K-FRAMES AND UNITARY REPRESENTATIONS

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In this paper, we propose to define an atomic system induced by a unitary representation π of a locally compact Hausdorff topological group on a Hilbert space. As a consequence, we give a K-frame corresponding to a unitary representation π, namely π-K-frame. Besides, the dual of π-K-frames are studied.

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1. INTRODUCTION AND BASIC DEFINITIONS

A family of local atoms with frame-like properties for a closed subspaces $\mathcal{H}_0$ of a separable Hilbert space $\mathcal{H}$ was introduced in [17]. In contrast to frames the building blocks for $\mathcal{H}_0$ do not necessarily belong to $\mathcal{H}_0$. This definition arises from sampling theory [16, 25, 26]. Atomic systems for a bounded linear operator $K \in B(\mathcal{H})$ as a generalization of families of local atoms, were introduced by Găvruţa [22]. Besides, Găvruţa [22] shows that this concept is equivalent to K-frames. We refer to [27] for more results on these concepts. In addition, the authors generalized these concepts and gave some new results in Hilbert spaces [11], Hilbert modules [12] and Banach spaces [13]. If $K = I_{\mathcal{H}}$, the identity operator on $\mathcal{H}$, then K-frames arise naturally as a generalization of the ordinary frames. For more details and applications of ordinary frames see [6–10, 15].

The concept of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure was proposed by G. Kaiser [23] and independently by Ali, Antoine and Gazeau [3]. These frames are known as continuous frames. Gabardo and Han in [19] called them frames associated with measurable spaces and in mathematical physics they are referred to as coherent states [3]. For more details and the basic definitions and some results the reader can refer to [2, 3, 5, 19, 23].

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On the other hand, Gabardo and Han [20] introduced the frame representations for group-like unitary systems. Also, Aldroubi, Larson, Tang and Weber proposed frames arising from the action of a unitary representation of a discrete countable abelian group [1].

In this paper, we are going to study atomic system and then K-frames arising from the action of a unitary representation of locally compact Hausdorff topological groups. In Section 2, atomic systems corresponding to a unitary representation π on a locally compact and Hausdorff group G are defined. As a result, π-K-frame, the K-frame corresponding to a unitary representation π, is introduced and its basic properties are studied. In Section 3, the dual of π-K-frames as a continuous K-frame is studied.

Let us recall some definitions and basic properties of atomic systems, K-frames and unitary representations that we need in the rest of the paper.

A sequence $\{u_j\}_{j\in\mathbb{N}}$ in the Hilbert space $\mathcal{H}$ is called an atomic system for the bounded linear operator K on $\mathcal{H}$ if

(i) the series $\sum_{j\in\mathbb{N}} c_j u_j$ converges for all $c = (c_j)_{j\in\mathbb{N}} \in l^2 := \{(b_j)_{j\in\mathbb{N}} : \sum_{j\in\mathbb{N}} |b_j|^2 < \infty\}$;

(ii) there exists $C > 0$ such that for every $t \in \mathcal{H}$ there exists $a_t = (a_j)_{j\in\mathbb{N}} \in l^2$ such that $\|a_t\|_{l^2} \le C\|t\|$ and $Kt = \sum_{j\in\mathbb{N}} a_j u_j$.

Găvruţa [22] shows that these concepts are equivalent to K-frames. A sequence $\{u_j\}_{j\in\mathbb{N}}$ in $\mathcal{H}$ is said to be a K-frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|K^*v\|^2 \le \sum_{j\in\mathbb{N}} |\langle v, u_j\rangle|^2 \le B\|v\|^2, \quad (v \in \mathcal{H}). \tag{1}$$

The constants A and B in (1) are called the lower and the upper bounds of $\{u_j\}_{j\in\mathbb{N}}$, respectively.

