THE INFLUENCE OF $\mathcal{F}_{hq}$-SUPPLEMENTED SUBGROUPS ON THE STRUCTURE OF FINITE GROUPS

M. EZZAT MOHAMED, M. I. ELASHIRY$^1$ and MOHAMMED M. AL-SHOMARANI$^2$

Communicated by Vasile Brînzănescu

Let $G$ be a finite group. A subgroup $H$ of a group $G$ is quasinormal in $G$ if it permutes with every subgroup of $G$. In this paper, we introduce the following definition: A subgroup $H$ is $\mathcal{F}_{hq}$-supplemented in $G$ if $G$ has a quasinormal subgroup $N$ such that $HN$ is a Hall subgroup of $G$ and $(H \cap N)H_G/H_G \leq Z_{\mathcal{F}}(G/H_G)$, where $H_G$ is the core of $H$ in $G$ and $Z_{\mathcal{F}}(G/H_G)$ is the hypercenter of $G/H_G$. Also, we study the structure of $G$ under assumption that all minimal subgroups are $\mathcal{F}_{hq}$-supplemented in $G$.

AMS 2010 Subject Classification: 20D10, 20D20.

Key words: finite groups, saturated formation, $\mathcal{F}_{hq}$-supplemented subgroup, Sylow subgroup, supersolvable group.

1. INTRODUCTION

All groups considered in this paper will be finite and $G$ always means a finite group. We use conventional notions and notations, as in Doerk and Hawkes [5].

Recall that a minimal subgroup of a group $G$ is a subgroup of prime order. For a $p$–group $P$, we denote $\Omega(P) = \Omega_1(P)$ if $p > 2$, and $\Omega(P) = \langle \Omega_1(P), \Omega_2(P) \rangle$ if $p = 2$, where $\Omega_i(P) = \langle x \in P : |x| = p^i \rangle$.

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation provided that (i) if $G \in \mathcal{F}$, then $G/N \in \mathcal{F}$, and (ii) if $G/N_1$ and $G/N_2 \in \mathcal{F}$, then $G/(N_1 \cap N_2) \in \mathcal{F}$ for arbitrary normal subgroups $N_1, N_2$ of $G$.

A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies $G \in \mathcal{F}$. Throughout this paper, $\mathcal{U}$ will denote the class of supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation. A formation $\mathcal{F}$ is said to be $S$–closed ($S_n$– closed) if it contains every subgroup (every normal subgroup, respectively) of all its

---

$^1$ Address for correspondence: Fayoum University, Faculty of Science, Department of Mathematics, Fayoum, Egypt. E-mail: mohaezzat@yahoo.com

$^2$ Address for correspondence: University of Northern Border, Faculty of Computing and Information Technology, Department of Mathematics, Rafha, P.O. 840, Saudi Arabia. E-mail: malshomrani@hotmail.com

MATH. REPORTS 21(71), 2 (2019), 135–144
groups. Let $[A]B$ stand for the semiprodouct of two groups $A$ and $B$. For a class $\mathcal{F}$ of groups, a chief factor $H/K$ of a group $G$ is called $\mathcal{F}$–central (see [6; Definition 2.4.3]) if $[H/K](G/C_G(H/K)) \in \mathcal{F}$. The symbol $Z_\mathcal{F}(G)$ denotes the $\mathcal{F}$–hypercenter of a group $G$, that is, the product of all such $H$ of $G$ whose $G$–chief factors are $\mathcal{F}$–central.

A subgroup $H$ of a group $G$ is quasinormal (or permutable) in $G$ if $HK = KH$ for all subgroups $K$ of $G$, or equivalently for all cyclic subgroups $K$ of $G$. Thus normal subgroups are always quasinormal, but not conversely. If $p$ is a prime, then any cyclic group $C_{p^n}$ extended by any cyclic group $C_{p^m}$ has all subgroups quasinormal (provided, when $p = 2$ and $n \geq 2$, the cyclic subgroup of order 4 in $C_{2^n}$ is central in the extension). The same is true if $C_{p^n}$ is replaced by any abelian $p$–group $H$ of finite exponent, with $C_{p^m}$ acting on $H$ as a group of power automorphisms (and elements of order 4 in $H$ are again central in the extension if $p = 2$). These results can be found in sections 2.3 and 2.4 of [12].

