# THE INFLUENCE OF $\mathcal{F}_{hq}$-SUPPLEMENTED SUBGROUPS ON THE STRUCTURE OF FINITE GROUPS

M. EZZAT MOHAMED, M. I. ELASHIRY[1] and MOHAMMED M. AL-SHOMARANI[2]

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Let $G$ be a finite group. A subgroup $H$ of a group $G$ is quasinormal in $G$if it permutes with every subgroup of $G$. In this paper, we introduce the following definition: A subgroup $H$ is $\mathcal{F}_{hq}$-supplemented in $G$ if $G$ has a quasinormal subgroup $N$ such that $HN$ is a Hall subgroup of $G$ and $(H \cap N)H_G/H_G \leq Z_{\mathcal{F}}(G/H_G)$, where $H_G$ is the core of $H$ in $G$ and $Z_{\mathcal{F}}(G/H_G)$ is the hypercenter of $G/H_G$. Also, we study the structure of $G$ under assumption that all minimal subgroups are $\mathcal{F}_{hq}$-supplemented in $G$.

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## 1. INTRODUCTION

All groups considered in this paper will be finite and $G$ always means a finite group. We use conventional notions and notations, as in Doerk and Hawkes [5].

Recall that a minimal subgroup of a group $G$ is a subgroup of prime order. For a $p-$group $P$, we denote $\Omega(P) = \Omega_1(P)$ if $p > 2$, and $\Omega(P) = \langle \Omega_1(P), \Omega_2(P) \rangle$ if $p = 2$, where $\Omega_i(P) = \left\langle x \in P : |x| = p^i \right\rangle$.

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation provided that (i) if $G \in \mathcal{F}$, then $G/N \in \mathcal{F}$, and (ii) if $G/N_1$ and $G/N_2 \in \mathcal{F}$, then $G/(N_1 \cap N_2) \in \mathcal{F}$ for arbitrary normal subgroups $N_1$, $N_2$ of $G$.

A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies $G \in \mathcal{F}$. Throughout this paper, $\mathfrak{U}$ will denote the class of supersolvable groups. Clearly, $\mathfrak{U}$ is a saturated formation. A formation $\mathcal{F}$ is said to be $S-$closed ($S_n-$ closed) if it contains every subgroup (every normal subgroup, respectively) of all its

[1] Address for correspondence: Fayoum University, Faculty of Science, Department of Mathematics, Fayoum, Egypt. E-mail: mohaezzat@yahoo.com

[2] Address for correspondence: University of Northern Border, Faculty of Computing and Information Technology, Department of Mathematics, Rafha, P.O. 840, Saudi Arabia. E-mail: malshomrani@hotmail.com

groups. Let $[A]B$ stand for the semiproduct of two groups $A$ and $B$. For a class $\mathcal{F}$ of groups, a chief factor $H/K$ of a group $G$ is called $\mathcal{F}-$central ( see [6; Definition 2.4.3]) if $[H/K](G/C_G(H/K)) \in \mathcal{F}$. The symbol $Z_{\mathcal{F}}(G)$ denotes the $\mathcal{F}-$heypercenter of a group $G$, that is, the product of all such $H$ of $G$ whose $G-$chief factors are $\mathcal{F}-$central.

A subgroup $H$ of a group $G$ is quasinormal (or permutable) in $G$ if $HK = KH$ for all subgroups $K$ of $G$, or equivalently for all cyclic subgroups $K$ of $G$. Thus normal subgroups are always quasinormal, but not conversely. If $p$ is a prime, then any cyclic group $C_{p^n}$ extended by any cyclic group $C_{p^m}$ has all subgroups quasinormal (provided, when $p = 2$ and $n \geqslant 2$, the cyclic subgroup of order 4 in $C_{2^n}$ is central in the extension). The same is true if $C_{p^n}$ is replaced by any abelian $p-$group $H$ of finite exponent, with $C_{p^m}$ acting on $H$ as a group of power automorphisms (and elements of order 4 in H are again central in the extension if $p = 2$). These results can be found in sections 2.3 and 2.4 of [12].

We say, following Kegel [9], that a subgroup of a group $G$ is $S$-quasinormal in $G$ if it permutes with every Sylow subgroup of $G$.

