MULTIVARIATE PERSPECTIVES

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We prove the multivariate perspective of an operator convex function of several variables is the unique extension of the corresponding multivariate regular operator mapping that preserves homogeneity and convexity.

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1. INTRODUCTION AND PRELIMINARIES

The concept of the operator perspective function is introduced in [2] by Effros for commuting operators where it is proved that the perspective of an operator convex function is operator jointly convex. A fully non-commutative perspective of the one variable function f is defined in [1] and the main results of [2] are generalized in [1] for the non-commutative case. Hence, the necessary and sufficient conditions for operator joint convexity (concavity) of the perspective and generalized perspective functions are established, cf. [9]. As an application of these results, Nikoufar *et al.* [7] gave the simplest proof of Lieb concavity theorem and Ando convexity theorem (see also [8]).

Hansen [4] introduced the notion of regular operator mappings of several variables generalizing the notion of the spectral function of Davis for functions of one variable. This setting is convenient for studying the mappings more general than those from the functional calculus, and it allows for Jensen type inequalities and multivariate perspectives. He generalized the notion of perspective of a regular mapping of several variables and defined the geometric mean for any number of operator variables. As a main application of the theory of regular operator mappings, Zhang [10] established operator concavity (convexity) of some functions of two or three variables by using perspectives of regular operator mappings of one or several variables and concavity (convexity) of the Fréchet differential mapping associated with some functions.

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Effros and Hansen [3] proved that the non-commutative perspective of an operator convex function of one variable is the unique extension of the corresponding commutative perspective that preserves homogeneity and convexity. In this paper, we consider this concept for functions of several variables and prove the multivariate perspective of an operator convex function of several variables is the unique extension of the corresponding multivariate regular operator mapping preserving homogeneity and convexity.

Throughout the paper, let $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})_{sa}$ denote the C^*-algebra of all linear bounded and the C^*-subalgebra of all linear bounded self–adjoint operators acting on a Hilbert space $\mathcal{H}$, respectively.

2. THE MAIN RESULTS

The notion of a regular mapping generalizes the notion of a spectral function of Davis for functions of one variable, the notion of a regular matrix mapping of two variables [5], and the notion of a regular operator mapping of two variables [3, Definition 2.1]. Hansen defined a regular mapping as follows [4, Definition 2.1]:

Definition 2.1. Let $F : \mathcal{D} \longrightarrow \mathcal{B}(\mathcal{H})$ be a mapping of k variables defined in a convex domain $\mathcal{D} \subseteq \mathcal{B}(\mathcal{H}) \times \ldots \times \mathcal{B}(\mathcal{H})$. The mapping F is called regular whenever

(i) The domain $\mathcal{D}$ is invariant under unitary transformations of $\mathcal{H}$ and

$$F(u^*x_1u, ..., u^*x_ku) = u^*F(x_1, ..., x_k)u \tag{2.1}$$

for every $x = (x_1, ..., x_k) \in \mathcal{D}$ and every unitary u on $\mathcal{H}$.

(ii) Let p and q be mutually orthogonal projections acting on $\mathcal{H}$ and take arbitrary k-tuples $(x_1, ..., x_k)$ and $(y_1, ..., y_k)$ of operators in $\mathcal{B}(\mathcal{H})$ such that the compressed tuples $(px_1p, ..., px_kp)$ and $(qy_1q, ..., qy_kq)$ are in the domain $\mathcal{D}$. Then, the k-tuple of diagonal block matrices

$$(px_1p + qy_1q, ..., px_kp + qy_kq)$$

is also in the domain $\mathcal{D}$ and

$$\begin{aligned} &F(px_1p + qy_1q, ..., px_kp + qy_kq) \\ &\quad = pF(px_1p, ..., px_kp)p + qF(qy_1q, ..., qy_kq)q. \end{aligned} \tag{2.2}$$

An operator function is a spectral function if and only if (2.1) and (2.2) are satisfied, cf. [4]. Note that by choosing q as the zero projection in the second condition in the above definition we obtain

$$F(px_1p, ..., px_kp) = pF(px_1p, ..., px_kp)p.$$

We may consider block matrices by the formula

$$F\left(\left(\begin{array}{cc} x_1 & 0 \\ 0 & y_1 \end{array}\right), \dots, \left(\begin{array}{cc} x_k & 0 \\ 0 & y_k \end{array}\right)\right) = \left(\begin{array}{cc} F(x_1, \dots, x_n) & 0 \\ 0 & F(y_1, \dots, y_k) \end{array}\right).$$