We say, following Kegel [9], that a subgroup of a group $G$ is $S$-quasinormal in $G$ if it permutes with every Sylow subgroup of $G$.

Agrawal [1] defined the generalized center $genz(G)$ of a group $G$ to be the subgroup $< g \in G : < g >$ is $S$-quasinormal in $G >$. The generalized hypercenter $genz_\infty(G)$, is the largest term of the chain

$$1 = genz_0(G) \leq genz_1(G) = genz(G) \leq genz_2(G) \leq .../,$$

where $genz_{i+1}(G)/genz_i(G) = genz(G/genz_i(G))$ for all $i > 0$.

Guo, Freng and Huang [8] introduced the following new concept. They defined that the subgroup $H$ of a group $G$ is said to be $\mathcal{F}_h$–normal if there exists a normal subgroup $K$ of $G$ such that $HK$ is a normal Hall subgroup of $G$ and $(H \cap K)H_G/H_G \leq Z_\mathcal{F}(G/H_G)$; the authors have obtained some interesting results (see [7]). The latter concept has since been improved by Li and Tang [10] by the following concept; let $\mathcal{F}$ be a class of groups. A subgroup $H$ of a group $G$ is said to be $\mathcal{F}_h$–supplemented in $G$ if there exists a normal subgroup $T$ of $G$ such that $HT$ is a Hall subgroup of $G$ and $(H \cap T)H_G/H_G \leq Z_\mathcal{F}(G/H_G)$. As a consequence, we now introduce an extension of the preceding concept, it means that we replace the quasinormality of subgroups instead of the normality of subgroups as in the following definition:

**Definition.** A subgroup $H$ of $G$ is $\mathcal{F}_{hq}$–supplemented in $G$ if $G$ has a quasinormal subgroup $N$ such that $HN$ is a Hall subgroup of $G$ and $(H \cap N)H_G/H_G \leq Z_\mathcal{F}(G/H_G)$, where $H_G = Core_G(H) = \cap_{g \in G} H^g$ is the maximal normal subgroup of $G$ which is contained in $H$.

The goal of this paper is to investigate the structure of $G$ under assump-
tion that all minimal subgroups are $F_{hq}$-supplemented in $G$. In fact, we have obtained significant new results on these groups; these results improve the past results in [10].

2. PRELIMINARY RESULTS

For the convenience of the reader, we start with several known lemmas found in that paper.

**Lemma 2.1.** Let $G$ be a group and $H \leq G$. Suppose that $F$ is a non-empty saturated formation and $Z = Z_F(G)$.

(a) If $H$ is a normal subgroup of $G$, then $HZ/H \leq Z_F(G/H)$.

(b) If $F$ is $S$-closed, then $Z \cap H \leq Z_F(H)$.

(c) If $G \in F$, then $Z = G$.

*Proof.* See ([7; Lemma 2.1]). □

**Lemma 2.2.** Let $G$ be a group and $H \leq K \leq G$. Then

(a) $H$ is $F_{hq}$-supplemented in $G$ if and only if $G$ has a quasinormal subgroup $N$ such that $HN$ is a Hall subgroup of $G$, $H_G \leq N$ and $(H/H_G) \cap (N/H_G) \leq Z_F(G/H_G)$.

(b) If $H$ is a normal subgroup of $G$ and $K$ is $F_{hq}$-supplemented in $G$, then $K/H$ is $F_{hq}$-supplemented in $G/H$.

(c) If $H$ is a normal subgroup of $G$, then the subgroup $HE/H$ is $F_{hq}$-supplemented in $G/H$ for every $F_{hq}$-supplemented in $G$ subgroup $E$ satisfying $(|H|, |E|) = 1$.

(d) If $H$ is $F_{hq}$-supplemented in $G$ and $F$ is $S$-closed, then $H$ is $F_{hq}$-supplemented in $K$.

