$$

where

$$\begin{aligned} W_0 =& -\tfrac{1}{2}\left(\kappa^2\left(1+\lambda^2\right)^2+\mu\right)T+\left(\kappa'\left(1+\lambda^2\right)^2+2\kappa\lambda\lambda'\left(1+\lambda^2\right)\right)N\\ & -\left(\lambda\kappa^2\left(1+\lambda^2\right)^2-4\kappa^2\lambda\left(1+\lambda^2\right)\right.\\ & \left.+\left(2\tfrac{\kappa'}{\kappa}\lambda\left(1+\lambda^2\right)\right)'+\left(2\lambda\left(1+\lambda^2\right)\right)''\right)B. \end{aligned} \tag{21}$$

Observe that when the timelike curve γ is a critical for the Sadowsky functional, b is a constant. Therefore, in the equation (21), W_0 is a constant for any $\Gamma\in\mathbb{R}_1^3$ in the equation (20).

By taking into account that

$$\left.\frac{\partial}{\partial\delta}\right|_{\delta=0}A_\delta\gamma(s)=\tilde{\Gamma}\times\gamma(s)=\underbrace{-<\tilde{\Gamma}\times\gamma,T>}_{u_1}T+\underbrace{<\tilde{\Gamma}\times\gamma,N>}_{u_2}N+\underbrace{<\tilde{\Gamma}\times\gamma,B>}_{u_3}B$$

for $\tilde{\Gamma}\in\mathbb{R}_1^3$ and $A_\delta\in SO_1(3)$, we find

$$\begin{aligned} u_2' &= <\tilde{\Gamma},B>+\kappa<\tilde{\Gamma},\gamma\times T>+\lambda\kappa<\tilde{\Gamma},\gamma\times B>,\\ u_3' &= -<\tilde{\Gamma},N>-\lambda\kappa<\tilde{\Gamma},\gamma\times N>,\\ u_3'' &= -\kappa<\tilde{\Gamma},T>-2\lambda\kappa<\tilde{\Gamma},B>-(\lambda\kappa)'<\tilde{\Gamma},\gamma\times N>\\ &\quad -\lambda\kappa^2<\tilde{\Gamma},\gamma\times T>-\lambda^2\kappa^2<\tilde{\Gamma},\gamma\times B>. \end{aligned} \tag{22}$$

Finally employing derivatives (22) into the equations (16), we conclude that

$$b = < \tilde{\Gamma}, W_1 >,$$

where

$$\begin{aligned} W_1 &= -2\lambda\kappa\left(1+\lambda^2\right)T + \left(\left(2\frac{\kappa'}{\kappa}\lambda\left(1+\lambda^2\right)\right) + \left(2\lambda\left(1+\lambda^2\right)\right)'\right)N \\ &\quad +\kappa\left(1+\lambda^2\right)\left(1-\lambda^2\right)B - \gamma \times W_0. \end{aligned}$$

Therefore W_1 becomes constant when the timelike curve γ is a critical for the modified Sadowsky functional.

Our next task is to show that elastic strips with timelike directrix are characterized by W_0 and W_1.

THEOREM 2 (First conservation law of elastic strips with timelike directrix). *F_γ is an elastic strip with timelike directrix if and only if the force vector $W_0 = a_1T + a_2N + a_3B$ is a constant, where*

$$a_1 = -\frac{1}{2}\left(\kappa^2\left(1+\lambda^2\right)^2 + \mu\right), \tag{23}$$

$$a_2 = \kappa'\left(1+\lambda^2\right)^2 + 2\kappa\lambda\lambda'\left(1+\lambda^2\right) \tag{24}$$

and

$$\begin{aligned} a_3 = &-\left(\lambda\kappa^2\left(1+\lambda^2\right)^2 - 4\kappa^2\lambda\left(1+\lambda^2\right)\right) \\ &-\left(\left(2\tfrac{\kappa'}{\kappa}\lambda\left(1+\lambda^2\right)\right)' + \left(2\lambda\left(1+\lambda^2\right)\right)''\right). \end{aligned} \tag{25}$$

