UNIQUENESS OF MEROMORPHIC FUNCTIONS SATISFYING A NON-LINEAR POLYNOMIAL DIFFERENTIAL EQUATION

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In this paper, with the notion of weighted sharing of values we study the uniqueness of meromorphic functions concerning general non-linear differential polynomials sharing a non-zero polynomial with certain degree. The results of the paper improve and generalise the recent results due to Cao and Zhang [7].

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1. INTRODUCTION AND PRELIMINARY RESULTS

A function $f(z)$ is called meromorphic if it is analytic in the complex plane $\mathbb{C}$ except at possible isolated poles. If no poles occur, then $f(z)$ reduces to an entire function.

In this paper, it is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna value distribution theory of meromorphic functions. For a meromorphic function $f(z)$ in the complex plane $\mathbb{C}$, we shall use the following standard notations of the value distribution theory:

$$T(r,f),\ m(r,f),\ N(r,f),\ \overline{N}(r,f),\ldots$$

(see, *e.g.*, [11, 24]). We adopt the standard notation $S(r,f)$ for any quantity satisfying the relation $S(r,f)=o(T(r,f))$ as $r\to\infty$ except possibly a set of finite linear measure.

We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\longrightarrow\infty$, outside of a possible exceptional set of finite linear measure.

Let $p\in\mathbb{N}$ and $a\in\mathbb{C}\cup\{\infty\}$. We use the notation $N(r,a;f\mid\geq p)$ ($\overline{N}(r,a;f\mid\geq p)$) to denote the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p. Again we use the notation $N(r,a;f\mid\leq p)$ ($\overline{N}(r,a;f\mid\leq p)$) to denote the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

For $a \in \mathbb{C} \cup \{\infty\}$ and $p \in \mathbb{N}$ we denote by $N_p(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots + \overline{N}(r, a; f \mid\geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

A meromorphic function a is said to be a small function of f if $T(r, a) = S(r, f)$, *i.e.*, if $T(r, a) = o(T(r, f))$ as $r \to \infty$ except possibly a set of finite linear measure.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. If $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities then we say that $f(z)$ and $g(z)$ share $a(z)$ with CM (counting multiplicities) and if we do not consider the multiplicities then we say that $f(z)$ and $g(z)$ share $a(z)$ with IM (ignoring multiplicities).

A finite value z_0 is called a fixed point of f if $f(z_0) = z_0$ or z_0 is a zero of $f(z) - z$.

In 1995, both W. Bergweiler and A. Eremenko (see [5], Theorem 2), H. H. Chen and M. L. Fang (see [6], Theorem 1) respectively, proved the following result.

Theorem A. *Let $f(z)$ be a transcendental meromorphic function and $n \in \mathbb{N}$. Then $f^n f' = 1$ has infinitely many solution.*

Corresponding to Theorem A, both Fang and Hua [9], Yang and Hua [23] obtained the following results.

Theorem B. *Let f and g be two non-constant entire (meromorphic) functions and let $n \in \mathbb{N}$ such that $n \geq 6$ ($n \geq 11$). If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $4(c_1 c_2)^{n+1} c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

To investigate the uniqueness result of entire or meromorphic functions having fixed points, Fang and Qiu [10] obtained the following result.

Theorem C. *Let f and g be two non-constant meromorphic (entire) functions and let $n \in \mathbb{N}$ such that $n \geq 11$ ($n \geq 6$). If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1 c_2)^{n+1} c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

During the last couple of years, a significant number of authors worked on the uniqueness problem of meromorphic functions when the non-linear differential polynomials generated by them share certain values or small functions (see [3, 4, 7–10, 16–21, 23, 26, 28, 29]).

It is instinctive to ask what happens if the first derivative f' in Theorem A is replaced by the general derivative $f^{(k)}$. By considering this problem, Xu et al. [21] or Zhang and Li [27] respectively, proved the following result.

THEOREM D. *Let f be a transcendental meromorphic function and let $k,\ n \in \mathbb{N}$ such that $n \geq 2$. Then $f^n f^{(k)}$ takes every finite non-zero value infinitely many times or has infinitely many fixed points.*

Regarding Theorem D, Y.H. Cao and X.B. Zhang [7] obtained the following results in 2012.

THEOREM E. *Let f and g be two transcendental meromorphic functions, whose zeros be of multiplicities at least k, where $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that $n > \max\{2k-1, k+\frac{4}{k}+4\}$. If $f^n f^{(k)} - z$ and $g^n g^{(k)} - z$ share 0 CM, f and g share ∞ IM, one of the following two conclusions holds:*

(i) $f^n f^{(k)} \equiv g^n g^{(k)}$;

(ii) $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, *where c_1, c_2 and c are constants such that* $4(c_1c_2)^{n+1}c^2 = -1$.

THEOREM F. *Let f and g be two non-constant meromorphic functions, whose zeros be of multiplicities at least k, where $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that $n > \max\{2k-1, k+\frac{4}{k}+4\}$. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share 1 CM, f and g share ∞ IM, one of the following two conclusions holds:*

(i) $f^n f^{(k)} \equiv g^n g^{(k)}$;

(ii) $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, *where c_3, c_4 and d are constants such that* $(-1)^k (c_3c_4)^{n+1} d^{2k} = 1$.

THEOREM G. *Let f and g be two non-constant meromorphic functions, whose zeros be of multiplicities at least $k+1$, where $k \in \mathbb{N}$ such that $1 \leq k \leq 5$. Let $n \in \mathbb{N}$ such that $n \geq 10$. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share 1 CM, $f^{(k)}$ and $g^{(k)}$ share 0 CM, f and g share ∞ IM, one of the following two conclusions holds:*

(i) $f \equiv tg$, *where t is a constant such that $t^{n+1} = 1$;*

(ii) $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, *where c_3, c_4 and d are constants such that* $(-1)^k (c_3c_4)^{n+1} d^{2k} = 1$.

Theorems E, F and G suggest the following questions as open problems.

Question 1. Can the lower bound of n be further reduced in Theorems E, F and G?

Question 2. Can the condition "Let f and g be two non-constant meromorphic functions, whose zeros be of multiplicities at least $k+1$, where $k \in \mathbb{N}$ such that $1 \leq k \leq 5$" in Theorem G be further weakened?

Question 3. Can one deduce generalized results in which Theorems E, F and G will be included?

2. MAIN RESULTS AND SOME DEFINITIONS

Throughout this paper, we always use $P(z)$ to denote an arbitrary non-constant polynomial of degree $n \in \mathbb{N}$ as follows,

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 = a_n (z - c_1)^{d_1} (z - c_2)^{d_2} \ldots (z - c_{s_1})^{d_{s_1}},$$

where $a_i (i = 0, 1, \ldots, n-1)$, $a_n \neq 0$ and $c_j (j = 1, 2, \ldots, s_1)$ are distinct finite complex numbers; $d_1, d_2, \ldots, d_{s_1} \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ such that

$$\sum_{i=1}^{s_1} d_i = n.$$

Let $d = \max\{d_1, d_2, \ldots, d_{s_1}\}$ and c be the corresponding zero of $P(z)$ of multiplicity d. We set an arbitrary non-zero polynomial $P_1(z)$ by

$$P_1(z) = a_n \prod_{\substack{i=1 \\ d_i \neq d}}^{s_1} (z - c_i)^{d_i} = b_{m_1} z^{m_1} + b_{m_1 - 1} z^{m_1 - 1} + \ldots + b_0,$$

where $a_n = b_{m_1}$ and $m_1 = n - d$. Obviously $P(z) = (z - c)^d P_1(z)$. We also use $P_2(z_1)$ as an arbitrary non-zero polynomial defined by

$$P_2(z_1) = a_n \prod_{\substack{i=1 \\ d_i \neq d}}^{s_1} (z_1 + c - c_i)^{d_i} = e_{m_1} z_1^{m_1} + e_{m_1 - 1} z_1^{m_1 - 1} + \ldots + e_0,$$

where $z_1 = z - c$ and $\deg(P_2(z_1)) = m_1 \geq 0$. Obviously

$$P(z) = z_1^d P_2(z_1). \tag{2.1}$$

Suppose $\Gamma_1 = m_2 + m_3$ and $\Gamma_2 = m_2 + 2m_3$, where m_2 is the number of simple zeros of $P_1(z)$ and m_3 is the number of multiple zeros of $P_1(z)$.