*Proof.* (a) Suppose that $H$ is $F_{hq}$-supplemented in $G$. Then $G$ has a quasinormal subgroup $N$ such that $HN$ is a Hall subgroup of $G$ and $(H \cap N)H_G/H_G \leq Z_F(G/H_G)$. Put $M = NH_G$. Clearly $M$ is quasinormal subgroup of $G$ and $HM = HN$ is a Hall subgroup of $G$. Also $(H/H_G) \cap (M/H_G) = (H \cap M)/H_G = (H \cap NH_G)/H_G = (H \cap N)H_G/H_G \leq Z_F(G/H_G)$. The converse is clear.

(b) Assume that $K$ is $F_{hq}$-supplemented in $G$. By (a), $G$ has a quasinormal subgroup $N$ such that $KN$ is a Hall subgroup of $G$, $K_G \leq N$ and $(K/K_G) \cap (N/K_G) \leq Z_F(G/K_G)$. Then $K/H$ is quasinormal in $G/K$ and clearly $H \leq K_G$. So $(K/H)(N/H) = KN/H$ is a Hall subgroup of $G/H$ and $((K/H)/(K_G/H)) \cap ((N/H)/(K_G/H)) = ((K/H)/(K/H)G/H) \cap ((N/H)/(K/H)_G/H) = ((K \cap N)/H)/(K/H)G/H \leq Z_F((G/H)/(K/H)_G/H)$. Hence $K/H$ is $F_{hq}$-supplemented in $G/H$. 
(c) Suppose that $E$ is $F_{hq}$-supplemented in $G$ and let $N$ be a quasinormal subgroup of $G$ such that $EN$ is a Hall subgroup of $G$ and $(E/E_G) \cap (N/E_G) \leq Z_F(G/E_G)$ by (a). Put $Z_F(G/E_G) = L/E_G$. We treat the following two cases:

Case 1. $H \leq N$.

Then $HEN = EHN = EN$ is a Hall subgroup of $G$ and so $HE \cap N = H(E \cap N) \leq HL$. Also

$$HL/HE_G = HE_GL/HE_G \cong L/(L \cap HE_G) = L/E_G(L \cap H)$$

so $HL/HE_G \leq Z_F(G/HE_G)$. Hence

$$(HE/HE_G) \cap (N/HE_G) = (HE \cap N)/HE_G = H(E \cap N)/HE_G \leq HL/HE_G \leq Z_F(G/HE_G).$$

By Lemma 2.1(a),


Then $(HE \cap N)(HE)_G/(HE)_G \leq Z_F(G/(HE)_G)$ and so $HE$ is $F_{hq}$-supplemented in $G$. Hence $HE/H$ is $F_{hq}$-supplemented in $G/H$, by (b).

Case 2. $H \nsubseteq N$.

Since $EN$ is a Hall subgroup of $G$ and $(|H|, |E|) = 1$, we have that $(HE/H)(NH/H) = HEN/H$ is a Hall subgroup of $G/H$ and since $(E \cap N)/E_G \leq L/E_G$, we have that $E \cap N \leq L$ and $(E \cap N)(HE)_G/(HE)_G \leq L(HE)_G/(HE)_G$. By Lemma 2.1(a),

$$((E \cap N)(HE)_G/E_G)/((HE)_G/E_G) \leq (L(HE)_G/E_G)/((HE)_G/E_G)$$

$$= Z_F(G/E_G)(HE)_G/E_G)/((HE)_G/E_G) \leq Z_F(G/E_G)/((HE)_G/E_G)).$$

So we have $(E \cap N)(HE)_G/(HE)_G \leq Z_F(G/(HE)_G)$. Since $(|H|, |E|) = 1$, we have that $(|HN : N|, |HN \cap E|) = 1$ and so $HN \cap E = N \cap E$. Hence

$$(HE/H \cap HN/H)(HE/H)_G/H/(HE/H)_G/H$$

$$= (H(E \cap N)/H)((HE)_G/H)/((HE)_G/H) \leq ((E \cap N)(HE)_G/H)/((HE)_G/H)$$

$$\leq (L(HE)_G/H)((HE)_G/H) \leq Z_F((G/H)/(HE)_G/H)).$$

Hence $HE/H$ is $F_{hq}$-supplemented in $G/H$.