Agrawal [1] defined the generalized center $genz(G)$ of a group $G$ to be the subgroup $< g \in G : \ < g > $ is $S$-quasinormal in $G >$. The generalized hypercenter $genz_\infty(G)$, is the largest term of the chain

$$1 = genz_0(G) \leq genz_1(G) = genz(G) \leq genz_2(G) \leq .../,$$

where $genz_{i+1}(G)/genz_i(G) = genz(G/genz_i(G))$ for all $i > 0$.

Guo, Freng and Huang [8] introduced the following new concept. They defined that the subgroup $H$ of a group $G$ is said to be $\mathcal{F}_h-$normal if there exists a normal subgroup $K$ of $G$ such that $HK$ is a normal Hall subgroup of $G$ and $(H \cap K)H_G/H_G \leqslant Z_{\mathcal{F}}(G/H_G)$; the authors have obtained some interesting results (see [7]). The latter concept has since been improved by Li and Tang [10] by the following concept; let $\mathcal{F}$ be a class of groups. A subgroup $H$ of a group $G$ is said to be $\mathcal{F}_h-$supplemented in $G$ if there exists a normal subgroup $T$ of $G$ such that $HT$ is a Hall subgroup of $G$ and $(H \cap T)H_G/H_G \leqslant Z_{\mathcal{F}}(G/H_G)$. As a consequence, we now introduce an extension of the preceding concept, it means that we replace the quasinormality of subgroups instead of the normality of subgroups as in the following defintion:

*Definition.* A subgroup $H$ of $G$ is $\mathcal{F}_{hq}$-supplemented in $G$ if $G$ has a quasinormal subgroup $N$ such that $HN$ is a Hall subgroup of $G$ and $(H \cap N)H_G/H_G \leq Z_{\mathcal{F}}(G/H_G)$, where $H_G = Core_G(H) = \bigcap_{g \in G} H^g$ is the maximal normal subgroup of $G$ which is contained in $H$.

The goal of this paper is to investigate the structure of $G$under assumption that all minimal subgroups are $\mathcal{F}_{hq}$-supplemented in $G$. In fact, we have obtained significant new results on these groups; these results improve the past results in [10].

## 2. PRELIMINARY RESULTS

For the convenience of the reader, we start with several known lemmas found in that paper.

LEMMA 2.1. *Let $G$ be a group and $H \le G$. Suppose that $\mathcal{F}$ is a non-empty saturated formation and $Z = Z_{\mathcal{F}}(G)$.*

(a) *If $H$ is a normal subgroup of $G$, then $HZ/H \le Z_{\mathcal{F}}(G/H)$.*

(b) *If $\mathcal{F}$ is $S$-closed, then $Z \cap H \le Z_{\mathcal{F}}(H)$.*

(c) *If $G \in \mathcal{F}$, then $Z = G$.*

*Proof.* See ([7; Lemma 2.1]). □

LEMMA 2.2. *Let $G$ be a group and $H \le K \le G$. Then*

(a) *$H$ is $\mathcal{F}_{hq}$-supplemented in $G$ if and only if $G$ has a quasinormal subgroup $N$ such that $HN$ is a Hall subgroup of $G$, $H_G \le N$ and $(H/H_G) \cap (N/H_G) \le Z_{\mathcal{F}}(G/H_G)$.*

(b) *If $H$ is a normal subgroup of $G$ and $K$ is $\mathcal{F}_{hq}$ -supplemented in $G$, then $K/H$ is $\mathcal{F}_{hq}$ -supplemented in $G/H$.*

(c) *If $H$ is a normal subgroup of $G$, then the subgroup $HE/H$ is $\mathcal{F}_{hq}$-supplemented in $G/H$ for every $\mathcal{F}_{hq}$-supplemented in $G$ subgroup $E$ satisfying $(|H|, |E|) = 1$.*

(d) *If $H$ is $\mathcal{F}_{hq}$-supplemented in $G$ and $\mathcal{F}$ is $S$-closed, then $H$ is $\mathcal{F}_{hq}$ -supplemented in $K$.*

*Proof.* (a) Suppose that $H$ is $\mathcal{F}_{hq}$ -supplemented in $G$. Then $G$ has a quasinormal subgroup $N$ such that $HN$ is a Hall subgroup of $G$ and $(H \cap N)H_G/H_G \le Z_{\mathcal{F}}(G/H_G)$. Put $M = NH_G$. Clearly $M$ is quasinormal subgroup of $G$ and $HM = HN$ is a Hall subgroup of $G$. Also $(H/H_G) \cap (M/H_G) = (H \cap M)/H_G = (H \cap NH_G)/H_G = (H \cap N)H_G/H_G \le Z_{\mathcal{F}}(G/H_G)$. The converse is clear.