We define $k^* \in \mathbb{N}$ as follows

$$k^* = \begin{cases} k, & if \;\; P_2(z_1) = e_i z_1^i \not\equiv 0 \\ k+1, & if \;\; P_2(z_1) \neq e_i z_1^i \not\equiv 0, \end{cases} \tag{2.2}$$

for $i \in \{0, 1, 2, \ldots, m_1\}$. Again we use $p(z)$ to denote a non-zero polynomial such that either $\deg(p) \leq n + m - 1$ or zeros of $p(z)$ are of multiplicities at most $n - 1$, where $m \in \mathbb{N}$, *i.e.*,

$$p(z) = d_n (z - z_1)^{l_1} (z - z_2)^{l_2} \ldots (z - z_t)^{l_t}, \tag{2.3}$$

where $d_n \neq 0$, $z_i (i = 1, 2, \ldots, t)$ are distinct complex numbers and $l_1, l_2, \ldots, l_t \in \mathbb{N} \cup \{0\}$. Here we see that either $\sum_{i=1}^{t} l_i \leq n + m - 1$ or $l_i \leq n - 1$ for all $i = 1, 2, \ldots, t$.

Before going to our main results, we need the following definition of weighted sharing.

Definition 2.1 ([13, 14]). Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k. We write f, g share (a, k) to mean that f, g share the value a with weight k.

Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively. If a is a small function we define that f and g share (a, l) if $f - a$ and $g - a$ share $(0, l)$.

In this paper, taking the possible answers of the above questions into background we obtain the following results which significantly improve and generalize Theorems E, F and G.

THEOREM 2.1. *Let f be a transcendental meromorphic function, such that zeros of $f - c$ be of multiplicities at least k^*, where k^* be defined as in (2.2) and m, $n \in \mathbb{N}$. Let $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f. If $n > m + \Gamma_1 + \frac{1}{k^*}$, then $P(f)(f^{(k)})^m - a$ has infinitely many zeros, where $P(z)$ is defined as in (2.1).*

THEOREM 2.2. *Let f, g be two transcendental meromorphic functions, such that zeros of $f - c$ and $g - c$ be of multiplicities at least k, where $k \in \mathbb{N}$. Let $P(z)$ and $p(z)$ be defined as in (2.1) and (2.3) respectively and m, $n \in \mathbb{N}$ such that*

$$n \geq k + 2(m + 2\Gamma_2) + \frac{3k + 7}{2k}.$$

If $P(f)(f^{(k)})^m - p$, $P(g)(g^{(k)})^m - p$ share $(0, k_1)$, where $k_1 = \left[\frac{m(k-1)+3}{n+m+(m-2)k-1}\right] + 3$ and f, g share $(\infty, 0)$, then one of the following cases holds:

(1) *$f - c \equiv t(g - c)$ for a constant t such that $t^s = 1$, where $s = GCD(n + m, \ldots, n + m - i, \ldots, m)$, $e_{m_1 - i} \neq 0$ for some $i \in \{0, 1, \ldots, m_1\}$,*
(2) *$P(f)(f^{(k)})^m \equiv P(g)(g^{(k)})^m$.*

THEOREM 2.3. *Let f, g be two transcendental meromorphic functions, such that zeros of $f - c$ and $g - c$ be of multiplicities at least k^*, where k^* be defined as in (2.2). Let $P(z)$ and $p(z)$ be defined as in (2.1) and (2.3)* *respectively and* $m,\ n\in\mathbb{N}$ *such that*

$$n\geq k+2(m+2\Gamma_2)+\frac{3k^*+7}{2k^*}.$$

If $P(f)(f^{(k)})^m-p$, $P(g)(g^{(k)})^m-p$ *share* $(0,k_1)$, *where* $k_1=\left[\frac{m(k-1)+3}{n+m+(m-2)k-1}\right]+3$, $f^{(k)}$, $g^{(k)}$ *share* $(0,\infty)$ *and* f, g *share* $(\infty,0)$, *then one of the following cases holds:*

(1) *If* $P_2(z_1)\equiv e_iz_1^i\not\equiv 0$, *for some* $i\in\{0,1,2,\ldots,m_1\}$, *then* $f-c\equiv t(g-c)$, *where* t *is a constant such that* $t^{d+m+i}=1$, *for some* $i\in\{0,1,2,\ldots,m_1\}$.

(2) *If* $P_2(z_1)\not\equiv e_iz_1^i\not\equiv 0$, *for* $i\in\{0,1,2,\ldots,m_1\}$ *and* f, g *share* $(c,0)$, *then* $f-c\equiv t(g-c)$ *for a constant* t *such that* $t^s=1$, *where* $s=GCD(n+m,\ldots,n+m-i,\ldots,1)$, $e_{m_1-i}\neq 0$ *for some* $i=0,1,2,\ldots,m_1-1$.

Remark 2.1. The results of the paper significantly rectify Theorems E, F and G in the following direction: "Conclusion (ii) does not occur in Theorems E, F and G". Actually in the statements of Theorems E, F and G, it is assumed that both f and g have zeros of multiplicities at least $k(\geq 1)$, but in the conclusion (ii) of Theorems E, F and G, we see that multiplicities of zeros of both f and g are equal to $k=0$ as they have no zeros.

Remark 2.2. The results of the paper, generalise Theorems E, F and G in different directions. For examples, we consider $P(f)$ instead of f^n and $(f^{(k)})^m$ instead of $f^{(k)}$ in Theorems E, F and G.

Remark 2.3. Theorem 2.3 improves Theorem G in the following direction: Theorem 2.3 holds for $k\geq 1$, but Theorem G holds for $1\leq k\leq 5$.

Remark 2.4. Let us take $d=n$, $c=0$, $P_2(z_1)=1$ and $m=1$. Then from Theorem 2.2 we can easily get a theorem which is the improvement of Theorems E and F.

Remark 2.5. Let us take $d=n$, $c=0$, $P_2(z_1)=1$ and $m=1$. Clearly $k^*=k$. Then from Theorem 2.3 we can easily get a theorem which is the improvement of Theorem G. Consequently, Theorem G holds when zeros of f and g are of multiplicities at least k, where $k\in\mathbb{N}$.

Remark 2.6. It is easy to see that the conditions "$f^{(k)}$, $g^{(k)}$ share $(0,\infty)$" and "f, g share $(c,0)$" in Theorem 2.3 are sharp by the following example.

Example 2.1. Let $P(z)=z^{n-1}\big((n+1)z-n\big)$, $f=\frac{1-h^n}{1-h^{n+1}}$ and $g=h\frac{1-h^n}{1-h^{n+1}}$, where $h(z)=e^z-1$ and $n\in\mathbb{N}$ with $n\geq 10$. Observe that f and g share (∞,∞) but f and g do not share 0. Note that

$$f'=\frac{h'h^{n-1}\big((n+1)h-h^{n+1}-n\big)}{(1-h^{n+1})^2}\ \text{and}\ g'=\frac{h'\big(1+nh^{n+1}-(n+1)h^n\big)}{(1-h^{n+1})^2}.$$

Clearly f' and g' do not share 0. Clearly $f^n(f-1) \equiv g^n(g-1)$, *i.e.*, $P(f)f' \equiv P(g)g'$. Therefore $P(f)f'$ and $P(g)g'$ share $(1,\infty)$, but $f \not\equiv tg$, $t \in \mathbb{C} \setminus \{0\}$.

Remark 2.7. The above example also shows that the conclusion (2) in Theorem 2.2 cannot be removed.

We now introduce the following definitions and notations which are necessary in the paper.