Proof. It suffices to show

$$W_0' = f_1N + f_2B, \tag{26}$$

since the force vector W_0 is a constant if and only if $f_1 = f_2 = 0$ in equation (26). We use Frenet equations (6) to obtain

$$W_0' = (a_1' + \kappa a_2)T + \left(a_2' + \kappa a_1 - \lambda\kappa a_3\right)N + \left(a_3' + \lambda\kappa a_2\right)B.$$

On the other hand, from (23) and (24), we find

$$a_2 = -\frac{1}{\kappa}a_1'.$$

It then follows that the coefficient of T in the equation (26) vanishes. Now, by using (23), (24) and (25), the coefficients of N and B can be stated as

$$a_2' + \kappa a_1 - \lambda\kappa a_3 = f_1, \tag{27}$$

$$a_3' + \lambda\kappa a_2 = f_2, \tag{28}$$

respectively. However Eq. (27) and (28) show that γ defines an elastic strip with timelike directrix if and only if $f_1 = f_2 = 0$. □

Theorem 3 (Second conservation law of elastic strips with timelike directrix). *F_γ is an elastic strip with timelike directrix if and only if the torque vector $W_1 = s_1 T + s_2 N + s_3 B - \gamma \times W_0$ is a constant, where*

$$\begin{aligned} s_1 &= -2\lambda\kappa\left(1+\lambda^2\right), \\ s_2 &= 2\tfrac{\kappa'}{\kappa}\lambda\left(1+\lambda^2\right) + \left(2\lambda\left(1+\lambda^2\right)\right)', \\ s_3 &= \kappa\left(1+\lambda^2\right)\left(1-\lambda^2\right). \end{aligned} \tag{29}$$

Moreover, if W_1 is a constant while γ does not define an elastic strip with timelike directrix, then $\|\gamma\|$ is conserved.

Proof. We take first the derivative of W_1 as follow

$$\begin{aligned} W_1' = \underbrace{(s_1' + \kappa s_2)}_{0} T &+ \Bigg(\underbrace{\kappa s_1 + s_2' - \lambda\kappa s_3}_{-a_3} - \underbrace{< T \times W_0, N >}_{-a_3}\Bigg) N \\ &+ \Bigg(\underbrace{s_3' + \lambda\kappa s_2}_{a_2} - \underbrace{< T \times W_0, B >}_{a_2}\Bigg) B - \gamma \times W_0' \end{aligned} \tag{30}$$

Note that the coefficients of T, N and B vanish by equations (29). Then the equation (30) reduces to

$$W_1' = -\gamma \times W_0'. \tag{31}$$

However Eq. (31) implies that W_1' is zero if and only if W_0 is a constant. This completes the first part of the proof as we already know that W_0 is constant if and only if γ defines an elastic strip with timelike directrix.

On the contrary, suppose that γ does not define an elastic strip with timelike directrix while $W_1' = 0$. Then, substituting $W_0' = f_1 N + f_2 B$ in Eq. (29), we obtain

$$W_1' = -\gamma \times (f_1 N + f_2 B) = 0;$$

hence, $\gamma \in Sp\{N, B\}$ that in turn implies

$$< \gamma, \gamma >' = 2 < \gamma, T >= 0$$

from which the claim follows. □

4. RELATIONS BETWEEN NEW TYPES ELASTIC STRIPS WITH TIMELIKE DIRECTRIX AND ELASTIC CURVES ON HYPERQUADRATICS

We begin this section by recalling hyperquadrics of Minkowski 3-space and elastic curves on hyperquadrics. We then introduce two new integrable systems of elastic strips with timelike directrix. We also state some relations between these strips and elastic curves on de Sitter 2-space and pseudohyperbolic 2-space. Finally, we establish a connection between semi−Riemannian Hopf cylinder and Willmore surface in anti de Sitter 3-space.