Hansen defined the perspective of a regular operator mapping of several variables as a generalization of the notion of perspective of functions of one variables, cf. [1, 4]. Denote by $\mathcal{D}_k^+$ the positive convex domain and define

$$\mathcal{D}_k^+ := \{(A_1, \dots, A_k) : A_1, \dots, A_k > 0\}$$

of strictly positive operators acting on an infinite dimensional Hilbert space $\mathcal{H}$ and

$$C(\mathcal{D}_k^+) := \{(A_1, \dots, A_k) \in \mathcal{D}_k^+ : A_1, \dots, A_k \text{ are compact and commutative}\}.$$

Definition 2.2. Let $F : \mathcal{D}_k^+ \longrightarrow \mathcal{B}(\mathcal{H})$ be a regular mapping. The perspective mapping $\mathcal{P}_F$ is defined in the domain $\mathcal{D}_{k+1}^+$ by setting

$$\mathcal{P}_F(A_1, \ldots, A_k, B) = B^{1/2} F(B^{-1/2} A_1 B^{-1/2}, \ldots, B^{-1/2} A_k B^{-1/2}) B^{1/2}$$

for strictly positive operators $A_1, \dots, A_k$ and B acting on a Hilbert space $\mathcal{H}$.

We know that the perspective of an operator convex function is operator convex as a function of two variables [1, Theorem 2.2]. Hansen generalized this result for a regular mapping of k operator variables [4, Theorem 3.2]. Indeed, he remarked that the perspective $\mathcal{P}_F$ of an operator convex and regular mapping $F : \mathcal{D}_k^+ \longrightarrow \mathcal{B}(\mathcal{H})$ is operator convex. In the following theorem, we confirm that the converse of this theorem holds. We apply the converse in the proof of Theorem 2.5.

THEOREM 2.3. *Let $F : \mathcal{D}_k^+ \longrightarrow \mathcal{B}(\mathcal{H})$ be a regular mapping. The perspective $\mathcal{P}_F$ is operator convex if and only if F is operator convex.*

Proof. Let $\mathcal{P}_F$ be operator convex. Then, the result comes from the fact that $F(A_1, \dots, A_k) = \mathcal{P}_F(A_1, \dots, A_k, 1)$. For the converse see [4, Theorem 3.2]. □

Our purpose is now to generalize the main result of [3] from the case of two variables to that of $k + 1$ operator variables. We are motivated by references [3, 4] and adopted the same concepts of regularity and perspectivity from [4] in connection with the main theorem of [3] for our setting.

THEOREM 2.4. *Let $(A_1, \dots, A_{k+1}) \longmapsto F(A_1, \dots, A_{k+1})$ be a regular mapping from $C(\mathcal{D}_{k+1}^+)$ into $\mathcal{B}(\mathcal{H})$ satisfying the conditions:*

(i) $F(tA_1, \dots, tA_{k+1}) = tF(A_1, \dots, A_{k+1}) \quad t > 0,$

(ii) $F(\dfrac{A_1 + B_1}{2}, \dots, \dfrac{A_{k+1} + B_{k+1}}{2}) \leq \dfrac{F(A_1, \ldots, A_{k+1}) + F(B_1, \dots, B_{k+1})}{2},$

(iii) $F(0, ..., 0) = 0$,

(iv) *the map* $(A_1, ..., A_k) \longmapsto F(A_1, \ldots, A_k, 1)$ *is continuous on bounded subsets in the strong operator topology where* 1 *is unit operator on* $\mathcal{H}$.