(b) Assume that $K$ is $\mathcal{F}_{hq}$-supplemented in $G$. By (a), $G$ has a quasinormal subgroup $N$ such that $KN$ is a Hall subgroup of $G$, $K_G \le N$ and $(K/K_G) \cap (N/K_G) \le Z_{\mathcal{F}}(G/K_G)$. Then $K/H$ is quasinormal in $G/K$ and clearly $H \le K_G$. So $(K/H)(N/H) = KN/H$ is a Hall subgroup of $G/H$ and $((K/H)/(K_G/H)) \cap ((N/H)/(K_G/H)) = ((K/H)/(K/H)_{G/H}) \cap ((N/H)/(K/H)_{G/H}) = ((K \cap N)/H)/(K/H)_{G/H} \le Z_{\mathcal{F}}((G/H)/(K/H)_{G/H})$. Hence $K/H$ is $\mathcal{F}_{hq}$ -supplemented in $G/H$.

(c) Suppose that $E$ is $\mathcal{F}_{hq}$-supplemented in $G$ and let $N$ be a quasinormal subgroup of $G$ such that $EN$ is a Hall subgroup of $G$ and $(E/E_G)\cap(N/E_G) \leq Z_{\mathcal{F}}(G/E_G)$ by (a). Put $Z_{\mathcal{F}}(G/E_G) = L/E_G$. We treat the following two cases:

*Case 1.* $H \leq N$.

Then $HEN = EHN = EN$ is a Hall subgroup of $G$ and so $HE \cap N = H(E \cap N) \leq HL$. Also

$$HL/HE_G = HE_GL/HE_G \cong L/(L \cap HE_G) = L/E_G(L \cap H)$$

so $HL/HE_G \leq Z_{\mathcal{F}}(G/HE_G)$. Hence

$$(HE/HE_G) \cap (N/HE_G) = (HE \cap N)/HE_G = H(E \cap N)/HE_G \leq HL/HE_G \leq Z_{\mathcal{F}}(G/HE_G).$$

By Lemma 2.1(a),

$$Z_{\mathcal{F}}(G/HE_G)((HE)_G/HE_G)/((HE)_G/HE_G) \leq Z_{\mathcal{F}}((G/HE_G)/((HE)_G/HE_G)).$$

Then $(HE \cap N)(HE)_G/(HE)_G \leq Z_{\mathcal{F}}(G/(HE)_G)$ and so $HE$ is $\mathcal{F}_{hq}$-supplemented in $G$. Hence $HE/H$ is $\mathcal{F}_{hq}$-supplemented in $G/H$, by (b).

*Case 2.* $H \nleq N$.

Since $EN$ is a Hall subgroup of $G$ and $(|H|, |E|) = 1$, we have that $(HE/H)(NH/H) = HEN/H$ is a Hall subgroup of $G/H$ and since $(E \cap N)/E_G \leq L/E_G$, we have that $E \cap N \leq L$ and $(E \cap N)(HE)_G/(HE)_G \leq L(HE)_G/(HE)_G$. By Lemma 2.1(a),

$$\begin{aligned}((E \cap N)(HE)_G/E_G)/((HE)_G/E_G) &\leq (L(HE)_G)/E_G)/((HE)_G/E_G) \\ &= Z_{\mathcal{F}}(G/E_G)(HE)_G)/E_G)/((HE)_G/E_G) \leq Z_{\mathcal{F}}((G/E_G)/((HE)_G/E_G)).\end{aligned}$$

So we have $(E \cap N)(HE)_G/(HE)_G \leq Z_{\mathcal{F}}(G/(HE)_G)$. Since $(|H|, |E|) = 1$, we have that $(|HN : N|, |HN \cap E|) = 1$ and so $HN \cap E = N \cap E$. Hence

$$\begin{aligned}&(HE/H \cap HN/H)(HE/H)_{G/H}/(HE/H)_{G/H} \\ &= (H(E\cap N)/H)((HE)_G/H)/((HE)_G/H) \leq ((E\cap N)(HE)_G/H)/((HE)_G/H) \\ &\leq (L(HE)_G/H)((HE)_G/H) \leq Z_{\mathcal{F}}((G/H)/((HE)_G/H)).\end{aligned}$$

Hence $HE/H$ is $\mathcal{F}_{hq}$-supplemented in $G/H$.