Recall that a unitary representation of a locally compact Hausdorff topological group G on a Hilbert space $\mathcal{H}_\pi$ is a homomorphism mapping π from a locally compact Hausdorff topological group G into the space of all unitary operators on $\mathcal{H}_\pi$, $\mathcal{U}(\mathcal{H}_\pi)$, for which $x \mapsto \pi(x)u$ is (strongly) continuous from G to $\mathcal{H}_\pi$ for all $u \in \mathcal{H}_\pi$. The left regular representation of G on $L^2(G)$ is defined as follows

$$(\pi_L(x)f)(y) = f(x^{-1}y), \quad \big(x, y \in G, \quad f \in L^2(G)\big).$$

Let π be a unitary representation of G on $\mathcal{H}_\pi$ and $L \in B(\mathcal{H}_\pi)$. The operator L is called intertwining operator, if $L\pi(x) = \pi(x)L$ holds, for all $x \in G$. The set of all such operators is denoted by $\mathcal{C}(\pi)$. An invariant subspace for π is a closed subspace M of $\mathcal{H}_\pi$ with the property that $\pi(x)M \subset M$ for all $x \in G$. The representation is said to be irreducible if there are exactly two trivial invariant subspaces ($\mathcal{H}_\pi$ and 0), otherwise this is reducible. For more details in unitary representations one can see [18].

Throughout this paper, G is a locally compact Hausdorff topological group with the left Haar measure μ, and π is a unitary representation of G on a Hilbert space $\mathcal{H}_\pi$.

2. π-K-FRAMES

In this section, K-frames induced by a unitary representation are studied. First we introduce an atomic system corresponding to a unitary representation π of a locally compact and Hausdorff group G.

Definition 2.1. Let $K \in B(\mathcal{H}_\pi)$ and $u \in \mathcal{H}_\pi$. $\pi(.)u$ is called a π-atomic system for K if the following conditions hold:

(i) $\int_G f(x)\langle \pi(x)u, v\rangle \mathrm{d}\mu(x)$, $v \in \mathcal{H}_\pi$ converges for all $f \in L^2(G)$;

(ii) for any $t \in \mathcal{H}_\pi$, there exists $g_t \in L^2(G)$ such that

$$\langle Kt, v\rangle = \int_G g_t(x)\langle \pi(x)u, v\rangle \mathrm{d}\mu(x),$$

where $\|g_t\| \leq C\|t\|$, and C is a positive constant.

Note that the condition (i) in this definition says that $\pi(.)u$ is the π-Bessel.

Now we give a characterization of π-atomic systems. The proof of the following theorem is similar to the discrete case of [21, Theorem 3] and we omit it.

THEOREM 2.2. *Let $K \in B(H)$ and $\pi(.)u$ be π-Bessel for G with respect to $\mathcal{H}_\pi$. Then the following statements are equivalent.*

(i) *$\pi(.)u$ is a π-atomic system for K;*

(ii) *there exist constants $0 < A \leq B < \infty$ such that*

$$A\|K^*v\|^2 \leq \int_G |\langle v, \pi(x)u\rangle|^2 \mathrm{d}\mu(x) \leq B\|v\|^2, \quad (v \in \mathcal{H}_\pi);$$

(iii) *$\pi(.)u$ is π-Bessel and there exists π-Bessel $\pi(.)v$ such that*

$$\langle Kt, w\rangle = \int_G \langle t, \pi(x)v\rangle\langle \pi(x)u, w\rangle \mathrm{d}\mu(x), \quad (w \in \mathcal{H}_\pi); \tag{2}$$

(iv) *$\int_G |\langle v, \pi(x)u\rangle|^2 \mathrm{d}\mu(x) < \infty, v \in \mathcal{H}_\pi$ and there exists π-Bessel $\pi(.)v$ such that*

$$\langle K^*t, w\rangle = \int_G \langle t, \pi(x)u\rangle\langle \pi(x)v, w\rangle \mathrm{d}\mu(x), \quad (w \in \mathcal{H}_\pi).$$

Now we are ready to introduce a K-frame corresponding to a unitary representation π.