We recall that de Sitter 2-space of radius $r > 0$ in $\mathbb{R}_1^3$ is the hyperquadric

$$S_1^2 = \{p \in \mathbb{R}_1^3 | < p, p >= r^2\}$$

with dimension 2 and index 1 [12]. As it is well-known, de Sitter 2−space S_1^2 is a timelike surface. Note also that a spacelike and a timelike elastic curves with geodesic curvature λ on S_1^2 satisfy the following equalities

$$\begin{array}{ll} (\lambda')^2 - \frac{1}{4}\lambda^4 + \left(1 + \frac{\sigma}{2}\right)\lambda^2 = A, & A = const \\ (\lambda')^2 - \frac{1}{4}\lambda^4 + \left(-1 + \frac{\sigma}{2}\right)\lambda^2 = A, & A = const \end{array} \tag{32}$$

respectively, where σ is a Lagrange multiplier [13].

The pseudohyperbolic 2-space of radius $r > 0$ in $\mathbb{R}_1^3$ is the hyperquadric

$$H^2 = H_0^2 = \{p \in \mathbb{R}_1^3 | < p, p >= -r^2\}$$

with dimension 2 and index 1 [12]. In this case, the pseudohyperbolic 2-space H^2 is a spacelike surface. A spacelike elastic curve with geodesic curvature λ on H^2 satisfies

$$(\lambda')^2 + \frac{1}{4}\lambda^4 - \left(1 + \frac{\sigma}{2}\right)\lambda^2 = A, \qquad A = const., \tag{33}$$

where σ is a Lagrange multiplier [13].

Now, we are ready to introduce two new types of elastic strips with timelike directrix.

Definition 3. An elastic strip with timelike directrix in Minkowski 3-space is called force-free strip with timelike directrix if $W_0 = 0$.

Definition 4. An elastic strip with timelike directrix in Minkowski 3-space is called a momentum strip with timelike directrix if

$$\langle W_1 + \gamma \times W_0, T \rangle \tag{34}$$

is a constant non-zero function.

Our next result provides a relation between momentum strips with timelike directrix in $\mathbb{R}_1^3$ and spacelike elastic curves on S_1^2.

THEOREM 4. *Let a timelike curve $\gamma : [0, \ell] \to \mathbb{R}_1^3$ define a momentum strip with timelike directrix with Lagrange multiplier μ. Then, the binormal vector B of γ corresponds to a spacelike elastic curve with Lagrange multiplier $-\frac{\mu}{4}$ in de Sitter 2-space S_1^2. Conversely for each such arc length parametrized* *curve* $B:[0,\tilde{\ell}]\to S_1^2$ *with non-vanishing, non-constant geodesic curvature* λ *and* $-T=B\times B'$ *the timelike curve*

$$\gamma(t)=\frac{1}{2}\int_0^t\left(1+\frac{1}{\lambda^2(s)}\right)T(s)\,\mathrm{d}s$$

defines an elastic momentum strip with timelike directrix with

$$S_\mu(\gamma(t))=2\int_0^\ell\left(1+\lambda^2\right)\mathrm{d}t-\mu\ell(\gamma).$$

Proof. Let $\gamma(s)$ define a momentum strip with timelike directrix. Once we choose

$$<W_1+\gamma\times W_0,T>=-s_1=4,$$

we obtain that

$$\kappa=\frac{2}{\lambda(1+\lambda^2)} \tag{35}$$

and

$$\begin{aligned}<W_0,W_0>=&-\tfrac{4}{\lambda^4(s)}-\tfrac{2}{\lambda^2(s)}\mu-\tfrac{1}{4}\mu^2+4\tfrac{\left(\lambda(s)'\right)^2}{\lambda^4(s)}\left(1+\lambda^2(s)\right)^2\\&+\tfrac{16}{\lambda^2(s)}\left(-1+\tfrac{4}{1+\lambda^2(s)}\right)^2.\end{aligned} \tag{36}$$