(d) Suppose that $H$ is $F_{hq}$-supplemented in $G$ and let $N$ be a quasinormal subgroup of $G$ such that $HN$ is a Hall subgroup of $G$ and $(H/H_G) \cap (N/H_G) \leq Z_F(G/H_G)$ by (a). Put $M = K \cap N$. For every subgroup $L$ of $K$, $LM = L(K \cap N) = K \cap LN = K \cap NL = (K \cap N)L = ML$ as $N$ is quasinormal in $G$. Then $M$ is quasinormal in $K$. Since $HN$ is a
Hall subgroup of $G$ and $H \leq K$, we have that $HM$ is a Hall subgroup of $K$. Also $M/H_G \cap H/H_G = (K \cap N \cap H)/H_G = K/H_G \cap Z_\mathcal{F}(G/H_G)$. Put $K/H_G \cap Z_\mathcal{F}(G/H_G) = R/H_G$. Since $\mathcal{F}$ is $S$-closed, we have by Lemma 2.1(b), $R/H_G \leq Z_\mathcal{F}(K/H_G)$. By Lemma 2.1(a), $(R/H_G)(H_K/H_G)/(H_K/H_G) \leq Z_\mathcal{F}(K/H_G)(H_K/H_G)/(H_K/H_G)$ and so $(M \cap H)H_K/H_K \leq Z_\mathcal{F}(K/H_K)$. Hence $H$ is $\mathcal{F}_{hq}$-supplemented in $K$. \hfill \Box

**Lemma 2.3.** (a) An $S$-quasinormal subgroup of $G$ is subnormal in $G$.  
(b) If $H$ is $S$-quasinormal Hall subgroup of $G$, then $H \triangleleft G$.  
(c) Let $H$ be a $p$-subgroup of $G$. Then $H$ is $S$-quasinormal in $G$ if and only if $O^p(G) \leq N_G(H)$.  
(d) If $K$ is a normal subgroup of a group $G$ and $H$ is $S$-quasinormal in $G$, then $H \cap K$ is $S$-quasinormal in $G$.

**Proof.** (a) See ([9; Satz 1, p. 209]).  
(b) By (a), $H$ is subnormal in $G$. Hence $H$ is subnormal Hall subgroup of $G$. This implies that $H \triangleleft G$.  
(c) See ([11; Lemma A, p. 287]).  
(d) Since $H$ is $S$-quasinormal in $G$, it follows by (a), that $H$ is subnormal in $G$ and so $HK$ and $H \cap K$ are subnormal in $G$. Let $P$ be an arbitrary Sylow subgroup of $G$. Clearly $H \cap P \in Syl(H)$, $K \cap P \in Syl(K)$, $HK \cap P \in Syl(HK)$ and $H \cap K \cap P \in Syl(H \cap K)$. Then $|(H \cap P)(K \cap P)| = \frac{|H \cap P||K \cap P|}{|H \cap K \cap P|} = |HK \cap P|$ and since $(H \cap P)(K \cap P) \leq HK \cap P$, it follows that $(H \cap P)(K \cap P) = HK \cap P$. Hence by [5; Lemma 1.2, p. 2], $(H \cap K)P = HP \cap KP$ and so $H \cap K$ is $S$-quasinormal in $G$. \hfill \Box

**Lemma 2.4.** (a) Let $p$ be the smallest prime dividing the order of $G$, and let $G_p$ be a Sylow $p$-subgroup of $G$. If $\Omega(G_p) \leq genz_\infty(G)$, then $G$ is $p$-nilpotent.  
(b) Let $P$ be a normal $p$-subgroup of $G$ such that $G/P$ is supersolvable. If $\Omega(P) \leq genz_\infty(G)$, then $G$ is supersolvable.

**Proof.** See ([3; Lemma 3.8 and Theorem 3.11, p. 2245-2246]). \hfill \Box

**Lemma 2.5.** Let $\mathcal{F}$ be a saturated formation containing $\Xi$ and let $K$ be a normal subgroup $K$ of $G$ such that $G/K \in \mathcal{F}$ and the cyclic subgroups of $K$ of prime order or order $4$ are $S$-quasinormal in $G$, then $G \in \mathcal{F}$.