Definition 2.2. We denote by $\overline{N}(r,a;f \mid= k)$ the reduced counting function of those a-points of f whose multiplicities exactly $k \in \mathbb{N}$, where $k \geq 2$.

Definition 2.3 ([1]). Let f and g be two non-constant meromorphic functions such that f and g share 1 IM. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g where $p > q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way, we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$.

Definition 2.4. *([14]).* Let f, g share a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

3. LEMMAS

Let F, G be two non-constant meromorphic functions. Henceforth, we shall denote by H and V the following two functions

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \tag{3.1}$$

and

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}. \tag{3.2}$$

Lemma 3.1 ([30]). *Let f be a non-constant meromorphic function and k, $p \in \mathbb{N}$. Then*

$$N_p(r,0;f^{(k)}) \leq N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 3.2 ([15]). *If $N(r,0;f^{(k)}\mid f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)}\mid f\neq 0)\leq k\overline{N}(r,\infty;f)+N_k(r,0;f)+S(r,f).$$

Lemma 3.3 ([22]). *Let f be a non-constant meromorphic function and $P(f)=a_0+a_1f+a_2f^2+\ldots+a_nf^n$, where $a_0,a_1,a_2\ldots,a_n$ are constants and $a_n\neq 0$. Then $T(r,P(f))=nT(r,f)+O(1)$.*

Lemma 3.4 ([24], Theorem 1.24). *Let f be a non-constant meromorphic function and let $k\in\mathbb{N}$. Suppose that $f^{(k)}\not\equiv 0$, then*

$$N(r,0;f^{(k)})\leq N(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 3.5 ([24]). *Let f_j $(j=1,2,3)$ be meromorphic functions where f_1 be non-constant. Suppose that*

$$\sum_{j=1}^{3}f_j\equiv 1\quad and\quad \sum_{j=1}^{3}N(r,0;f_j)+2\sum_{j=1}^{3}\overline{N}(r,\infty;f_j)<(\lambda+o(1))T(r),$$

as $r\to+\infty$, $r\in I$, $\lambda<1$ and $T(r)=\max_{1\leq j\leq 3}T(r,f_j)$. Then $f_2\equiv 1$ or $f_3\equiv 1$.

Lemma 3.6 ([11,25]). *Let f be a non-constant meromorphic function and let $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r,a_i)=S(r,f)$, $i=1,2$. Then*

$$T(r,f)\leq\overline{N}(r,\infty;f)+\overline{N}(r,a_1;f)+\overline{N}(r,a_2;f)+S(r,f).$$

Lemma 3.7 ([11]). *Suppose that f is a non-constant meromorphic function, $k(\geq 2)$ is an integer. If*

$$N(r,\infty,f)+N(r,0;f)+N(r,0;f^{(k)})=S\big(r,\frac{f'}{f}\big),$$

then $f(z)=e^{az+b}$, where $a\neq 0$, b are constants.

Lemma 3.8. *Let f be a transcendental meromorphic function and k, m, n $\in\mathbb{N}$. Then $P(f)(f^{(k)})^m$ is non-constant, where $P(z)$ is defined as in (2.1).*

Proof. Let $\Psi=P(f)(f^{(k)})^m$. If possible, suppose Ψ is constant. Then $\overline{N}(r,0;P(f))=0$ and $\overline{N}(r,\infty;f)=0$. If $P_1(z)$ is a non-constant polynomial, by the Second fundamental theorem we arrive at a contradiction. Next we suppose $P(z)=a_n(z-c)^n$. Let $f_1=f-c$. Therefore $\Psi=a_nf_1^n(f_1^{(k)})^m$. Also we see that

$$\left(\frac{1}{f_1}\right)^{n+m} \equiv a_n \frac{(f_1^{(k)})^m}{f_1^m}\frac{1}{\Psi}.$$

Then by Lemma 3.3 we have

$$\begin{aligned}(n+m)\,T(r,f_1) &\leq T\Big(r,\frac{(f_1^{(k)})^m}{f_1^m}\Big)+T(r,\frac{1}{\Psi})+O(1)\\ &\leq N\Big(r,\infty;\frac{(f_1^{(k)})^m}{f_1^m}\Big)+S(r,f_1)\\ &\leq m\,[N_k(r,0;f_1)+k\,\overline{N}(r,\infty;f_1)]+S(r,f_1)\\ &\leq mk\,[\overline{N}(r,0;f_1)+\overline{N}(r,\infty;f_1)]+S(r,f_1)=S(r,f_1),\end{aligned}$$

which is impossible. Hence Ψ is non-constant. This completes the lemma. □

Lemma 3.9. *Let f, g be two transcendental meromorphic functions, whose zeros be of multiplicities at least k, where $k \in \mathbb{N}$ and let m, $n \in \mathbb{N}$. Suppose $P(f)(f^{(k)})^m - p$, $P(g)(g^{(k)})^m - p$ share $(0,\infty)$ and f, g share $(\infty,0)$, where $p(z)$ and $P(z)$ are defined as in (2.3) and (2.1) respectively. Then $P(f)(f^{(k)})^m P(g)(g^{(k)})^m \not\equiv p^2$.*

Proof. Suppose

$$P(f)(f^{(k)})^m P(g)(g^{(k)})^m \equiv p^2. \tag{3.3}$$

Since f and g share $(\infty,0)$, from (3.3) it follows that f and g are transcendental entire functions.

Suppose $P_1(z)$ is a non-constant polynomial. For the sake of simplicity, we may assume that $P_1(z)=a_n(z-c_{m_1})^{m_1}$, where $d+m_1=n$. Obviously $c \neq c_{m_1}$.

From (3.3) we see that

$$N(r,c;f)=O(\log r)\ \ \textit{and}\ \ N(r,c_{m_1};f)=O(\log r).$$

Now by the second fundamental theorem we have

$$\begin{aligned}T(r,f) \leq \overline{N}(r,c;f)+\overline{N}(r,c_{m_1};f)+\overline{N}(r,\infty;f)+S(r,f)\\ =O(\log r)+S(r,f)=S(r,f),\end{aligned}$$

which is impossible, since f is a transcendental entire function. Therefore $P(z)$ must be of the form $a_n(z-c)^n$ and so from (3.3) we have

$$a_n^2(f-c)^n(f^{(k)})^m(g-c)^n(g^{(k)})^m \equiv p^2.$$

Let $f_1 = f - c$, $g_1 = g - c$ and $p_1^2 = \frac{p^2}{a_n^2}$. Then from above we have

$$f_1^n(f_1^{(k)})^m g_1^n(g_1^{(k)})^m \equiv p_1^2. \tag{3.4}$$

We consider the following cases.

Case 1. Let $\deg(p_1) = l(\geq 1)$.

From (3.4), it follows that $N(r, 0; f_1) = O(\log r)$ and $N(r, 0; g_1) = O(\log r)$. Let

$$F_1 = \frac{f_1^n(f_1^{(k)})^m}{p_1} \quad \textit{and} \quad G_1 = \frac{g_1^n(g_1^{(k)})^m}{p_1}. \tag{3.5}$$

Clearly from (3.4) we get

$$F_1 G_1 \equiv 1. \tag{3.6}$$

If $F_1 \equiv d_1 G_1$, where d_1 is a non-zero constant, then F_1 is a constant, which is impossible by Lemma 3.8. Hence $F_1 \not\equiv d_1 G_1$. Let

$$\Phi = \frac{f_1^n(f_1^{(k)})^m - p_1}{g_1^n(g_1^{(k)})^m - p_1}. \tag{3.7}$$

Since f_1 and g_1 are transcendental entire functions, it follows that $f_1^n(f_1^{(k)})^m - p_1 \neq \infty$ and $g_1^n(g_1^{(k)})^m - p_1 \neq \infty$. Also since $f_1^n(f_1^{(k)})^m - p_1$ and $g_1^n(g_1^{(k)})^m - p_1$ share $(0, \infty)$, we deduce from (3.7) that

$$\Phi \equiv e^{\beta}, \tag{3.8}$$

where β is an entire function. Let $f_{11} = F_1$, $f_{12} = -e^{\beta} G_1$ and $f_{13} = e^{\beta}$. Here f_{11} is transcendental. Now from (3.7) and (3.8), we have

$$f_{11} + f_{12} + f_{13} \equiv 1.$$

Hence by Lemma 3.4 we get

$$\begin{aligned} \sum_{j=1}^{3} N(r, 0; f_{1j}) + 2\sum_{j=1}^{3} \overline{N}(r, \infty; f_{1j}) &\leq N(r, 0; F_1) + N(r, 0; e^{\beta} G_1) + O(\log r) \\ &\leq (\lambda + o(1))T(r), \end{aligned}$$

as $r \to +\infty$, $r \in I$, $\lambda < 1$ and $T(r) = \max_{1 \leq j \leq 3} T(r, f_{1j})$.