(d) Suppose that $H$ is $\mathcal{F}_{hq}$-supplemented in $G$ and let $N$ be a quasinormal subgroup of $G$ such that $HN$ is a Hall subgroup of $G$ and $(H/H_G) \cap (N/H_G) \leq Z_{\mathcal{F}}(G/H_G)$ by (a). Put $M = K \cap N$. For every subgroup $L$ of $K$, $LM = L(K \cap N) = K \cap LN = K \cap NL = (K \cap N)L = ML$ as $N$ is quasinormal in $G$. Then $M$ is quasinormal in $K$. Since $HN$ is a

Hall subgroup of $G$ and $H \leq K$, we have that $HM$ is a Hall subgroup of $K$. Also $M/H_G \cap H/H_G = (K \cap N \cap H)/H_G = K/H_G \cap Z_{\mathcal{F}}(G/H_G)$. Put $K/H_G \cap Z_{\mathcal{F}}(G/H_G) = R/H_G$. Since $\mathcal{F}$ is $S$-closed, we have by Lemma 2.1(b), $R/H_G \leq Z_{\mathcal{F}}(K/H_G)$. By Lemma 2.1(a), $(R/H_G)(H_K/H_G)/(H_K/H_G) \leq Z_{\mathcal{F}}(K/H_G)(H_K/H_G)/(H_K/H_G) \leq Z_{\mathcal{F}}(K/H_G)/(H_K/H_G)$ and so $(M \cap H)H_K/H_K \leq Z_{\mathcal{F}}(K/H_K)$. Hence $H$ is $\mathcal{F}_{hq}$ -supplemented in $K$. □

Lemma 2.3. (a) *An $S$-quasinormal subgroup of $G$ is subnormal in $G$.*

(b) *If $H$ is $S$-quasinormal Hall subgroup of $G$, then $H \lhd G$.*

(c) *Let $H$ be a $p$-subgroup of $G$. Then $H$ is $S$-quasinormal in $G$ if and only if $O^p(G) \leq N_G(H)$.*

(d) *If $K$ is a normal subgroup of a group $G$ and $H$ is $S$-quasinormal in $G$, then $H \cap K$ is $S$-quasinormal in $G$.*

*Proof.* (a) See ([9; Satz 1, p. 209]).

(b) By (a), $H$ is subnormal in $G$. Hence $H$ is subnormal Hall subgroup of $G$. This implies that $H \lhd G$.

(c) See ([11; Lemma A, p. 287]).

(d) Since $H$ is $S$-quasinormal in $G$, it follows by (a), that $H$ is subnormal in $G$ and so $HK$ and $H \cap K$ are subnormal in $G$. Let $P$ be an arbitrary Sylow subgroup of $G$. Clearly $H \cap P \in Syl(H)$, $K \cap P \in Syl(K)$, $HK \cap P \in Syl(HK)$ and $H \cap K \cap P \in Syl(H \cap K)$. Then $|(H \cap P)(K \cap P)| = \frac{|H \cap P||K \cap P|}{|H \cap K \cap P|} = |HK \cap P|$ and since $(H \cap P)(K \cap P) \leq HK \cap P$, it follows that $(H \cap P)(K \cap P) = HK \cap P$. Hence by [5; Lemma 1.2, p. 2], $(H \cap K)P = HP \cap KP$ and so $H \cap K$ is $S$-quasinormal in $G$. □

Lemma 2.4. (a) *Let $p$ be the smallest prime dividing the order of $G$, and let $G_p$ be a Sylow $p$-subgroup of $G$. If $\Omega(G_p) \leq genz_\infty(G)$, then $G$ is $p$-nilpotent.*

(b) *Let $P$ be a normal $p$-subgroup of $G$ such that $G/P$ is supersolvable. If $\Omega(P) \leq genz_\infty(G)$, then $G$ is supersolvable.*

*Proof.* See ([3; Lemma 3.8 and Theorem 3.11, p. 2245-2246]). □

Lemma 2.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathfrak{U}$ and let $K$ be a normal subgroup $K$ of $G$ such that $G/K \in \mathcal{F}$ and the cyclic subgroups of $K$ of prime order or order 4 are $S$-quasinormal in $G$, then $G \in \mathcal{F}$.*

*Proof.* See ([2; Theorem 1, p.2773]). □

Lemma 2.6. *Let $K$ be a normal subgroup $K$ of a group $G$ with $G/K$ contained in a saturated formation $\mathcal{F}$. If $\Omega(P) \leq Z_{\mathcal{F}}(G)$, where $P$ is a Sylow $p$-subgroup of $K$, then $G/O_{p'}(K) \in \mathcal{F}$.*

*Proof.* See ([4]). □

## 3. RESULTS

We proceed now to the first main results.