**Proof.** See ([2; Theorem 1, p.2773]). \hfill \Box

**Lemma 2.6.** Let $K$ be a normal subgroup $K$ of a group $G$ with $G/K$ contained in a saturated formation $\mathcal{F}$. If $\Omega(P) \leq Z_\mathcal{F}(G)$, where $P$ is a Sylow $p$-subgroup of $K$, then $G/O_p'(K) \in \mathcal{F}$.

**Proof.** See ([4]). \hfill \Box
3. RESULTS

We proceed now to the first main results.

**Lemma 3.1.** Let $p$ be the smallest prime dividing the order of $G$ and let the cyclic subgroup of $G$ of order $p$ or 4 be $\mathfrak{U}_{hq}$-supplemented in $G$. Then $G$ is $p$-nilpotent.

*Proof.* Suppose the result is false and let $G$ be a counter-example of minimal order. Suppose that $G_p$ is a Sylow $p$-subgroup of $G$. Then we have:

1. $O_p'(G) = 1$.

If not, then by Lemma 2.2(c), it is easy to see that the cyclic subgroup of $G_pO_p'(G)/O_p'(G)$ of order $p$ or order 4 is $\mathfrak{U}_{hq}$-supplemented in $G/O_p'(G)$. The minimality of $G$ implies that $G/O_p'(G)$ is $p$-nilpotent and hence $G$ is $p$-nilpotent; a contradiction.

2. $p = 2$.

Suppose $p > 2$. If the cyclic subgroup of $G$ of order $p$ is normal of $G$, then $\Omega_1(G_p) \leq Z_\U(G) \leq \text{genz}_\infty(G)$. Hence by Lemma 2.4(a), $G$ is $p$-nilpotent; a contradiction. Thus we may assume that there exists a subgroup $H$ of $G$ of order $p$ such that $H$ is not normal in $G$. By hypothesis, $H$ is $\mathfrak{U}_{hq}$-supplemented in $G$. By Lemma 2.2(a), $G$ has a quasinormal subgroup $N$ of $G$ such that $HN$ is a Hall subgroup of $G$ and $H/HG \cap N/HG \leq Z_\U(G/HG)$. Since $H$ is not normal in $G$, we have that $H_G = 1$ and so $H \cap N \leq Z_\U(G)$. If $H \cap N = H \leq Z_\U(G)$, then $\Omega_1(G_p) \leq Z_\U(G) \leq \text{genz}_\infty(G)$. Hence by Lemma 2.4(a), $G$ is $p$-nilpotent; a contradiction. Thus $H \cap N = 1$. Clearly $N < G$. Since $N$ is a quasinormal subgroup of $G$, we have by Lemma 2.3(a), that $N$ is a subnormal in $G$ and since $N < G$, we have a normal subgroup $M$ of $G$ containing $N$. If $N$ is not $p$-subgroup of $G$, then also $M$ is not $p$-subgroup of $G$. Since the class of supersolvable group is $S$-closed, we have by Lemma 2.2(d), that the cyclic subgroup of $M$ of order $p$ is $\mathfrak{U}_{hq}$-supplemented in $M$. Then $M$ is $p$-nilpotent, by the minimality of $G$. Hence $O_p'(M) \neq 1$. Since $M$ is a normal subgroup of $G$, we have that, $1 < O_p'(M) \leq O_p'(G)$; a contradiction. Thus $N$ is $p$-subgroup of $G$. Since $HN$ is a Hall subgroup of $G$, we have that $HN = G_p$ and since $H \cap N = 1$, we have $N$ is a maximal subgroup of $G_p$. So $N \triangleleft G_p$. Let $G_q$ be an arbitrary Sylow $q$-subgroup of $G$, with $q > p$. Since $N$ is a quasinormal subgroup in $G$, we have that $NG_q \leq G$ and so $N$ is a quasinormal Hall in $NG_q$. Then by Lemma 2.3(b), $N \triangleleft NG_q$ i.e., $G_q \leq N_G(N)$. Thus $O^p(G) \leq N_G(N)$ and since $N \triangleleft G_p$, we have that $N \triangleleft G$. Now consider the group $G/N$. Clearly $G_p/N$ is a Sylow $p$-subgroup of $G/N$ of order $p$. By Burnside’s theorem, $G/N$ is $p$-nilpotent. Then $G/N$ has a normal Hall $p$-subgroup $K/N$ and so $K$ is a proper normal subgroup of $G$. Since the class of supersolvable groups is $S$-closed, we have by Lemma 2.2(d), that the cyclic subgroups of $K$ of order
is $\mathsf{U}_{hq}$-supplemented in $K$. Then $K$ is $p$-nilpotent, by the minimality of $G$. Hence $O_{p'}(K) \neq 1$. So $1 < O_{p'}(K) \leq O_{p'}(G)$; a contradiction. 