So by Lemma 3.5, we get either $e^{\beta} G_1 \equiv -1$ or $e^{\beta} \equiv 1$. But here the only possibility is that $e^{\beta} G_1 \equiv -1$, *i.e.*, $g_1^n(g_1^{(k)})^m \equiv -e^{-\beta} p_1$ and so from (3.4) we obtain $F_1 \equiv e^{\gamma_1} G_1$, *i.e.*,

$$f_1^n(f_1^{(k)})^m \equiv e^{\gamma_1} g_1^n(g_1^{(k)})^m,$$

where γ_1 is a non-constant entire function. Then from (3.4) we get

$$f_1^n(f_1^{(k)})^m \equiv d_2 e^{\frac{1}{2}\gamma_1} p_1 \text{ and } g_1^n(g_1^{(k)})^m \equiv d_2 e^{-\frac{1}{2}\gamma_1} p_1, \text{ where } d_2 = \pm 1. \tag{3.9}$$

This shows that $f_1^n(f_1^{(k)})^m$ and $g_1^n(g_1^{(k)})^m$ share $(0,\infty)$. Since $N(r,0;f_1) = O(\log r)$ and $N(r,0;g_1) = O(\log r)$, so we can take

$$f_1(z) = h_1(z)e^{\alpha(z)} \text{ and } g_1(z) = h_2(z)e^{\beta(z)}, \tag{3.10}$$

where h_1 and h_2 are non-constant polynomials and α, β are two non-constant entire functions.

We deduce from (3.4) and (3.10) that either both α and β are transcendental entire functions or both α and β are polynomials. Now we consider the following sub-cases.

Sub-case 1.1. Let $k \geq 2$.

First we suppose both α and β are transcendental entire functions.

Let $\alpha_1 = \alpha' + \frac{h_1'}{h_1}$ and $\beta_1 = \beta' + \frac{h_2'}{h_2}$. Clearly both α_1 and β_1 are transcendental. Note that

$$S(r,\alpha_1) = S\Big(r, \frac{f_1'}{f_1}\Big) \text{ and } S(r,\beta_1) = S\Big(r, \frac{g_1'}{g_1}\Big).$$

Moreover, we see that

$$N(r,0;f_1^n(f_1^{(k)})^m) \leq N(r,0;p_1^2) = O(\log r) \text{ and } N(r,0;g_1^n(g_1^{(k)})^m) \leq N(r,0;p_1^2) = O(\log r).$$

From these and using (3.10) we have

$$N(r,\infty;f_1) + N(r,0;f_1) + N(r,0;f_1^{(k)}) = S(r,\alpha_1) = S\Big(r, \frac{f_1'}{f_1}\Big), \tag{3.11}$$

$$N(r,\infty;g_1) + N(r,0;g_1) + N(r,0;g_1^{(k)}) = S(r,\beta_1) = S\Big(r, \frac{g_1'}{g_1}\Big). \tag{3.12}$$

Then from (3.11), (3.12) and Lemma 3.7 we must have $f_1(z) = e^{a_3 z + b_3}$ and $g_1(z) = e^{c_3 z + d_3}$ where $a_3 \neq 0$, b_3, $c_3 \neq 0$ and d_3 are constants. But this is impossible because zeros of f_1 and g_1 are of multiplicities at least k.

Next we suppose α and β are both polynomials.

Now from (3.4) we get $\alpha + \beta \equiv C \in \mathbb{C}$ *i.e.*, $\alpha' \equiv -\beta'$. Therefore $\deg(\alpha) = \deg(\beta)$.

We deduce from (3.10) that

$$f_1^n(f_1^{(k)})^m \equiv A_1 h_1^n[h_1(\alpha')^k + P_{k-1}(\alpha', h_1')]^m e^{(n+m)\alpha} \equiv p_1 e^{(n+m)\alpha}, \tag{3.13}$$

$$g_1^n(g_1^{(k)})^m \equiv B_1 h_2^n[h_2(\beta')^k + Q_{k-1}(\beta', h_2')]^m e^{(n+m)\beta} \equiv p_1 e^{(n+m)\beta}, \tag{3.14}$$

where A_1, B_1 are non-zero constants, $P_{k-1}(\alpha', h_1')$ and $Q_{k-1}(\beta', h_2')$ are differential polynomials in α', h_1' and β', h_2' respectively.

By virtue of polynomial p_1, from (3.13) and (3.14) we arrive at a contradiction.

Sub-case 1.2. Let $k = 1$.

Suppose that α and β are transcendental. Then from (3.4) and (3.10) we get

$$(h_1 h_2)^n \big(h_1\alpha' + h_1'\big)^m \big(h_2\beta' + h_2'\big)^m e^{(n+m)(\alpha+\beta)} \equiv p_1^2. \tag{3.15}$$

Let $\alpha + \beta = \gamma$ and $l = n + m$. From (3.15), it is clear that γ is not a constant. Now from (3.15) we get

$$(h_1 h_2)^n \big(h_1\alpha' + h_1'\big)^m \big(h_2(\gamma' - \alpha') + h_2'\big)^m e^{l\gamma} \equiv p_1^2. \tag{3.16}$$

We have $T(r, \gamma') = m(r, l\gamma') + O(1) = m(r, \frac{(e^{l\gamma})'}{e^{l\gamma}}) = S(r, e^{l\gamma})$. Thus from (3.16) we get

$$\begin{aligned} T(r, e^{l\gamma}) &\leq T\left(r, \frac{p_1^2}{(h_1h_2)^n\big(h_1\alpha' + h_1'\big)^m\big(h_2(\gamma' - \alpha') + h_2'\big)^m}\right) + O(1) \\ &\leq m\,T(r, \alpha') + m\,T(r, \gamma' - \alpha') + O(\log r) + O(1) \\ &\leq 2m\,T(r, \alpha') + S(r, \alpha') + S(r, e^{l\gamma}), \end{aligned}$$

which implies that $T(r, e^{l\gamma}) = O(T(r, \alpha'))$ and so $S(r, e^{l\gamma})$ can be replaced by $S(r, \alpha')$. Thus we get $T(r, \gamma') = S(r, \alpha')$ and so γ' is a small function with respect to α'. In view of (3.16) and by Lemma 3.6 we get

$$\begin{aligned} & T(r, \alpha') \\ \leq\ & \overline{N}(r, \infty; \alpha') + \overline{N}(r, 0; h_1\alpha' + h_1') + \overline{N}(r, 0; h_2(\gamma' - \alpha') + h_2') + S(r, \alpha') \\ \leq\ & O(\log r) + S(r, \alpha'), \end{aligned}$$

which shows that α' is a polynomial and so α is a polynomial. Similarly, we can prove that β is also a polynomial. This contradicts the fact that α and β are transcendental.

Thus α and β are both polynomials. Consequently from (3.4), we can conclude that $\alpha + \beta$ is a constant and so $\alpha' + \beta' \equiv 0$. We deduce from (3.4) that

$$f_1^n(f_1')^m \equiv h_1^n(h_1\alpha' + h_1')^m e^{(n+m)\alpha} \equiv p_1 e^{(n+m)\alpha}, \tag{3.17}$$

$$g_1^n(g_1')^m \equiv h_2^n(h_2\beta' + h_2')^m e^{(n+m)\beta} \equiv p_1 e^{(n+m)\beta}. \tag{3.18}$$

By virtue of the polynomial p_1, from (3.17) and (3.18) we arrive at a contradiction.