If we multiply each term of the equation (36) with $\frac{1}{16}$, we can rearrange (36) in the following

$$\begin{aligned}&-\tfrac{1}{4}\tfrac{1}{\lambda^4(s)}-\tfrac{1}{8}\tfrac{1}{\lambda^2(s)}\mu+\tfrac{1}{4}\tfrac{(\lambda'(s))^2}{\lambda^4(s)}\left(1+\lambda^2(s)\right)^2+\\&\tfrac{1}{\lambda^2(s)}\left(-1+2\tfrac{2}{1+\lambda^2(s)}\right)^2=\tfrac{1}{16}\left(<W_0,W_0>+\tfrac{1}{4}\mu^2\right).\end{aligned} \tag{37}$$

Now, we change the parameter as $\widetilde{\lambda}(t)=\lambda(s(t))$ such that $\widetilde{\lambda}'(t)\,t'(s)=\lambda'(s(t))$, where $t'(s)=\frac{2}{1+\lambda^2(s)}$. Thus we can reformulate the equation (37) as follows

$$\begin{aligned}&-\tfrac{1}{4}\tfrac{1}{\widetilde{\lambda}^4(t)}+\tfrac{1}{\widetilde{\lambda}^2(t)}\left(-1+2t'(s)-\tfrac{1}{8}\mu\right)\\&+\tfrac{1}{4}\left(\tfrac{\widetilde{\lambda}'(t)t'(s)}{\widetilde{\lambda}^2(s)}\right)^2 4\left(\tfrac{1}{t'(s)}\right)^2=\tfrac{1}{16}\left(<W_0,W_0>+\tfrac{1}{4}\mu^2\right).\end{aligned}$$

By using Frenet equation (6) it is apparent that t and $\frac{1}{\widetilde{\lambda}}$ are respectively the arc length parameter and curvature of B. So we calculate

$$-\frac{1}{4}\left(\frac{1}{\widetilde{\lambda}}\right)^4+\left(\frac{1}{\widetilde{\lambda}}\right)^2\left(1+\frac{\left(-\frac{\mu}{4}\right)}{2}\right)+\left(\left(\frac{1}{\widetilde{\lambda}}\right)'\right)^2=\frac{1}{16}\left(<W_0,W_0>+\frac{1}{4}\mu^2\right). \tag{38}$$

One can easily see that $\frac{1}{16}\left(< W_0, W_0 > +\frac{1}{4}\mu^2\right)$ is a constant and Eq. (38) has at least one non-zero solution. From Eq. (32), we show that binormal B of the timelike curve γ corresponds to spacelike elastic curves with Lagrange multiplier $-\frac{\mu}{4}$ on S_1^2 of radius 1.

Conversely, let B be such an arc length parametrized spacelike elastic curve on de Sitter 2-space S_1^2 with non-vanishing, non-constant geodesic curvature λ. Also, assume that B is the binormal of a timelike curve γ with Frenet frame $\{T, N, B\}$ in $\mathbb{R}_1^3$. Then the Darboux trihedron of B is $\{-N, B, -T\}$ and derivative formulas of the Darboux frame are

$$\begin{aligned} B' &= -N, \\ -N' &= \lambda T - B, \\ -T' &= \lambda N. \end{aligned} \tag{39}$$

If we consider the spacelike curve B having a reparametrization with an arc length s with $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1+\lambda^2}{2\lambda^2} = \frac{1}{2}\left(1 + \frac{1}{\lambda^2}\right)$, there is a timelike curve

$$\gamma(t) = \frac{1}{2}\int_0^t \left(1 + \frac{1}{\lambda^2(s)}\right) T(s)\,\mathrm{d}s.$$

Now from the stated equations in (39), the curvature and modified torsion of γ can be written as

$$\kappa_\gamma = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3} = \frac{2}{\frac{1}{\lambda}\left(1 + \frac{1}{\lambda}\right)} \text{ and } \lambda_\gamma = \frac{\tau}{\kappa} = -\frac{1}{\lambda}.$$