LEMMA 3.1. *Let $p$ be the smallest prime dividing the order of $G$ and let the cyclic subgroup of $G$ of order $p$ or 4 be $\mathfrak{U}_{hq}$-supplemented in $G$. Then $G$ is $p$-nilpotent.*

*Proof.* Suppose the result is false and let $G$ be a counter-example of minimal order. Suppose that $G_p$ is a Sylow $p$-subgroup of $G$. Then we have:

(1) $O_{p'}(G) = 1$.

If not, then by Lemma 2.2(c), it is easy to see that the cyclic subgroup of $G_pO_{p'}(G)/O_{p'}(G)$ of order $p$ or order 4 is $\mathfrak{U}_{hq}$-supplemented in $G/O_{p'}(G)$. The minimality of $G$ implies that $G/O_{p'}(G)$ is $p$-nilpotent and hence $G$ is $p$-nilpotent; a contradiction.

(2) $p = 2$.

Suppose $p > 2$. If the cyclic subgroup of $G$ of order $p$ is normal of $G$, then $\Omega_1(G_p) \leq Z_{\mathfrak{U}}(G) \leq genz_\infty(G)$. Hence by Lemma 2.4(a), $G$ is $p$-nilpotent; a contradiction. Thus we may assume that there exists a subgroup $H$ of $G$ of order $p$ such that $H$ is not normal in $G$. By hypothesis, $H$ is $\mathfrak{U}_{hq}$-supplemented in $G$. By Lemma 2.2(a), $G$ has a quasinormal subgroup $N$ of $G$ such that $HN$ is a Hall subgroup of $G$ and $H/H_G \cap N/H_G \leq Z_{\mathfrak{U}}(G/H_G)$. Since $H$ is not normal in $G$, we have that $H_G = 1$ and so $H \cap N \leq Z_{\mathfrak{U}}(G)$. If $H \cap N = H \leq Z_{\mathfrak{U}}(G)$, then $\Omega_1(G_p) \leq Z_{\mathfrak{U}}(G) \leq genz_\infty(G)$. Hence by Lemma 2.4(a), $G$ is $p$-nilpotent; a contradiction. Thus $H \cap N = 1$. Clearly $N < G$. Since $N$ is a quasinormal subgroup of $G$, we have by Lemma 2.3(a), that $N$ is a subnormal in $G$ and since $N < G$, we have a normal subgroup $M$ of $G$ containing $N$. If $N$ is not $p$-subgroup of $G$, then also $M$ is not $p$-subgroup of $G$. Since the class of supersolvable group is $S$-closed, we have by Lemma 2.2(d), that the cyclic subgroup of $M$ of order $p$ is $\mathfrak{U}_{hq}$-supplemented in $M$. Then $M$ is $p$-nilpotent, by the minimality of $G$. Hence $O_{p'}(M) \neq 1$. Since $M$ is a normal subgroup of $G$, we have that, $1 < O_{p'}(M) \leq O_{p'}(G)$; a contradiction. Thus $N$ is $p$-subgroup of $G$. Since $HN$ is a Hall subgroup of $G$, we have that $HN = G_p$ and since $H \cap N = 1$, we have $N$ is a maximal subgroup of $G_p$. So $N \lhd G_p$. Let $G_q$ be an arbitrary Sylow $q$-subgroup of $G$, with $q > p$. Since $N$ is a quasinormal subgroup in $G$, we have that $NG_q \leq G$ and so $N$ is a quasinormal Hall in $NG_q$. Then by Lemma 2.3(b), $N \lhd NG_q$ *i.e.*, $G_q \leq N_G(N)$. Thus $O^p(G) \leq N_G(N)$ and since $N \lhd G_p$, we have that $N \lhd G$. Now consider the group $G/N$. Clearly $G_p/N$ is a Sylow $p$-subgroup of $G/N$ of order $p$. By Burnside's theorem, $G/N$ is $p$-nilpotent. Then $G/N$ has a normal Hall $p'$-subgroup $K/N$ and so $K$ is a proper normal subgroup of $G$. Since the class of supersolvable groups is $S$-closed, we have by Lemma 2.2(d), that the cyclic subgroups of $K$ of order

$p$ is $\mathfrak{U}_{hq}$-supplemented in $K$. Then $K$ is $p$-nilpotent, by the minimality of $G$. Hence $O_{p'}(K) \neq 1$. So $1 < O_{p'}(K) \leq O_{p'}(G)$; a contradiction.