Definition 2.3. Let $K \in B(\mathcal{H}_\pi)$ and $u \in \mathcal{H}_\pi$. $\{\pi(x)u\}_{x\in G}$ (or simply $\pi(.)u$) is said to be a π-K-frame with respect to G for $\mathcal{H}_\pi$ if there exist $A, B > 0$ such that

$$A||K^*v||^2 \leq \int_G |\langle v, \pi(x)u\rangle|^2 \mathrm{d}\mu(x) \leq B||v||^2, \quad (v \in \mathcal{H}_\pi).$$

The elements A and B are called the lower and upper frame bounds, respectively.

If $A = B = \lambda$, then the π-K-frame $\pi(.)u$ is said to be a λ-tight π-K-frame. In the special case $A = B = 1$, it is called a Parseval π-K-frame. If $\pi(.)u$ possesses an upper frame bound, but not necessarily a lower frame bound, we called it a π-K-Bessel (or π-Bessel).

Let $\pi(.)u$ be π-Bessel. Then it is well known that the analysis operator $T_u : \mathcal{H}_\pi \to L^2(G)$ of $\pi(.)u$ defined by

$$(T_u v)(x) = \langle v, \pi(x)u\rangle, \quad (u, v \in \mathcal{H}_\pi,\ x \in G),$$

is bounded. Also its adjoint, the synthesis operator, is as follows

$$\langle T_u^* g, w\rangle = \int_G g(x)\langle \pi(x)u, w\rangle \mathrm{d}\mu(x), \tag{3}$$

for every $w \in \mathcal{H}_\pi$ and $g \in L^2(G)$.

The operator $S_u := T_u^* T_u$ is called the frame operator of $\pi(.)u$, which is of the form

$$\langle S_u v, w\rangle = \int_G \langle v, \pi(x)u\rangle\langle \pi(x)u, w\rangle \mathrm{d}\mu(x), \qquad (v, w \in \mathcal{H}_\pi).$$

The frame operator S_u is bounded, positive and $AI \leq S_u \leq BI$. The following characterization of continuous frames has been given by Gabardo and Han [19].

LEMMA 2.4. *Let (X, ν) be a measure space and $\mathcal{H}$ a Hilbert space. Then a mapping $F : X \to \mathcal{H}$ is a continuous frame with lower and upper bounds A and B, respectively, if and only if $T_F : \mathcal{H} \to L^2(X)$ defined by $T_F u(x) = \langle u, F(x)\rangle$ is bounded by B and bounded below, with lower bound A.*

As a result of this lemma, one can see that for a family $\{\pi(x)u\}_{x\in G}$, the operator T_u^* defined by (3) is bounded and onto if and only if $\pi(.)u$ is a π-frame with respect to G for $\mathcal{H}_\pi$.

Recall that for Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote by $B(H_1, H_2)$ the space of all bounded linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$ and for $L \in B(H_1, H_2)$ we denote by $R(L)$ the range of L. Now we give a lemma for our next results.

LEMMA 2.5 ([14]). *Let $L_1 \in B(H_1, H)$ and $L_2 \in B(H_2, H)$. Then the following statements are equivalent:*

(i) $R(L_1) \subset R(L_2)$;

(ii) $L_1 L_1^* \leq C L_2 L_2^*$ *for some* $C \geq 0$.

By analogy with the discrete frame, we give some characterizations of π-K-frames.

THEOREM 2.6. *Let $\pi(.)u$ be π-Bessel. Then the operator T_u^* defined by (3) is bounded and $R(K) \subset R(T_u^*)$ if and only if $\pi(.)u$ is a π-K-frame.*

Proof. By using Lemmas 2.4 and 2.5 and the fact that

$$\|T_u v\|^2 = \int_G |\langle v, \pi(x)u\rangle|^2 \mathrm{d}\mu(x), \quad (v \in \mathcal{H}_\pi),$$

we can prove this theorem easily. □

THEOREM 2.7. *Let $\pi(.)u$ be π-Bessel. Then $\pi(.)u$ is a π-K-frame if and only if there exists $A, B > 0$ such that $AKK^* \leq S_u \leq BI$, where S_u is the π-frame operator of $\pi(.)u$. Moreover in this case $||K|| \leq \sqrt{\frac{B}{A}}$.*