(3) Final contradiction.

If the cyclic subgroup of $G$ of order 2 or 4 is normal in $G$, then $\Omega_2(G) \leq Z_\mathsf{U}(G) \leq \operatorname{genz}_\infty(G)$. Hence by Lemma 2.4(a), $G$ is $p$-nilpotent; a contradiction. Thus we may assume that there exists a subgroup $H$ of $G$ of order 2 or 4 such that $H$ is not normal in $G$. If $|H| = 2$, then $H \cap G = 1$. By hypothesis, $H$ is $\mathsf{U}_{hq}$-supplemented in $G$. By Lemma 2.2(a), $G$ has a quasinormal subgroup $N$ of $G$ such that $HN$ is a Hall subgroup of $G$ and $H \cap N = Z_\mathsf{U}(G)$. If $H \cap N = 1$, then $O_{p'}(G) \neq 1$, by repeating the proof of (2). Thus $H \cap N = H \leq Z_\mathsf{U}(G)$ and so $\Omega_1(G_p) \leq Z_\mathsf{U}(G)$. If the cyclic subgroup of $G$ of order 4 is normal in $G$, then $\Omega_2(G) \leq Z_\mathsf{U}(G) \leq \operatorname{genz}_\infty(G)$. Hence by Lemma 2.4(a), $G$ is $p$-nilpotent; a contradiction. Thus there exists a subgroup $L$ of $G$ of order 4 such that $L$ is not normal in $G$. By hypothesis, $L$ is $\mathsf{U}_{hq}$-supplemented in $G$. By Lemma 2.2(a), $G$ has a quasinormal subgroup $T$ of $G$ such that $LT$ is a Hall subgroup of $G$ and $L/L_G \cap T/L_G \leq Z_\mathsf{U}(G/L_G)$. Since $L$ is not normal in $G$, we have that $|L_G| = 2$ or 1. If $|L_G| = 2$, then $L_G \leq Z_\mathsf{U}(G)$. Hence $Z_\mathsf{U}(G/L_G) = Z_\mathsf{U}(G)/L_G$. So $L \cap T \leq Z_\mathsf{U}(G)$. Also if $|L_G| = 1$, then $L \cap T \leq Z_\mathsf{U}(G)$. If $L = L \cap T \leq Z_\mathsf{U}(G)$, then $\Omega_2(G) \leq Z_\mathsf{U}(G) \leq \operatorname{genz}_\infty(G)$. Hence by Lemma 2.4(a), $G$ is $p$-nilpotent; a contradiction. Thus $L \cap T$ is a proper subgroup of $L$. So $T$ is also a proper subgroup of $G$. If $T$ is not $p$-subgroup of $G$, then $O_{p'}(G) \neq 1$ by repeating the proof of (2); a contradiction. Thus $T$ is $p$-subgroup of $G$. Since $LT$ is a Hall subgroup of $G$, we have that $LT = G_p$. Since $L \cap T$ is a proper subgroup of $L$, we have that $T$ is a proper subgroup of $G_p$ and since $T$ is quasinormal subgroup in $G$, we have by Lemma 2.3(c), that $O^p(G) \leq N_G(T)$. If $N_G(T) = G$, then $T \leq G$. Since $G_p/T$ is cyclic, we have by Burnside’s theorem, $G/T$ is $p$-nilpotent. By repeating the proof of (2), we have that $1 < O_{p'}(G)$; a contradiction. Thus we may assume that $O_p(G) \leq N_G(T) < G$. Since the class of supersolvable groups is $S$-closed, we have by Lemma 2.2(d), that the cyclic subgroups of $O^p(G)$ of order 2 or 4 is $\mathsf{U}_{hq}$-supplemented in $O^p(G)$. By the minimality of $G$, $O^p(G)$ is $p$-nilpotent and also $G$; a final contradiction. \[ \square \]

As a consequence, we also obtain improvement of Corollary 4.2 in [9].