Case 2. Let p_1 be a non-zero constant, say b.

Then from (3.4) we get $f_1^n(f_1^{(k)})^m g_1^n(g_1^{(k)})^m \equiv (b/a_n)^2$, where f_1 and g_1 are transcendental entire functions. Clearly f_1 and g_1 have no zeros. But this is impossible because zeros of f_1 and g_1 are of multiplicities at least k. This completes the lemma. □

LEMMA 3.10 ([12]). *Let f and g be two non-constant meromorphic functions. Suppose that f and g share $(0,\infty)$ and (∞,∞), $f^{(k)}$ and $g^{(k)}$ share $(0,\infty)$ for $k = 1, 2, \ldots, 6$. Then f and g satisfy one of the following cases*

(i) $f \equiv tg$, *where $t(\neq 0)$ is a constant,*

(ii) $f(z) = e^{az+b}$, $g(z) = e^{cz+d}$, *where a, b, c and d are constants such that $ac \neq 0$,*

(iii) $f(z) = \frac{a}{1-be^{\alpha(z)}}$, $g(z) = \frac{a}{e^{-\alpha(z)}-b}$, *where a, b are non-zero constants and $\alpha(z)$ is a non-constant entire function,*

(iv) $f(z) = a(1-be^{cz})$, $g(z) = d(e^{-cz}-b)$, *where a, b, c and d are non-zero constants.*

LEMMA 3.11. *Let f and g be two transcendental meromorphic functions such that zeros of $f-c$ and $g-c$ be of multiplicities at least k^*, where k^* be defined as in (2.2) and m, $n \in \mathbb{N}$. Suppose $f^{(k)}$, $g^{(k)}$ share $(0,\infty)$ and f, g share $(\infty, 0)$. If $f_1^d P_2(f_1)(f_1^{(k)})^m \equiv g_1^d P_2(g_1)(g_1^{(k)})^m$, where $f_1 = f-c$ and $g_1 = g-c$, then one of the following cases holds:*

(1) *If $P_2(z_1) \equiv e_i z_1^i \not\equiv 0$, for some $i \in \{0, 1, 2, \ldots, m_1\}$, then $f-c \equiv t(g-c)$, where t is a constant such that $t^{d+m+i} = 1$, for some $i \in \{0, 1, 2, \ldots, m_1\}$.*

(2) *If $P_2(z_1) \not\equiv e_i z_1^i \not\equiv 0$, for $i \in \{0, 1, 2, \ldots, m_1\}$ and f, g share $(c, 0)$, then $f-c \equiv t(g-c)$ for a constant t such that $t^s = 1$, where $s = GCD(n+m, \ldots, n+m-i, \ldots, 1)$, $e_{m_1-i} \neq 0$ for some $i = 0, 1, 2, \ldots, m_1-1$.*

Proof. Suppose

$$f_1^d P_2(f_1)(f_1^{(k)})^m \equiv g_1^d P_2(g_1)(g_1^{(k)})^m, \tag{3.19}$$

$$i.e., \quad \frac{P_2(f_1)}{P_2(g_1)} \equiv \frac{g_1^d(g_1^{(k)})^m}{f_1^d(f_1^{(k)})^m}. \tag{3.20}$$

From (3.19) we see that $f_1^{(k)}$ and $g_1^{(k)}$ share (∞,∞). Again since $f_1^{(k)}$ and $g_1^{(k)}$ share $(0,\infty)$, it follows that f_1 and g_1 share $(0,\infty)$ also. Since f_1 and g_1 share $(0,\infty)$, (∞,∞), it follows that $f_1 = e^{\gamma} g_1$, where γ is an entire function. We now consider following two cases.

Case 1. Suppose $P_2(z_1)$ is a non-zero monomial.

Let $P_2(z_1)\equiv e_iz_1^i\not\equiv 0$, for some $i\in\{0,1,2,\ldots,m_1\}$. Then from (3.19) we have

$$f_1^{n_1}(f_1^{(k)})^m\equiv g_1^{n_1}(g_1^{(k)})^m, \tag{3.21}$$

$$i.e.,\ \frac{f_1^{n_1}}{g_1^{n_1}}\equiv\frac{(g_1^{(k)})^m}{(f_1^{(k)})^m}, \tag{3.22}$$

where $n_1=d+i$, for some $i\in\{0,1,2,\ldots,m_1\}$. Let $h_1=\frac{f_1}{g_1}$ and $h_2=\frac{f_1^{(k)}}{g_1^{(k)}}$. Then $h_1\neq 0,\infty$ and $h_2\neq 0,\infty$. From (3.22) we see that

$$h_1^{n_1}h_2^m\equiv 1. \tag{3.23}$$

First we suppose h_1 is a non-constant entire function. Consequently h_2 is also a non-constant entire function. Let $F_1=h_1^{n_1}$ and $G_1=h_2^m$. Also from (3.23) we get

$$F_1G_1\equiv 1. \tag{3.24}$$

Clearly $F_1\not\equiv dG_1$, where d is a non-zero constant, otherwise F_1 will be a constant and so h_1 will be a constant.

Since $F_1\neq 0,\infty$ and $G_1\neq 0,\infty$ then there exist two non-constant entire functions α and β such that $F_1=e^{\alpha}$ and $G_1=e^{\beta}$. Now from (3.24) we see that $\alpha+\beta=C\in\mathbb{C}$. Therefore $\alpha'=-\beta'$. Note that $F_1'=\alpha'e^{\alpha}$ and $G_1'=\beta'e^{\beta}$. This shows that F_1' and G_1' share $(0,\infty)$. Note that $F_1\neq 0,\infty$, $G_1\neq 0,\infty$ and $F_1\not\equiv dG_1$, where d is a non-zero constant. Now in view of Lemma 3.10 we have $F_1(z)=c_1e^{az}$ and $G_1(z)=c_2e^{-az}$, where a,c_1,c_2 are non-zero constants such that $c_1c_2=1$. Since $\left(\frac{f_1(z)}{g_1(z)}\right)^{n_1}=c_1e^{az}$ and $\left(\frac{f_1^{(k)}(z)}{g_1^{(k)}(z)}\right)^{m}=c_2e^{-az}$, it follows that

$$\frac{f_1(z)}{g_1(z)}=t_1e^{\frac{a}{n_1}z}=t_1e^{d_1z} \tag{3.25}$$

and

$$\frac{f_1^{(k)}(z)}{g_1^{(k)}(z)}=t_2e^{-\frac{a}{m}z}=t_2e^{d_2z}, \tag{3.26}$$

where d_1,d_2,t_1,t_2 are non-zero constant such that $t_1^{n_1}=c_1$, $t_2^m=c_2$, $d_1=\frac{a}{n_1}$ and $d_2=-\frac{a}{m}$. Let

$$\Phi_1=\frac{f_1^{(k+1)}}{f_1^{(k)}}-\frac{g_1^{(k+1)}}{g_1^{(k)}}. \tag{3.27}$$

From (3.26), we see that

$$\Phi_1(z) = d_2. \tag{3.28}$$

Again from (3.25) we see that

$$f_1^{(j)}(z) = t_1 \sum_{i=0}^{j} {}^jC_i (e^{d_1 z})^{(j-i)} g_1^{(i)}(z),$$

where we define $g_1^{(0)}(z) = g_1(z)$. Consequently we have

$$\begin{aligned} f_1^{(k+1)}(z) = t_1[&d_1^{k+1} e^{d_1 z} g_1(z) + (k+1) d_1^k e^{d_1 z} g_1'(z) + \ldots \\ &+ \frac{k(k+1)}{2} d_1^2 e^{d_1 z} g_1^{(k-1)}(z) + (k+1) d_1 e^{d_1 z} g_1^{(k)}(z) + e^{d_1 z} g_1^{(k+1)}(z)] \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} f_1^{(k)}(z) = t_1[&d_1^k e^{d_1 z} g_1(z) + k d_1^{k-1} e^{d_1 z} g_1'(z) + \ldots \\ &+ \frac{(k-1)k}{2} d_1^2 e^{d_1 z} g_1^{(k-2)}(z) + k d_1 e^{d_1 z} g_1^{(k-1)}(z) + e^{d_1 z} g_1^{(k)}(z)]. \end{aligned} \tag{3.30}$$