Proof. Since $\pi(.)u$ is a π-K-frame with respect to G for $\mathcal{H}_\pi$, so

$$A||K^*v||^2 \leq \int_G |\langle v, \pi(x)u\rangle|^2 \mathrm{d}\mu(x) \leq B||v||^2, \qquad (v \in \mathcal{H}_\pi)$$

if and only if

$$A\langle KK^*v, v\rangle \leq \langle S_u v, v\rangle \leq B\langle v, v\rangle \quad (v \in \mathcal{H}_\pi)$$

For the last part, one can see that $AKK^* \leq BI$, hence $||K|| \leq \sqrt{\frac{B}{A}}$. □

COROLLARY 2.8. *Let $\pi(.)u$ be π-Bessel. Then $\pi(.)u$ is a π-K-frame if and only if $R(K) \subset R(S_u^{\frac{1}{2}})$, where S_u is the π-frame operator of $\pi(.)u$.*

Now we state the stability of π-K-frame.

PROPOSITION 2.9. *Let $\pi(.)u$ be a π-K-frame with lower and upper frame bounds A and B, respectively, and $L \in B(\mathcal{H}_\pi)$ such that $L \in \mathcal{C}(\pi)$ then $\pi(.)Lu$ is a π-LK-frame with lower and upper frame bounds A and $B||L||^2$, respectively, and its π-frame operator is $S_u' = LS_uL^*$, where S is the π-frame operator for $\pi(.)u$.*

Proof. Let $v \in \mathcal{H}_\pi$ then

$$\int_G |\langle v, \pi(x)Lu\rangle|^2 \mathrm{d}\mu(x) = \int_G |\langle v, L\pi(x)u\rangle|^2 \mathrm{d}\mu(x) = \int_G |\langle L^*v, \pi(x)u\rangle|^2 \mathrm{d}\mu(x),$$

so we have

$$\begin{aligned} A||(LK)^*v||^2 = A||K^*L^*v||^2 &\leq \int_G |\langle v, \pi(x)Lu\rangle|^2 \mathrm{d}\mu(x) \\ &\leq B||L^*v||^2 \leq B||L||^2||v||^2. \end{aligned}$$

For π-frame operator of $\pi(.)Lu$ we have

$$LS_uL^*v = \int_G \langle L^*v, \pi(x)u\rangle L\pi(x)u\mathrm{d}\mu(x) = \int_G \langle v, \pi(x)Lu\rangle \pi(x)Lu\mathrm{d}\mu(x).$$

Hence $S' = LS_uL^*$. □

Corollary 2.10. *Assume that $K \in B(\mathcal{H}_\pi) \cap \mathcal{C}(\pi)$. Let $\pi(.)u$ be a π-frame with lower and upper frame bounds A and B, respectively, then $\pi(.)Ku$ is a π-K-frame with lower and upper frame bounds A and $B||K||^2$, respectively.*

We obtain the following necessary and sufficient condition under which every π-K-frame is a π-frame in Hilbert spaces.

Theorem 2.11. *Suppose that π is irreducible, $\pi(.)u$ is a π-K-frame, and $KK^* \in \mathcal{C}(\pi)$ then $\pi(.)u$ is a π-frame.*

Proof. Since π is irreducible and $KK^* \in \mathcal{C}(\pi)$, by Schur's Lemma, there exists a constant C such that $KK^* = CI$. Thus, $||K^*v||^2 = C||v||^2$. Hence every π-K-frame is a π-frame by these hypotheses. □

2.1. π-K-DUALS

In this section, the dual of a π-K-frame as a continuous K-frame is studied. Some results about the dual of K-frames can be found in [24].