(3) Final contradiction.

If the cyclic subgroup of $G$ of order 2 or 4 is normal in $G$, then $\Omega_2(G) \leq Z_{\mathfrak{U}}(G) \leq genz_{\infty}(G)$. Hence by Lemma 2.4(a), $G$ is $p$-nilpotent; a contradiction. Thus we may assume that there exists a subgroup $H$ of $G$ of order 2 or 4 such that $H$ is not normal in $G$. If $|H| = 2$, then $H_G = 1$. By hypothesis, $H$ is $\mathfrak{U}_{hq}$-supplemented in $G$. By Lemma 2.2(a), $G$ has a quasinormal subgroup $N$ of $G$ such that $HN$ is a Hall subgroup of $G$ and $H \cap N \leq Z_{\mathfrak{U}}(G)$. If $H \cap N = 1$, then $O_{p'}(G) \neq 1$, by repeating the proof of (2). Thus $H \cap N = H \leq Z_{\mathfrak{U}}(G)$ and so $\Omega_1(G_p) \leq Z_{\mathfrak{U}}(G)$. If the cyclic subgroup of $G$ of order 4 is normal in $G$, then $\Omega_2(G) \leq Z_{\mathfrak{U}}(G) \leq genz_{\infty}(G)$. Hence by Lemma 2.4(a), $G$ is $p$-nilpotent; a contradiction. Thus there exists a subgroup $L$ of $G$ of order 4 such that $L$ is not normal in $G$. By hypothesis, $L$ is $\mathfrak{U}_{hq}$-supplemented in $G$. By Lemma 2.2(a), $G$ has a quasinormal subgroup $T$ of $G$ such that $LT$ is a Hall subgroup of $G$ and $L/L_G \cap T/L_G \leq Z_{\mathfrak{U}}(G/L_G)$. Since $L$ is not normal in $G$, we have that $|L_G| = 2$ or 1. If $|L_G| = 2$, then $L_G \leq Z_{\mathfrak{U}}(G)$. Hence $Z_{\mathfrak{U}}(G/L_G) = Z_{\mathfrak{U}}(G)/L_G$. So $L \cap T \leq Z_{\mathfrak{U}}(G)$. Also if $|L_G| = 1$, then $L \cap T \leq Z_{\mathfrak{U}}(G)$. If $L = L \cap T \leq Z_{\mathfrak{U}}(G)$, then $\Omega_2(G) \leq Z_{\mathfrak{U}}(G) \leq genz_{\infty}(G)$. Hence by Lemma 2.4(a), $G$ is $p$-nilpotent; a contradiction. Thus $L \cap T$ is a proper subgroup of $L$. So $T$ is also a proper subgroup of $G$. If $T$ is not $p$-subgroup of $G$, then $O_{p'}(G) \neq 1$ by repeating the proof of (2); a contradiction. Thus $T$ is $p$-subgroup of $G$. Since $LT$ is a Hall subgroup of $G$, we have that $LT = G_p$. Since $L \cap T$ is a proper subgroup of $L$, we have that $T$ is a proper subgroup of $G_p$ and since $T$ is quasinormal subgroup in $G$, we have by Lemma 2.3(c), that $O^p(G) \leq N_G(T)$. If $N_G(T) = G$, then $T \lhd G$. Since $G_p/T$ is cyclic, we have by Burnside's theorem, $G/T$ is $p$-nilpotent. By repeating the proof of (2), we have that $1 < O_{p'}(G)$; a contradiction. Thus we may assume that $O^p(G) \leq N_G(T) < G$. Since the class of supersolvable groups is $S$-closed, we have by Lemma 2.2(d), that the cyclic subgroups of $O^p(G)$ of order 2 or 4 is $\mathfrak{U}_{hq}$-supplemented in $O^p(G)$. By the minimality of $G$, $O^p(G)$ is $p$-nilpotent and also $G$; a final contradiction. □

As a consequence, we also obtain improvement of Corollary 4.2 in [9].