Now from (3.27), (3.29) and (3.30) we have

$$\Phi_1 = \frac{F_2 - G_2 + (k+1) d_1 g_1^{(k)} g_1^{(k)} - k d_1 g_1^{(k-1)} g_1^{(k+1)}}{F_3 + g_1^{(k)} g_1^{(k)}}, \tag{3.31}$$

where

$$F_2 = d_1^{k+1} g_1 g_1^{(k)} + (k+1) d_1^k g_1' g_1^{(k)} + \ldots + \frac{k(k+1)}{2} d_1^2 g_1^{(k-1)} g_1^{(k)},$$

$$G_2 = d_1^k g_1 g_1^{(k+1)} + k d_1^{k-1} g_1' g_1^{(k+1)} + \ldots + \frac{(k-1)k}{2} d_1^2 g_1^{(k-2)} g_1^{(k+1)}$$

and

$$F_3 = d_1^k g_1 g_1^{(k)} + \ldots + k d_1 g_1^{(k-1)} g_1^{(k)}.$$

Let z_p be a zero of $g_1(z)$ of multiplicity $p(\geq k)$. Then the Taylor expansion of g_1 about z_p is

$$g_1(z) = b_p(z - z_p)^p + b_{p+1}(z - z_p)^{p+1} + b_{p+2}(z - z_p)^{p+2} + \ldots, b_p \neq 0. \tag{3.32}$$

We now consider the following two sub-cases.

Sub-case 1.1. Suppose $p = k$. Then

$$g_1^{(k)}(z) = k! b_k + (k+1)! b_{k+1}(z - z_k) + \ldots \tag{3.33}$$

and

$$g_1^{(k+1)}(z) = (k+1)!b_{k+1} + (k+2)!b_{k+2}(z-z_k) + \dots. \tag{3.34}$$

Now from (3.31), (3.33) and (3.34) we have

$$\Phi_1(z_k) = d_1 \frac{(k+1)(k!)^2 b_k^2}{(k!)^2 b_k^2} = d_1(k+1). \tag{3.35}$$

Therefore we arrive at a contradiction from (3.28) and (3.35).

Sub-case 1.2. Suppose $p \geq k+1$. Then

$$g_1^{(k-2)}(z) = p(p-1)\dots(p-k+3)b_p(z-z_p)^{p-k+2} + \dots$$

$$g_1^{(k-1)}(z) = p(p-1)\dots(p-k+2)b_p(z-z_p)^{p-k+1} + \dots$$

$$g_1^{(k)}(z) = p(p-1)\dots(p-k+1)b_p(z-z_p)^{p-k} + \dots$$

and

$$g_1^{(k+1)}(z) = p(p-1)\dots(p-k)b_p(z-z_p)^{p-k-1} + \dots.$$

Therefore

$$g_1^{(k)}(z)g_1^{(k)}(z) = Kb_p^2(z-z_p)^{2p-2k} + \dots \tag{3.36}$$

$$g_1^{(k-1)}(z)g_1^{(k+1)}(z) = \frac{p-k}{p-k+1}Kb_p^2(z-z_p)^{2p-2k} + \dots, \tag{3.37}$$

where $K = [p(p-1)\dots(p-k+1)]^2$. Also

$$F_2(z) = O((z-z_p)^{2p-2k+1}), \;\; G_2(z) = O((z-z_p)^{2p-2k+1})$$
$$\text{and} \;\; F_3(z) = O((z-z_p)^{2p-2k+1}).$$

Now from (3.31), (3.36) and (3.37) we have

$$\Phi_1(z_p) = \frac{(k+1)d_1Kb_p^2 - kd_1\frac{p-k}{p-k+1}Kb_p^2}{Kb_p^2} = d_1\frac{p+1}{p-k+1}. \tag{3.38}$$

Therefore we arrive at a contradiction from (3.28) and (3.38).

Thus in either cases one can easily say that g_1 has no zeros. Since f_1 and g_1 share $(0,\infty)$, it follows that both f_1 and g_1 have no zeros. But this is impossible because zeros of f_1 and g_1 are of multiplicities at least $k(\geq 1)$. Hence h_1 is constant. Then from (3.21) we get $h_1^{n_1+m} = 1$. Therefore we have $f - c \equiv t(g-c)$, where t is a constant such that $t^{d+m+i} = 1$, for some $i \in \{0,1,2,\dots,m_1\}$.

Case 2. Suppose $P_2(z_1)$ is not a monomial.

For the sake of simplicity, we assume that

$$P_2(z_1) = e_{m_1} z_1^{m_1} + e_{m_1-1} z_1^{m_1-1} + \ldots + e_1 z_1 + e_0; \; e_0 \neq 0.$$

Let $h_1^* = \frac{P_2(f_1)}{P_2(g_1)}$ and $h_2^* = \frac{f_1^d (f_1^{(k)})^m}{g_1^d (g_1^{(k)})^m}$. Then $h_1^* \neq 0, \infty$ and $h_2^* \neq 0, \infty$. From (3.20) we see that

$$h_1^* h_2^* \equiv 1. \tag{3.39}$$

We now consider the following two sub-cases.

Sub-case 2.1. Suppose h_1^* is a non-zero constant, say b. Then we have

$$\begin{aligned} & e_{m_1} f_1^{m_1} + e_{m_1-1} f_1^{m_1-1} + \ldots + e_1 f_1 + e_0 \\ \equiv \;& b(e_{m_1} g_1^{m_1} + e_{m_1-1} g_1^{m_1-1} + \ldots + e_1 g_1 + e_0). \end{aligned} \tag{3.40}$$

Let $b = 1$. Then from (3.19) we have

$$f_1^d (f_1^{(k)})^m \equiv g_1^d (g_1^{(k)})^m.$$

Then by Case 1 we have $f_1 \equiv t g_1$, where t is a constant such that $t^{d+m} = 1$, *i.e.*, $t^{n+m-m_1} = 1$.

Again from (3.40) we have

$$e_{m_1} g_1^{m_1} (t^{m_1} - 1) + e_{m_1-1} g_1^{m_1-1} (t^{m_1-1} - 1) + \ldots + e_1 g_1 (t-1) \equiv 0,$$

i.e.,

$$e_{m_1} g_1^{m_1} (t^{n+m} - 1) + e_{m_1-1} g_1^{m_1-1} (t^{n+m-1} - 1) + \ldots + e_1 g_1 (t-1) \equiv 0.$$

This shows that $f - c \equiv t(g - c)$ for a constant t such that $t^s = 1$, where $s = GCD(n+m, \ldots, n+m-i, \ldots, 1)$, $e_{m_1-i} \neq 0$ for some $i = 0, 1, 2, \ldots, m_1-1$.

Let $b \neq 1$. Since $f_1 = e^{\gamma} g_1$, from (3.40) we have

$$\begin{aligned} & e_{m_1} g_1^{m_1} (e^{m_1\gamma} - b) + e_{m_1-1} g_1^{m_1-1} (e^{(m_1-1)\gamma} - b) + \ldots \\ & + e_1 g_1 (e^{\gamma} - b) \equiv e_0 (b - 1). \end{aligned} \tag{3.41}$$

Note that $g_1 \not\equiv d$, where $d \in \mathbb{C}$. Then from (3.41) we see that g_1 has no zero. But this is impossible because zeros of g_1 are of multiplicities at least $k+1$.

Sub-case 2.2. Suppose h_1^* is non-constant.