Definition 2.12. Let $\pi(.)u$ and $\pi(.)v$ be two π-Bessel families which satisfy (2). Then we say that $\pi(.)v$ is the π-K-dual of $\pi(.)u$ for G with respect to $\mathcal{H}_\pi$.

We write π-dual instead of π-I-dual, when I is the identity operator on $\mathcal{H}_\pi$. Note that $S_u^{-1}\pi(.)u$ is the (standard) π-dual of $\pi(.)u$ and $\pi(.)v$ is a π-K-dual of $\pi(.)u$ if and only if $T_u^*T_v = K$.

Proposition 2.13. *Let $\pi(.)u$ be a π-K-dual of $\pi(.)v$. Then $\pi(.)u$ and $\pi(.)v$ are K-frame and K^*-frame, respectively.*

Proof. Let $t \in \mathcal{H}_\pi$, then

$$\begin{aligned} ||Kt|| &= \sup_{||w||=1} |\langle Kt, w\rangle| \\ &= \sup_{||w||=1} \left| \int_G \langle w, \pi(x)u\rangle \langle \pi(x)v, t\rangle \mathrm{d}\mu(x) \right| \end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{\|w\|=1}\left(\int_G |\langle w,\pi(x)u\rangle|^2 \mathrm{d}\mu(x)\right)^{\frac{1}{2}}\left(\int_G |\langle w,\pi(x)v\rangle|^2 \mathrm{d}\mu(x)\right)^{\frac{1}{2}}\\
&\leq \sqrt{B}\sup_{\|w\|=1}\|w\|\left(\int_G |\langle w,\pi(x)v\rangle|^2 \mathrm{d}\mu(x)\right)^{\frac{1}{2}}\\
&\leq \sqrt{B}\left(\int_G |\langle w,\pi(x)v\rangle|^2 \mathrm{d}\mu(x)\right)^{\frac{1}{2}},
\end{aligned}
$$

where B is an upper bound of $\pi(.)u$. It means that $\pi(.)v$ is a K^*-frame. By the similar argument we can show that $\pi(.)u$ is a K-frame. □

In the following theorem, π-K-duals in any irreducible representation are studied.

THEOREM 2.14. *Let π be an irreducible representation and $\pi(.)v$ be a π-K-dual of $\pi(.)u$. Then K must be a scalar multiple of the identity.*

Proof. For $x, y \in G$ and $g \in L^2(G)$, we have

$$
\begin{aligned}
\pi(x)T_u^* g &= \int_G g(y)\pi(x)\pi(y)u\mathrm{d}\mu(y)\\
&= \int_G g(y)\pi(xy)u\mathrm{d}\mu(y)\\
&= \int_G g(x^{-1}y)\pi(y)u\mathrm{d}\mu(y)\\
&= \int_G (\pi_L(x)g)(y)\pi(y)u\mathrm{d}\mu(y)\\
&= T_u^*(\pi_L(x)g).
\end{aligned}
$$

On the other hand, for $v \in \mathcal{H}_\pi$ we have

$$
\begin{aligned}
\big(T_u\pi(x)v\big)(y) &= \langle \pi(x)v, \pi(y)u\rangle\\
&= \langle v, \pi(x^{-1}y)u\rangle\\
&= \big(T_u v\big)(x^{-1}y)\\
&= \big(\pi_L(x)T_u v\big)(y).
\end{aligned}
$$

So

$$T_v^*\pi_L(x)T_u = \pi(x)T_v^*T_u.$$

and

$$T_v^*T_u\pi(x) = T_v^*\pi_L(x)T_u.$$

These imply that

$$T_v^* T_u \pi(x) = \pi(x) T_v^* T_u.$$

By Schur's Lemma, there exists a constant λ such that $T_v^* T_u = \lambda I_{\mathcal{H}_\pi}$ and so $K = T_u^* T_v = \lambda I_{\mathcal{H}_\pi}$, with the following λ

$$\lambda \langle w, w \rangle = \langle T_v^* T_u w, w \rangle, \quad (w \in \mathcal{H}_\pi),$$

that is, $\lambda = \frac{1}{\|w\|^2} \int_G \langle w, \pi(x)u \rangle \langle \pi(x)v, w \rangle \mathrm{d}\mu(x)$. □

In a unimodular group, two π-frame vectors and their π-dual vectors have an interesting relation.