Then there exists an operator convex function $f : \mathbb{R}_+^k \longrightarrow \mathbb{R}$ *such that*

$$F(t_1.1, ..., t_k.1, 1) = f(t_1, ..., t_k)1, \quad t_1, ..., t_k > 0.$$

Furthermore,

$$F(A_1, ..., A_{k+1}) = \mathcal{P}_f(A_1, ..., A_{k+1}).$$

Proof. By using the regularity of F we see that

$$u^*F(t_1.1, ..., t_k.1, 1)u = F(u^*t_1.1u, ..., u^*t_k.1u, u^*u) = F(t_1.1, ..., t_k.1, 1)$$

for every unitary u in $\mathcal{B}(\mathcal{H})$. Thus, $F(t_1.1, ..., t_k.1, 1)$ commutes with every unitary in $\mathcal{B}(\mathcal{H})$. Since $\mathcal{B}(\mathcal{H})$ is factor, there exists a function $f : \mathbb{R}_+^k \longrightarrow \mathbb{R}$ such that

$$F(t_1.1, ..., t_k.1, 1) = f(t_1, ..., t_k)1, \quad t_1, ..., t_k > 0.$$

Let $A_1, ..., A_k \in C(\mathcal{D}_k^+)$ be finite rank operators and assume without loss of generality that they have the same rank. Then they are simultaneously diagonalizable by a unitary operator. Hence, there exist mutually perpendicular projections $p_1, ..., p_n$ such that $\sum_{j=1}^n p_j = 1$ and $A_i = \sum_{j=1}^n \lambda_{ij} p_j$, where λ_{ij} are the distinct eigenvalues of A_i. So,

$$\begin{aligned} F(A_1, ..., A_k, 1) &= F(\sum_{j=1}^n \lambda_{1j} p_j, ..., \sum_{j=1}^n \lambda_{kj} p_j, \sum_{j=1}^n p_j) \\ &= \sum_{j=1}^n p_j F(\lambda_{1j} p_j, ..., \lambda_{kj} p_j, p_j) p_j \\ &= \sum_{j=1}^n p_j F(\lambda_{1j}.1, ..., \lambda_{kj}.1, 1) p_j \\ &= \sum_{j=1}^n f(\lambda_{1j}, ..., \lambda_{kj}) p_j \\ &= f(A_1, ..., A_k). \end{aligned} \tag{2.3}$$

Note that regularity of F entails regularity of f. Define $D := (1 - CC^*)^{1/2}$, $E := (1 - C^*C)^{1/2}$ and consider the unitary block matrices

$$U = \begin{pmatrix} C & D \\ E & -C^* \end{pmatrix}, \quad V = \begin{pmatrix} C & -D \\ E & C^* \end{pmatrix},$$

where C is a contraction. We notice that

$$\frac{1}{2}U^* \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} U + \frac{1}{2}V^* \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} V = \begin{pmatrix} C^*AC & 0 \\ 0 & DAD \end{pmatrix} \tag{2.4}$$

for every operator $A \in \mathcal{B}(\mathcal{H})$. By using (2.4) and regularity of F we have

$$
\begin{aligned}
&\begin{pmatrix} F(C^*A_1C,...,C^*A_{k+1}C) & 0 \\ 0 & F(DA_1D,...,DA_{k+1}D) \end{pmatrix} \\
&= F\left(\begin{pmatrix} C^*A_1C & 0 \\ 0 & DA_1D \end{pmatrix},...,\begin{pmatrix} C^*A_{k+1}C & 0 \\ 0 & DA_{k+1}D \end{pmatrix}\right) \\
&= F\left(\frac{1}{2}U^*\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}U + \frac{1}{2}V^*\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}V,\right. \\
&\qquad \left. ..., \frac{1}{2}U^*\begin{pmatrix} A_{k+1} & 0 \\ 0 & 0 \end{pmatrix}U + \frac{1}{2}V^*\begin{pmatrix} A_{k+1} & 0 \\ 0 & 0 \end{pmatrix}V\right) \\
&\le \frac{1}{2}F\left(U^*\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}U,...,U^*\begin{pmatrix} A_{k+1} & 0 \\ 0 & 0 \end{pmatrix}U\right) \\
&\qquad + \frac{1}{2}F\left(V^*\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}V,...,V^*\begin{pmatrix} A_{k+1} & 0 \\ 0 & 0 \end{pmatrix}V\right) \\
&\quad = \frac{1}{2}U^*F\left(\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix},...,\begin{pmatrix} A_{k+1} & 0 \\ 0 & 0 \end{pmatrix}\right)U \\
&\qquad + \frac{1}{2}V^*F\left(\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix},...,\begin{pmatrix} A_{k+1} & 0 \\ 0 & 0 \end{pmatrix}\right)V \\
&\quad = \frac{1}{2}U^*\begin{pmatrix} F(A_1,...,A_{k+1}) & 0 \\ 0 & F(0,...,0) \end{pmatrix}U \\
&\qquad + \frac{1}{2}V^*\begin{pmatrix} F(A_1,...,A_{k+1}) & 0 \\ 0 & F(0,...,0) \end{pmatrix}V \\
&\quad = \begin{pmatrix} C^*F(A_1,...,A_{k+1})C & 0 \\ 0 & DF(A_1,...,A_{k+1})D \end{pmatrix}.
\end{aligned}
$$