Substituting κ_γ and τ_γ in the equation (34), we conclude that the timelike curve γ defines a momentum strip with timelike directrix. Similarly, the substitution of $\frac{1}{\lambda}$ for λ in the equation (38) yields

$$(\lambda')^2 - \frac{1}{4}\lambda^4 + \left(1 - \frac{\mu}{8}\right)\lambda^2 = \frac{1}{16}\left(< W_0, W_0 > +\frac{1}{4}\mu^2\right). \tag{40}$$

We conclude that $< W_0, W_0 >$ is conserved, since B is a spacelike elastic curve with Lagrange multiplier $-\frac{\mu}{4}$. Note also that we get

$$< W_0, W_1 >= -2\left(4\lambda^2 + \mu\right) + 2\left(\frac{1}{\lambda} - \lambda\right) = -2\mu - 8.$$

On the other hand, from (26) and (31), we find

$$\begin{aligned} 0 &=< W_0, W_1 >' \\ &=< f_1 N + f_2 B, b_1 > + < W_0, -\gamma \times W_0 > \\ &=< f_1 N + f_2 B, b_1 + \gamma \times W_0 > \\ &= f_2 s_3. \end{aligned}$$

Since λ is a non-constant solution of (40), we must have $f_2 = 0$. Moreover, $f_1 = 0$ by a similar calculation as

$$0 =< W_0, W_0 >' = 2 < f_1 N + f_2 B, W_1 >= 2a_2 f_1. \quad \square$$

Now we suppose that γ defines a force-free strip with timelike directrix. By Theorem 2, we have

$$W_0 = a_1 T + a_2 N + a_3 B = 0,$$

where T, N and B are component of the Frenet frame. This implies that

$$a_1 = -\frac{1}{2}\left(\kappa^2 \left(1+\lambda^2\right)^2 + \mu\right) = 0 \Rightarrow \mu < 0.$$

We can assume that $\mu = -1$. If we take the first derivative of a_1, we get

$$0 = \kappa' \left(1+\lambda^2\right)^2 + 2\kappa\lambda\lambda' \left(1+\lambda^2\right) = a_2$$

and

$$\kappa = \frac{1}{(1+\lambda^2)}. \tag{41}$$

These assumptions together with consecutive calculations make the proof of the following lemma obvious. Therefore it is omitted.

Lemma 2. *If γ is a timelike curve with non-constant modified torsion $\lambda = \frac{\tau}{\kappa}$, then the following conditions are equivalent:*
(i) *γ defines a force-free strip with timelike directrix,*
(ii) *$W_1 = -2\lambda T + 2\lambda' \left(1+\lambda^2\right) N + \left(1-\lambda^2\right) B$ is a constant,*
(iii) *$a_1 = 0$ and $< J, J >$ is conserved, where $J = s_1 T + s_2 N + s_3 B$.*

Theorem 5. *Let a timelike curve γ define a force-free strip with timelike directrix. Then, the tangent vector T of γ is a spacelike elastic curve with Lagrange multiplier 1 in pseudohyperbolic 2-space H^2. Conversely, for each such arc length parametrized curve $T : [0, \ell] \to H^2$ with geodesic curvature λ, the timelike curve*

$$\gamma(t) = \int_0^t \left(1+\lambda^2(s)\right) T(s)\,\mathrm{d}s \tag{42}$$

defines a force–free strip with timelike directrix with

$$S_{-1}(\gamma) = 2\int_0^\ell \left(1+\lambda^2\right) \mathrm{d}t = 2\ell(\gamma).$$

Proof. Suppose that a timelike curve γ defines a force-free strip with timelike directrix with arc length parameter s. From (iii) of Lemma 2, we have

$$< W_1, W_1 >= \left(1+\lambda^2\right)^2 4\lambda'^2 + 1 - 6\lambda^2 + \lambda^4. \tag{43}$$