Therefore h_2^* is also a non-constant entire function. Note that $h_1^* \not\equiv d_0^* h_2^*$, where d_0^* is a non-zero constant. Since $h_1^* \neq 0, \infty$ and $h_2^* \neq 0, \infty$, then there exist two non-constant entire functions α^* and β^* such that $h_1^* = e^{\alpha^*}$ and $h_2^* = e^{\beta^*}$. Now from (3.39) we see that $(\alpha^*)' = -(\beta^*)'$. Therefore $(h_1^*)'$ and $(h_2^*)'$ share $(0, \infty)$. Now in view of Lemma 3.10 we get $h_1^*(z) = c_1^* e^{az}$ and $h_2^*(z) = c_2^* e^{-az}$, where a, c_1, c_2 are non-zero complex constants such that

$c_1c_2 = 1$. Therefore we have

$$\begin{aligned}(3.42)\qquad & e_{m_1}g_1^{m_1}(e^{m_1\gamma}-c_1^*e^{az})+e_{m_1-1}g_1^{m_1-1}(e^{(m_1-1)\gamma}-c_1^*e^{az})+\dots\\ & +e_1g_1(e^{\gamma}-c_1^*e^{az})\equiv e_0(c_1^*e^{az}-1).\end{aligned}$$

Note that $c_1^*e^{az}-1$ has only simple zeros. Also from (3.42), we see that zeros of g_1 are also the zeros of $c_1^*e^{az}-1$. Since all the zeros of g_1 are of multiplicities at least $k+1$, from (3.42) we arrive at a contradiction. This completes the lemma. □

Lemma 3.12. *Let f, g be two non-constant meromorphic functions such that zeros of $f-c$ and $g-c$ be of multiplicities at least k^*, where k^* be defined as in (2.2) and $F=P(f)(f^{(k)})^m$, $G=P(g)(g^{(k)})^m$, where $k,\ m,n\in\mathbb{N}$ such that $n+m+mk>2k+1$ and $P(z)$ be defined as in (2.1). Suppose $H\not\equiv 0$. If F, G share $(1,k_1)$ and f, g share $(\infty,0)$ then*

$$\begin{aligned}\overline{N}(r,\infty;f) &\le \frac{k^*\Gamma_1+k^*+1}{k^*(n+m+(m-2)k-1)}\,[T(r,f)+T(r,g)]\\ &\quad+\frac{1}{n+m+(m-2)k-1}\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

Proof. Let $f_1=f-c$ and $g_1=g-c$. If ∞ is a Picard exceptional value of f and g, then the lemma follows immediately. Next suppose ∞ is not a Picard exceptional value of f and g. Since $H\not\equiv 0$, it follows that $F\not\equiv G$. We claim that $V\not\equiv 0$. If possible suppose $V\equiv 0$. Then by integration we obtain $1-\frac{1}{F}\equiv A(1-\frac{1}{G})$. Note that if z_* is a pole of f then it is a pole of g. Hence from the definition of F and G we have $\frac{1}{F(z_*)}=0$ and $\frac{1}{G(z_*)}=0$. So $A=1$ and hence $F\equiv G$, which is impossible. Let z_0 be a pole of f with multiplicity q and a pole of g with multiplicity r. Clearly z_0 is a pole of F with multiplicity $(n+m)q+mk$ and a pole of G with multiplicity $(n+m)r+mk$. Since f and g share $(\infty,0)$, from the definition of V it is clear that z_0 is a zero of V with multiplicity at least $n+m+mk-1$. Now in view of Lemma 3.2 and the definition of V we have

$$\begin{aligned}&[n+m+mk-1]\,\overline{N}(r,\infty;f)\\ \le\;& N(r,0;V)\\ \le\;& N(r,\infty;V)+S(r,f)+S(r,g)\\ \le\;& \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ \le\;& \overline{N}(r,0;P_1(f))+\overline{N}(r,0;f_1)+\overline{N}(r,0;f_1^{(k)}|f_1\neq 0)+\overline{N}(r,0;P_1(g))\\ &+\overline{N}(r,0;g_1)+\overline{N}(r,0;g_1^{(k)}|g_1\neq 0)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ \le\;& \overline{N}(r,0;P_1(f))+\overline{N}(r,0;f_1)+k\,\overline{N}(r,\infty;f_1)+N_k(r,0;f_1)\end{aligned}$$

$$
\begin{aligned}
& +\overline{N}(r,0;P_1(g)) + \overline{N}(r,0;g_1) + k\,\overline{N}(r,\infty;g_1) + N_k(r,0;g_1) \\
& +\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\
\leq\ & \overline{N}(r,0;P_1(f)) + \overline{N}(r,0;P_1(g)) + \frac{k^*+1}{k^*}\left[N(r,0;f_1) + N(r,0;g_1)\right] \\
& +2k\,\overline{N}(r,\infty;f) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\
\leq\ & \frac{k^*\,\Gamma_1 + k^* + 1}{k^*}\left[T(r,f) + T(r,g)\right] + 2k\,\overline{N}(r,\infty;f) + \overline{N}_*(r,1;F,G) \\
& +S(r,f) + S(r,g).
\end{aligned}
$$

Hence the lemma follows. □

LEMMA 3.13. *Let f be a non-constant meromorphic function and $F = P(f)(f^{(k)})^m$, where k, m, $n \in \mathbb{N}$ and $P(z)$ be defined as in (2.1). Then*

$$(n-m)T(r,f) \leq T(r,F) - m\,N(r,\infty;f) - N(r,0;(f^{(k)})^m) + S(r,f).$$

Proof. Note that

$$
\begin{aligned}
N(r,\infty;F) &= N(r,\infty;P(f)) + N(r,\infty;(f^{(k)})^m) \\
&= N(r,\infty;P(f)) + m\,N(r,\infty;f) + mk\overline{N}(r,\infty;f),
\end{aligned}
$$

i.e.,

$$N(r,\infty;P(f)) = N(r,\infty,F) - m\,N(r,\infty;f) - mk\,\overline{N}(r,\infty,f) + S(r,f).$$

Also

$$
\begin{aligned}
& m(r,P(f)) \\
=\ & m\big(r,\frac{F}{(f^{(k)})^m}\big) \\
\leq\ & m(r,F) + m\big(r,\frac{1}{(f^{(k)})^m}\big) + S(r,f) \\
=\ & m(r,F) + T(r,(f^{(k)})^m) - N(r,0;(f^{(k)})^m) + S(r,f) \\
=\ & m(r,F) + N(r,\infty;(f^{(k)})^m) + m(r,(f^{(k)})^m) - N(r,0;(f^{(k)})^m) + S(r,f) \\
\leq\ & m(r,F) + m\,N(r,\infty;f) + m\,k\overline{N}(r,\infty;f) + m\big(r,\frac{(f^{(k)})^m}{f^m}\big) + m(r,f^m) \\
& -N(r,0;(f^{(k)})^m) + S(r,f) \\
=\ & m(r,F) + m\,T(r,f) + m\,k\overline{N}(r,\infty;f) - N(r,0;(f^{(k)})^m) + S(r,f).
\end{aligned}
$$

Now

$$
\begin{aligned}
n\,T(r,f) &= N(r,\infty;P(f)) + m(r,P(f)) \\
&\leq T(r,F) + m\,T(r,f) - m\,N(r,\infty;f) - N(r,0;(f^{(k)})^m) + S(r,f),
\end{aligned}
$$

i.e., $(n-m)\,T(r,f)\leq T(r,F)-m\,N(r,\infty;f)-N(r,0;(f^{(k)})^m)+S(r,f)$.