PROPOSITION 2.15. *Let G be unimodular, $\pi(.)v$ be a π-dual of $\pi(.)u$ and $\pi(.)w$ be a π-dual of $\pi(.)t$ for G with respect to $\mathcal{H}_\pi$. Then $\langle t, w \rangle = \langle u, v \rangle$.*

Proof. Since G is unimodular, the left and right Haar measure coincide and then we have

$$\begin{aligned} \langle t, w \rangle &= \int_G \langle t, \pi(x)v \rangle \langle \pi(x)u, w \rangle \mathrm{d}\mu(x) \\ &= \int_G \langle \pi(x^{-1})t, v \rangle \langle u, \pi(x^{-1})w \rangle \mathrm{d}\mu(x) \\ &= \int_G \langle u, \pi(x)w \rangle \langle \pi(x)t, v \rangle \mathrm{d}\mu(x) \\ &= \langle u, v \rangle \end{aligned}$$

Note that the third equality holds since G is unimodular. □

In the following theorem, by using π-dual, we show that the range of the analysis operator of a π-frame is a reproducing kernel Hilbert space and in particular is closed. Recall that a Hilbert space $\mathcal{H}$ of complex-valued functions on a set Ω is called a reproducing kernel Hilbert space if the evaluation functionals $E_z(f) = f(z)$, $z \in \Omega$, $f \in \mathcal{H}$, are bounded linear functionals (see [4] for more details).

THEOREM 2.16. *For a π-frame $\pi(.)u$, the range $R(T_u)$ of T_u is a reproducing kernel Hilbert space.*

Proof. First we show that $R(T_u)$ is a Hilbert space. In so doing, it is enough to show that $R(T_u)$ is a closed subspace of $L^2(G)$. Closedness of the range $R(T_u)$ of the analysis operator follows immediately from the fact that T_u is bounded from below.

Now let $f \in R(T_u)$. Then there exists $w \in \mathcal{H}_\pi$ such that $f = T_u w$. For $y \in G$ and $w \in \mathcal{H}_\pi$ we have

$$(T_u w)(y) = \langle w, \pi(y)u \rangle$$

$$\begin{aligned} &= \int_G \langle w, \pi(x)u\rangle \langle S_u^{-1}\pi(x)u, \pi(y)u\rangle \mathrm{d}\mu(x) \\ &= \int_G (T_u w)(x)\langle \pi(y^{-1})S_u^{-1}\pi(x)u, u\rangle \mathrm{d}\mu(x) \\ &= \int_G f(x)\langle \pi(y^{-1})S_u^{-1}\pi(x)u, u\rangle \mathrm{d}\mu(x), \end{aligned}$$

that is, $f(y) = \int_G f(x)\langle \pi(y^{-1})S_u^{-1}\pi(x)u, u\rangle \mathrm{d}\mu(x)$ which implies that $R(T_u)$ is a reproducing kernel Hilbert space with kernel $\langle \pi(y^{-1})S_u^{-1}\pi(x)u, u\rangle$. □

REFERENCES

[1] A. Aldroubi, D.R. Larson, W-S. Tang and E. Weber, *Geometric aspects of frame representations of abelian groups.* Trans. Amer. Math. Soc. **356** (2004), 4767–4186.

[2] S.T. Ali, J.P. Antoine and J.P. Gazeau, *Coherent States, Wavelets and Their Generalizations.* Graduate Texts in Contemporary Physics, Springer-Verlag, New York, 2000.

[3] S.T. Ali, J.P. Antoine and J.P. Gazeau, *Continuous frames in Hilbert spaces.* Ann. Physics **222** (1993), 1–37.

[4] N. Aronszajn, *Theory of reproducing kernels.* Trans. Amer. Math. Soc. **68** (1950), 337–404.