In particular,

$$F(C^*A_1C,...,C^*A_{k+1}C) \le C^*F(A_1,...,A_{k+1})C \tag{2.5}$$

for contractions C. From (2.5) it follows that

$$(C^*)^{-1}F(C^*A_1C,...,C^*A_{k+1}C)C^{-1} \le F(A_1,...,A_{k+1}), \tag{2.6}$$

whence using (2.5) we obtain

$$
\begin{aligned}
F(A_1,...,A_{k+1}) &= F((C^*)^{-1}C^*A_1CC^{-1},...,(C^*)^{-1}C^*A_{k+1}CC^{-1}) \\
&\le (C^*)^{-1}F(C^*A_1C,...,C^*A_{k+1}C)C^{-1}. \qquad (2.7)
\end{aligned}
$$

From (2.6) and (2.7) we deduce that

$$C^*F(A_1, ..., A_{k+1})C = F(C^*A_1C, ..., C^*A_{k+1}C).$$

Then, by setting $C = A_{k+1}^{-1/2}$ we obtain

$$A_{k+1}^{-1/2}F(A_1, ..., A_k, A_{k+1})A_{k+1}^{-1/2} = F(A_{k+1}^{-1/2}A_1A_{k+1}^{-1/2}, ..., A_{k+1}^{-1/2}A_kA_{k+1}^{-1/2}, 1).$$

If A_i, $i = 1, ..., k$ are strictly positive and finite rank operators, then so are $A_{k+1}^{-1/2}A_iA_{k+1}^{-1/2}$ and hence (2.3) entails

$$F(A_{k+1}^{-1/2}A_1A_{k+1}^{-1/2}, ..., A_{k+1}^{-1/2}A_kA_{k+1}^{-1/2}, 1) = f(A_{k+1}^{-1/2}A_1A_{k+1}^{-1/2}, ..., A_{k+1}^{-1/2}A_kA_{k+1}^{-1/2}).$$

The continuity condition in (iv) now ensures

$$F(A_{k+1}^{-1/2}A_1A_{k+1}^{-1/2}, ..., A_{k+1}^{-1/2}A_kA_{k+1}^{-1/2}, 1) = f(A_{k+1}^{-1/2}A_1A_{k+1}^{-1/2}, ..., A_{k+1}^{-1/2}A_kA_{k+1}^{-1/2})$$

for strictly positive operators A_i defined on $\mathcal{H}$. Therefore,

$$\begin{aligned} F(A_1, ..., A_{k+1}) &= A_{k+1}^{1/2}F(A_{k+1}^{-1/2}A_1A_{k+1}^{-1/2}, ..., A_{k+1}^{-1/2}A_kA_{k+1}^{-1/2}, 1)A_{k+1}^{1/2} \\ &= A_{k+1}^{1/2}f(A_{k+1}^{-1/2}A_1A_{k+1}^{-1/2}, ..., A_{k+1}^{-1/2}A_kA_{k+1}^{-1/2})A_{k+1}^{1/2} \\ &= \mathcal{P}_f(A_1, ..., A_{k+1}). \quad \square \end{aligned}$$

THEOREM 2.5. *Suppose that the regular mapping $F : C(\mathcal{D}_{k+1}^+) \longrightarrow \mathcal{B}(\mathcal{H})$ satisfying the conditions (i)-(iv) of Theorem 2.4. Then, F is operator convex if and only if f is operator convex.*

Proof. In view of Theorem 2.4, there exists an operator convex function $f : \mathbb{R}_+^k \longrightarrow \mathbb{R}$ such that

$$F(A_1, ..., A_{k+1}) = \mathcal{P}_f(A_1, ..., A_{k+1}). \tag{2.8}$$

If F is operator convex, then $\mathcal{P}_f$ is operator convex by (2.8) and so is f by Theorem 2.3. The converse comes from Theorem 2.3 and (2.8). $\square$

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