**Theorem 3.2.** If the cyclic subgroups of $G$ of prime order or order 4 are $\mathsf{U}_{hq}$-supplemented in $G$, then $G$ is supersolvable.

**Proof.** Suppose the result is false and let $G$ be a counter-example of minimal order. Lemma 3.1 implies that $G$ is $r$-nilpotent, where $r$ is the smallest prime dividing the order of $G$. Then $G = G_r K$, where $G_r$ is a Sylow $r$-subgroup of $G$ and $K$ is a normal Hall $r'$-subgroup of $G$. Since the class of supersolvable
is $S$-closed, we have by Lemma 2.2(d), the hypothesis of the theorem satisfies over $K$. Then $K$ is supersolvable by the minimality of $G$. Hence $K$ has a characteristic Sylow $q$-subgroup $G_q$ and $q$ is the largest prime dividing the order of $K$. Since $K \triangleleft G$, we have that $G_q \triangleleft G$ and since $K$ is a Hall $r'$-subgroup of $G$, we have $G_q$ is a Sylow $q$-subgroup of $G$. Now consider the factor group $G/G_q$. By Lemma 2.2(c), the hypothesis satisfies $G/G_q$. Then $G/G_q$ is supersolvable by the minimality of $G$. Thus $G^{\Omega} \leq G_q$, where $G^{\Omega}$ is supersolvable residual of $G$. If the cyclic subgroups of $G^{\Omega}$ of order $q$ are normal in $G$, then $\Omega_1(G^{\Omega}) \leq Z_{\Omega}(G) \leq genz_{\infty}(G)$. Hence by Lemma 2.4(b), $G$ is supersolvable; a contradiction. Thus there exists a subgroup $H$ of $G^{\Omega}$ of order $q$ is not normal in $G$. By hypothesis, $H$ is $\Omega_q$-supplemented in $G$. By Lemma 2.2(a), $G$ has a quasinormal subgroup $N$ of $G$ such that $HN$ is a Hall subgroup of $G$ and $H/HN \cap N/HG \leq Z_{\Omega}(G/HG)$. Since $H$ is not normal in $G$, we have that $H_G = 1$ and so $H \cap N \leq Z_{\Omega}(G)$. If $H \leq N$, then $H = H \cap N \leq Z_{\Omega}(G)$. Hence $\Omega_1(G^{\Omega}) \leq Z_{\Omega}(G) \leq genz_{\infty}(G)$. Hence by Lemma 2.4(b), $G$ is supersolvable; a contradiction. Thus $H \cap N = 1$. Since $HN$ is a Hall subgroup of $G$, we have that $G_q \leq HN$ and $N \cap G_q$ is a maximal subgroup of $G_q$. Since $N$ is quasinormal in $G$, we have that $N$ is $S$-quasinormal in $G$ and since $G_q$ is a normal subgroup of $G$, we have by Lemma 2.3(d), that $N \cap G_q$ is $S$-quasinormal in $G$. Then $O^q(G) \leq N_G(N \cap G_q)$ by Lemma 2.3(c) and since $N \cap G_q$ is a maximal subgroup of $G_q$, we have that $G_q \leq N_G(N \cap G_q)$. Then $G = G_qO^q(G) \leq N_G(N \cap G_q)$. Hence $N \cap G_q \triangleleft G$. Since $G/G_q$ is supersolvable, we have that $(G/(N \cap G_q))/(G_q/(N \cap G_q)) \cong G/G_q$ is supersolvable and since $G_q/(N \cap G_q)$ is cyclic of order $q$, we have that $G/(N \cap G_q)$ is supersolvable. Then $G^{\Omega} \leq N \cap G_q$. Hence $H \leq N \cap G_q \leq N$; a final contradiction. \hfill $\square$

Now we prove an extension of Corollary 4.3 in [10].

**Theorem 3.3.** Let $F$ be an $S$-closed saturated formation containing $\Omega$. Then $G \in F$ if and only if there exists a normal subgroup $K$ of $G$ such that $G/K \in F$ and the cyclic subgroups of $K$ of prime order or order $4$ are $\Omega_{hq}$-supplemented in $G$.