THEOREM 3.2. *If the cyclic subgroups of $G$ of prime order or order 4 are $\mathfrak{U}_{hq}$-supplemented in $G$, then $G$ is supersolvable.*

*Proof.* Suppose the result is false and let $G$ be a counter-example of minimal order. Lemma 3.1 implies that $G$ is $r$-nilpotent, where $r$ is the smallest prime dividing the order of $G$. Then $G = G_rK$, where $G_r$ is a Sylow $r$-subgroup of $G$ and $K$ is a normal Hall $r'$-subgroup of $G$. Since the class of supersolvable is $S$-closed, we have by Lemma 2.2(d), the hypothesis of the theorem satisfies over $K$. Then $K$ is supersolvable by the minimality of $G$. Hence $K$ has a chracteristic Sylow $q$-subgroup $G_q$ and $q$ is the largest prime dividing the order of $K$. Since $K \lhd G$, we have that $G_q \lhd G$ and since $K$ is a Hall $r'$-subgroup of $G$, we have $G_q$ is a Sylow $q$-subgroup of $G$. Now consider the factor group $G/G_q$. By Lemma 2.2(c), the hypothesis satisfies $G/G_q$. Then $G/G_q$ is supersolvable by the minimality of $G$. Thus $G^{\mathfrak{U}} \leq G_q$, where $G^{\mathfrak{U}}$ is supersolvable residual of $G$. If the cyclic subgroups of $G^{\mathfrak{U}}$ of order $q$ are normal in $G$, then $\Omega_1(G^{\mathfrak{U}}) \leq Z_{\mathfrak{U}}(G) \leq genz_{\infty}(G)$. Hence by Lemma 2.4(b), $G$ is supersolvable; a contradiction. Thus there exists a subgroup $H$ of $G^{\mathfrak{U}}$ of order $q$ is not normal in $G$. By hypothesis, $H$ is $\mathfrak{U}_q$-supplemented in $G$. By Lemma 2.2(a), $G$ has a quasinormal subgroup $N$ of $G$ such that $HN$ is a Hall subgroup of $G$ and $H/H_G \cap N/H_G \leq Z_{\mathfrak{U}}(G/H_G)$. Since $H$ is not normal in $G$, we have that $H_G = 1$ and so $H \cap N \leq Z_{\mathfrak{U}}(G)$. If $H \leq N$, then $H = H \cap N \leq Z_{\mathfrak{U}}(G)$. Hence $\Omega_1(G^{\mathfrak{U}}) \leq Z_{\mathfrak{U}}(G) \leq genz_{\infty}(G)$. Hence by Lemma 2.4(b), $G$ is supersolvable; a contradiction. Thus $H \cap N = 1$. Since $HN$ is a Hall subgroup of $G$, we have that $G_q \leq HN$ and $N \cap G_q$ is a maximal subgroup of $G_q$. Since $N$ is quasinormal in $G$, we have that $N$ is $S$-quasinormal in $G$ and since $G_q$ is a normal subgroup of $G$, we have by Lemma 2.3(d), that $N \cap G_q$ is $S$-quasinormal in $G$. Then $O^q(G) \leq N_G(N \cap G_q)$ by Lemma 2.3(c) and since $N \cap G_q$ is a maximal subgroup of $G_q$, we have that $G_q \leq N_G(N \cap G_q)$. Then $G = G_q O^q(G) \leq N_G(N \cap G_q)$. Hence $N \cap G_q \lhd G$. Since $G/G_q$ is supersolvable, we have that $(G/(N \cap G_q))/(G_q/(N \cap G_q)) \cong G/G_q$ is supersolvable and since $G_q/(N \cap G_q)$ is cyclic of order $q$, we have that $G/(N \cap G_q)$ is supersolvable. Then $G^{\mathfrak{U}} \leq N \cap G_q$. Hence $H \leq N \cap G_q \leq N$; a final contradiction. □

Now we prove an extension of Corollary 4.3 in [10].

THEOREM 3.3. *Let $\mathcal{F}$ be an $S$-closed saturated formation containing $\mathfrak{U}$. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup $K$ of $G$ such that $G/K \in \mathcal{F}$ and the cyclic subgroups of $K$ of prime order or order 4 are $\mathfrak{U}_{hq}$-supplemented in $G$.*