On the other hand, if we choose the arclength t as $t'(s) = \frac{1}{1+\lambda^2}$ for the tangent vector T of γ, and write $\widetilde{\lambda}(t) = \lambda(s(t))$ in (43), then by Lemma 2, we obtain

$$\left(\widetilde{\lambda}'\right)^2 + \frac{1}{4}\widetilde{\lambda}^4 - \left(1+\frac{1}{2}\right)\widetilde{\lambda}^2 = \frac{1}{4}\left(< W_1, W_1 > -1\right)$$

where $\frac{1}{4}(< W_1, W_1 > -1) = const.$ Note also that $\widetilde{\lambda}$ is geodesic curvature of T as a consequence of Frenet equations (6). From (33), the tangent vector T of γ corresponds to a spacelike elastic curve with Lagrange multiplier 1 on H^2 of radius 1.

Conversely let T be such an arc length parametrized elastic curve with geodesic curvature λ on pseudohyperbolic 2-space H^2 as well as it is the tangent vector of a timelike curve γ with Frenet frame $\{T, N, B\}$ in $\mathbb{R}^3_1$. Then the Darboux trihedron of T is $\{N, -T, B\}$ and derivative formulas of the Darboux frame are

$$\begin{aligned} N' &= -T + \lambda B, \\ -T' &= N, \\ B' &= -\lambda N. \end{aligned} \tag{44}$$

If we consider the spacelike curve T having a reparametrization with an arc length s, it has $\frac{\mathrm{d}s}{\mathrm{d}t} = 1 + \lambda^2$, the following provides a timelike curve:

$$\gamma(t) = \int_0^t \left(1 + \lambda^2(s)\right) T(s)\,\mathrm{d}s.$$

By using equations (44), the curvature and modified torsion of γ can be stated as

$$\kappa_\gamma = \frac{1}{1+\lambda^2} \text{ and } \lambda_\gamma = -\lambda. \tag{45}$$

Substituting (45) in the equation (23), we conclude that

$$a_1 = -\frac{1}{2}\left(\kappa^2\left(1+\lambda^2\right)^2 - 1\right) = 0.$$

On the other hand, for the curve $\widetilde{\gamma}(s) = \gamma(t(s))$, we obtain

$$\frac{< J, J > -1}{4} = \left(\lambda'\right)^2 + \frac{1}{4}\lambda^4 - \left(1+\frac{1}{2}\right)\lambda^2. \tag{46}$$

Equation (46) shows that $< J, J >$ is conserved, since T is a spacelike elastic curve with Lagrange multiplier 1 in H^2. It then follows from Lemma 2 that the timelike curve γ defines a force-free strip with timelike directrix. □

Pinkall shows that each closed spherical elastic curve with Lagrange multiplier 1 corresponds to a Willmore torus in S^3 by using standard Hopf fibre $\pi : S^3 \to S^2$. He finds that there is the unique Willmore-Hopf tori coming from an elastica in S^2 with constant curvature (actually a geodesic) [19]. Barros *et al.* also obtain Pinkall's result by a straightforward computation in anti-de Sitter 3-space H_1^3. In the anti-de Sitter world, a Hopf torus M_β is a Willmore surface in H_1^3 if and only if β is an elastic curve in H^2 with $\lambda = -4$. However, there is no (Lorentzian) Willmore Hopf torus in H_1^3, because there is no closed $(-4)-$elastic curve in H^2 [1]. On the other hand, Barros *et al.* showed that a Hopf cylinder M_β is a Willmore surface in H_1^3 if and only if β is an elastic curve in H^2 [1, 2]. This latter fact implies that if the tangent vector T of a timelike curve is a spacelike elastic curve in H^2, we can reparametrize all lifted to H_1^3 along the curve (42) that in turn provides the following result.

Corollary 1. *Let the timelike curve* (42) *define a force-free strip with timelike directrix. Then, the semi-Riemannian Hopf cylinder associated to the tangent image T of the timelike curve* (42) *is a Willmore surface in anti-de Sitter 3-space H_1^3.*

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