**Proof.** If $G \in F$, then the result holds with $K = 1$. \hfill $\square$

The converse, suppose the result is false and let $G$ be a counter-example of minimal order. Since $F$ is $S$-closed, we have by Lemma 2.2(d), the cyclic subgroups of $K$ of prime order or order $4$ are $\Omega_{hq}$-supplemented in $K$. Then $K$ is supersolvable by Theorem 3.2. Hence $K$ has a characteristic Sylow $p$-subgroup $K_p$, where $p$ is the largest prime dividing the order of $K$. Since $K \triangleleft G$, we have that $K_p \triangleleft G$. Then $(G/K_p)/(K/K_p) \cong G/K \in F$. By hypothesis and Lemma 2.2(c), the cyclic subgroups of $K/K_p$ of prime order or
order 4 are $\U_hq$-supplemented in $G/K_p$. Then $G/K_p \in \mathcal{F}$ by the minimality of $G$. If $p = 2$, then $K_pG_q \leq G$, for every Sylow $q$-subgroup $G_p$ of $G$ with $q > 2$. By Lemma 3.1, $K_pG_q$ is 2-nilpotent. Then $K_pG_q = K_p \times G_q$. Since $K_p$ is a normal 2-subgroup of $G$, we have that the cyclic subgroups of $K_p$ of order 2 or 4 are $S$-quasinormal in $G$. Then $G \in \mathcal{F}$ by Lemma 2.5; a contradiction. Thus we may assume that $p > 2$. Clearly $1 \neq G^F \leq K_p$ as $G/K_p \in \mathcal{F}$. If the subgroups of $G^F$ of order $p$ lay in $Z_F(G)$, then $G \in \mathcal{F}$ by Lemma 2.6; a contradiction. Thus there exists a subgroup $H$ of $G^F$ of order $p$ such that $H \not\leq Z_F(G)$. Then $H \not\leq Z_U(G)$ as $U \subseteq \mathcal{F}$. Hence $H_G = 1$. By hypothesis $H$ is $\U$-supplemented in $G$. By Lemma 2.1(a), $G$ has a quasinormal subgroup $N$ of $G$ such that $HN$ is a Hall subgroup of $G$ and $H \cap N \leq Z_G(G)$. Since $HN$ is a Hall subgroup of $G$, we have that $G^F \leq G_p \leq HN$, where $G_p$ is a Sylow $p$-subgroup of $G$. Also since $H \not\leq Z_U(G)$, we have that $H \cap N = 1$. Then $G^F \cap N$ is a maximal $p$-subgroup of $G^F$. Since $N$ is a quasinormal subgroup of $G$, we have that $N$ is $S$-quasinormal subgroup of $G$ and since $G^F \not\leq G$, we have by Lemma 2.3(d), that $G^F \cap N$ is $S$-quasinormal subgroup of $G$. Then $O^p(G) \leq N_G(G^F \cap N)$. Clearly $G^F \cap N \vartriangleleft N$ and since $G^F \cap N$ is a maximal subgroup of $G^F$, we have that $G^F \cap N \vartriangleleft G^F$. Then $G_p \leq HN = G^F N \leq N_G(G^F \cap N).$ Hence $G = G_pO^p(G) \leq N_G(G^F \cap N)$ i.e., $G^F \cap N \vartriangleleft G$. Since $(G/G^F \cap N)/(G^F/(G^F \cap N) \cong G/G^F \in \mathcal{F}$ and since $G^F/(G^F \cap N)$ is cyclic of order $p$, we have that $G^F/(G^F \cap N) \leq Z_F(G/(G^F \cap N))$. Hence $G/G^F \cap N \in \mathcal{F}$ and so $H \leq G^F \leq G^F \cap N \leq N$; a final contradiction.

Acknowledgements. This paper was supported by the Deanship of Scientific Research of Northern Border University under grant 8-087-435.

REFERENCES


Received 29 November 2016

Northern Border University,
Faculty of Arts and Science,
Rafha, P.O. 840,
Saudi Arabia

King Abdulaziz University,
Faculty of Science,
Mathematics Department,
P. O. Box 80203,
Jeddah 21589,
Saudi Arabia