*Proof.* If $G \in \mathcal{F}$, then the result holds with $K = 1$. □

The converse, suppose the result is false and let $G$ be a counter-example of minimal order. Since $\mathcal{F}$ is $S$-closed, we have by Lemma 2.2(d), the cyclic subgroups of $K$ of prime order or order 4 are $\mathfrak{U}_{hq}$-supplemented in $K$. Then $K$ is supersolvable by Theorem 3.2. Hence $K$ has a characteristic Sylow $p$-subgroup $K_p$, where $p$ is the largest prime dividing the order of $K$. Since $K \lhd G$, we have that $K_p \lhd G$. Then $(G/K_p)/(K/K_p) \cong G/K \in \mathcal{F}$. By hypothesis and Lemma 2.2(c), the cyclic subgroups of $K/K_p$ of prime order or order 4 are $\mathfrak{U}_{hq}$-supplemented in $G/K_p$. Then $G/K_p \in \mathcal{F}$ by the minimality of $G$. If $p = 2$, then $K_pG_q \leq G$, for every Sylow $q$-subgroup $G_p$ of $G$ with $q > 2$. By Lemma 3.1, $K_pG_q$ is 2-nilpotent. Then $K_pG_q = K_p \times G_q$. Since $K_p$ is a normal 2-subgroup of $G$, we have that the cyclic subgroups of $K_p$ of order 2 or 4 are $S$-quasinormal in $G$. Then $G \in \mathcal{F}$ by Lemma 2.5; a contradiction. Thus we may assume that $p > 2$. Clearly $1 \neq G^{\mathcal{F}} \leq K_p$ as $G/K_p \in \mathcal{F}$. If the subgroups of $G^{\mathcal{F}}$ of order $p$ lay in $Z_{\mathcal{F}}(G)$, then $G \in \mathcal{F}$ by Lemma 2.6; a contradiction. Thus there exists a subgroup $H$ of $G^F$ of order $p$ such that $H \nleq Z_{\mathcal{F}}(G)$. Then $H \nleq Z_{\mathfrak{U}}(G)$ as $\mathfrak{U} \subseteq \mathcal{F}$. Hence $H_G = 1$. By hypothesis $H$ is $\mathfrak{U}_{hq}$-supplemented in $G$. By Lemma 2.1(a), $G$ has a quasinormal subgroup $N$ of $G$ such that $HN$ is a Hall subgroup of $G$ and $H \cap N \leq Z_{\mathfrak{U}}(G)$. Since $HN$ is a Hall subgroup of $G$, we have that $G^{\mathcal{F}} \leq G_p \leq HN$, where $G_p$ is a Sylow $p$-subgroup of $G$. Also since $H \nleq Z_{\mathfrak{U}}(G)$, we have that $H \cap N = 1$. Then $G^{\mathcal{F}} \cap N$ is a maximal $p$-subgroup of $G^{\mathcal{F}}$. Since $N$ is a quasinormal subgroup of $G$, we have that $N$ is $S$-quasinormal subgroup of $G$ and since $G^{\mathcal{F}} \lhd G$, we have by Lemma 2.3(d), that $G^{\mathcal{F}} \cap N$ is $S$-quasinormal subgroup of $G$. Then $O^p(G) \leq N_G(G^{\mathcal{F}} \cap N)$. Clearly $G^{\mathcal{F}} \cap N \lhd N$ and since $G^{\mathcal{F}} \cap N$ is a maximal subgroup of $G^{\mathcal{F}}$, we have that $G^{\mathcal{F}} \cap N \lhd G^{\mathcal{F}}$. Then $G_p \leq HN = G^{\mathcal{F}}N \leq N_G(G^{\mathcal{F}} \cap N)$. Hence $G = G_pO^p(G) \leq N_G(G^{\mathcal{F}} \cap N)$ *i.e.*, $G^{\mathcal{F}} \cap N \lhd G$. Since $(G/(G^{\mathcal{F}} \cap N))/(G^{\mathcal{F}}/(G^{\mathcal{F}} \cap N) \cong G/G^{\mathcal{F}} \in \mathcal{F}$ and since $G^{\mathcal{F}}/(G^{\mathcal{F}} \cap N)$ is cyclic of order $p$, we have that $G^{\mathcal{F}}/(G^{\mathcal{F}} \cap N) \leq Z_{\mathcal{F}}(G/(G^{\mathcal{F}} \cap N))$. Hence $G/G^{\mathcal{F}} \cap N \in \mathcal{F}$ and so $H \leq G^{\mathcal{F}} \leq G^{\mathcal{F}} \cap N \leq N$; a final contradiction.

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*Northern Border University,*
*Faculty of Arts and Science,*
*Rafha, P.O. 840,*
*Saudi Arabia*

*King Abdulaziz University,*
*Faculty of Science,*
*Mathematics Department,*
*P. O. Box 80203,*
*Jeddah 21589,*
*Saudi Arabia*