[5] A. Askari-Hemmat, M.A. Dehghan and M. Radjabalipour, *Generalized frames and their redundancy.* Proc. Amer. Math. Soc. **129** (2001), *4*, 1143–1147.

[6] B.G. Bodmannand and V.I. Paulsen, *Frames, graphs and erasures.* Linear Algebra Appl. **404** (2005), 118–146.

[7] H. Bolcskel, F. Hlawatsch and H.G. Feichtinger, *Frame-theoretic analysis of oversampled filter banks.* IEEE Trans. Signal Process. **46** (1998), 3256–3268.

[8] E.J. Candes and D.L. Donoho, *New tight frames of curvelets and optimal representations of objects with piecewise $\mathbb{C}^2$ singularities.* Commun. Pure Appl. Anal. **56** (2004), 216–266.

[9] P.G. Casazza, *The art of frame theory.* Taiwanese J. Math. **4** (2000), *2*, 129–202.

[10] O. Christensen, *Frames and Bases: An Introductory Course.* Appl. Numer. Harmon. Anal., Birkhäuser, Boston, 2008.

[11] B. Dastourian and M. Janfada, *G-frames for operators in Hilbert spaces.* Sahand Commun. Math. Anal. **8** (2017), *1*, 1–21.

[12] B. Dastourian and M. Janfada, *∗-frames for operators on Hilbert modules.* Wavel. Linear Algebra **3** (2016), 27–43.

[13] B. Dastourian and M. Janfada, *Frames for operators in Banach spaces via semi-inner products.* Int. J. Wavelets Multiresolut. Inf. Process. **14** (2016), *3*, 1650011 (17 pages).

[14] R.G. Douglas, *On majorization, factorization and range inclusion of operators on Hilbert space.* Proc. Amer. Math. Soc. **17** (1966), *2*, 413–415.

[15] Y.C. Eldar and T. Werther, *General framework for consistent sampling in Hilbert spaces.* Int. J. Wavelets Multiresolut. Inf. Process. **3** (2005), *3*, 347–359.

[16] H.G. Feichtinger and K. Grochenig, *Irregular sampling theorems and series expansion of band-limited functions.* Math. Anal. Appl. **167** (1992), 530–556.

[17] H.G. Feichtinger and T. Werther, *Atomic systems for subspaces.* In: L. Zayed (Ed.). Proceedings SampTA, Orlando, FL, 163–165, 2001.

[18] G.B. Folland, *A Course in Abstract Harmonic Analysis.* Studies in Advanced Mathematics. Boca Raton, CRC Press, 1995.

[19] J.P. Gabardo and D. Han, *Frames associated with measurable space.* Adv. Comp. Math. **18** (2003), 127–147.

[20] J.P. Gabardo and D. Han, *Frame representations for group-like unitary operator systems.* J. Operator Theory **49** (2003), 223-244.

[21] L. Găvruţa, *Atomic decompositions for operators in reproducing kernel Hilbert spaces.* Math. Rep. (Bucur.) **17(67)** (2015), *3*, 303–314.

[22] L. Găvruţa, *Frames for operators.* Appl. Comput. Harmon. Anal. **32** (2012), 139–144.

[23] G. Kaiser, *A Friendly Guide to Wavelets.* Birkhäuser, Boston, 1994.

[24] F.A. Neyshaburi and A.A. Arefijamaal, *Some constructions of K-frames and their duals.* Rocky Mountain J. Math. **47** (2017), *6*, 1749–1764.

[25] M. Pawlak and U. Stadtmuller, *Recovering band-limited signals under noise.* IEEE Trans. Inform. Theory **42** (1994), 1425–1438.

[26] T. Werther, *Reconstruction from irregular samples with improved locality.* Masters thesis, University of Vienna, 1999.

[27] X. Xiao, Y. Zhu and L. Găvruţa, *Some properties of K-frames in Hilbert spaces.* Results Math. **63** (2013), *3–4*, 1243–1255.

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