This completes the proof. $\square$

Lemma 3.14. *Let f, g be two transcendental meromorphic functions, such that zeros of $f-c$ and $g-c$ be of multiplicities at least k^*, where k^* be defined as in (2.2) and $m,n\in\mathbb{N}$ such that $n>2\Gamma_1+\frac{2}{k^*}+m+k+1$. Let $F=\frac{P(f)(f^{(k)})^m}{p_1}$, $G=\frac{P(g)(g^{(k)})^m}{p_1}$, where $p_1(z)$ be any non-zero polynomial and $P(z)$ be defined as in (2.1). If f, g share $(\infty,0)$ and $H\equiv 0$ then one of the following three cases holds*

(1) $P(f)(f^{(k)})^mP(g)(g^{(k)})^m\equiv p_1^2$, *where* $P(f)(f^{(k)})^m-p_1$ *and* $P(g)(g^{(k)})^m-p_1$ *share* $(0,\infty)$,
(2) $f-c\equiv t(g-c)$ *for a constant t such that $t^s=1$, where* $s=GCD(n+m,\ldots,n+m-i,\ldots,m)$, $e_{m_1-i}\neq 0$ *for some* $i\in\{0,1,\ldots,m_1\}$,
(3) $P(f)(f^{(k)})^m\equiv P(g)(g^{(k)})^m$.

Proof. Since $H\equiv 0$, on integration, we get

$$\frac{F'}{(F-1)^2}=A\frac{G'}{(G-1)^2},$$

where A is a non-zero constant, *i.e.*,

$$\frac{\left(\frac{F_1-p_1}{p_1}\right)'}{\left(\frac{F_1-p_1}{p_1}\right)^2}=A\frac{\left(\frac{G_1-p_1}{p_1}\right)'}{\left(\frac{G_1-p_1}{p_1}\right)^2},$$

where $F_1=P(f)(f^{(k)})^m$ and $G_1=P(g)(g^{(k)})^m$. This shows that $\frac{F_1-p_1}{p_1}$ and $\frac{G_1-p_1}{p_1}$ share $(0,\infty)$. Therefore $P(f)(f^{(k)})^m-p_1$ and $P(g)(g^{(k)})^m-p_1$ share $(0,\infty)$. Finally, on integration we have

$$\frac{1}{F-1}\equiv\frac{bG+a-b}{G-1}, \tag{3.43}$$

where a,b are constants and $a\neq 0$. We now consider the following cases.

Case 1. Let $b\neq 0$ and $a\neq b$.

If $b=-1$, then from (3.43) we have $F\equiv\frac{-a}{G-a-1}$. Therefore

$$\overline{N}(r,a+1;G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)+S(r,f).$$

So in view of Lemma 3.13 and the second fundamental theorem we get

$$\begin{aligned}&(n-m)\,T(r,g)\\ \leq\;& T(r,P(g)(g^{(k)})^m)-m\,N(r,\infty;g)-N(r,0;(g^{(k)})^m)+S(r,g)\\ \leq\;& T(r,G)-m\,N(r,\infty;g)-N(r,0;(g^{(k)})^m)+S(r,g)\end{aligned}$$

$$\begin{aligned}
&\leq \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,a+1;G)-m\,N(r,\infty;g)\\
&\quad -N(r,0;(g^{(k)})^m)+S(r,g)\\
&\leq \overline{N}(r,0;P_1(g))+\overline{N}(r,0;g-c)+\overline{N}(r,0;(g^{(k)})^m)+\overline{N}(r,\infty;f)\\
&\quad -N(r,0;(g^{(k)})^m)+S(r,g)\\
&\leq \overline{N}(r,\infty;g)+\Gamma_1\,T(r,g)+\frac{1}{k^*}\,T(r,g)+S(r,g)\\
&\leq N(r,\infty;g)+(\Gamma_1+\frac{1}{k^*})\,T(r,g)+S(r,g)\leq(\Gamma_1+\frac{1}{k^*}+1)\,T(r,g)+S(r,g),
\end{aligned}$$

which is contradiction since $n>\Gamma_1+\frac{1}{k^*}+m+1$.

If $b\neq-1$, from (3.43) we obtain $F-(1+\frac{1}{b})\equiv\frac{-a}{b^2[G+\frac{a-b}{b}]}$ and so

$$\overline{N}(r,\frac{b-a}{b};G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)+S(r,f).$$

Using Lemma 3.13 and the same argument as used in the case when $b=-1$ we can get a contradiction.

Case 2. Let $b\neq 0$ and $a=b$.

If $b=-1$, then from (3.43) we have $FG\equiv 1$, *i.e.*, $P(f)(f^{(k)})^mP(g)(g^{(k)})^m\equiv p_1^2$, where $P(f)(f^{(k)})^m-p_1$ and $P(g)(g^{(k)})^m-p_1$ share $(0,\infty)$.

If $b\neq-1$, from (3.43) we have $\frac{1}{F}\equiv\frac{bG}{(1+b)G-1}$. Therefore

$$\overline{N}(r,\frac{1}{1+b};G)=\overline{N}(r,0;F).$$

So in view of Lemmas 3.1, 3.13 and the second fundamental theorem we get

$$\begin{aligned}
&(n-m)\,T(r,g)\\
&\leq T(r,G)-m\,N(r,\infty;g)-N(r,0;(g^{(k)})^m)+S(r,g)\\
&\leq \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,\frac{1}{1+b};G)-m\,N(r,\infty;g)\\
&\quad -N(r,0;(g^{(k)})^m)+S(r,g)\\
&\leq \overline{N}(r,0;P(g))+\overline{N}(r,0;(g^{(k)})^m)+\overline{N}(r,0;F)-N(r,0;(g^{(k)})^m)+S(r,g)\\
&\leq \overline{N}(r,0;g-c)+\overline{N}(r,0;P_1(g))+\overline{N}(r,0;f-c)+\overline{N}(r,0;P_1(f))\\
&\quad +\overline{N}(r,0;f_1^{(k)}|f_1\neq 0)+S(r,g)\\
&\leq (\Gamma_1+\frac{1}{k^*})\,\{T(r,f)+T(r,g)\}+N_k(r,0,f)+k\overline{N}(r,\infty;f)+S(r,g)\\
&\leq N(r,0;f)+k\overline{N}(r,\infty;f)+(\Gamma_1+\frac{1}{k^*})\,\{T(r,f)+T(r,g)\}\\
&\quad +S(r,f)+S(r,g)
\end{aligned}$$

$$\leq \ (\Gamma_1 + \frac{1}{k^*})\ \{T(r,f) + T(r,g)\} + T(r,f) + k\, T(r,f) + S(r,g).$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$. So for $r \in I$ we have

$$(n-m)\, T(r,g) \leq (2\Gamma_1 + \frac{2}{k^*} + k + 1)\, T(r,g) + S(r,g),$$

which is a contradiction since $n > 2\Gamma_1 + \frac{2}{k^*} + m + k + 1$.

Case 3. Let $b = 0$. From (3.43) we obtain

$$F \equiv \frac{G + a - 1}{a}. \tag{3.44}$$

If $a \neq 1$ then from (3.44) we obtain $\overline{N}(r, 1-a; G) = \overline{N}(r, 0; F)$. We can similarly deduce a contradiction as in Case 2. Therefore $a = 1$ and from (3.44) we obtain $F \equiv G$, *i.e.*, $f_1^d P_2(f_1)(f_1^{(k)})^m \equiv g_1^d P_2(g_1)(g_1^{(k)})^m$, where $f_1 = f - c$ and $g_1 = g - c$. This gives

$$\begin{aligned} f_1^d &\left[e_{m_1} f_1^{m_1} + e_{m_1-1} f_1^{m_1-1} + \ldots + e_1 f_1 + e_0\right] (f_1^{(k)})^m \\ &\equiv g_1^d \left[e_{m_1} g_1^{m_1} + e_{m_1-1} g_1^{m_1-1} + \ldots + e_1 g_1 + e_0\right] (g_1^{(k)})^m. \end{aligned} \tag{3.45}$$

Let $h = \frac{f_1}{g_1}$. If h is a constant, by putting $f_1 = h g_1$ in (3.45) we get

$$\begin{aligned} e_{m_1} g_1^{m_1}(h^{n+m} - 1) + e_{m_1-1} g_1^{m_1-1}(h^{n+m-1} - 1) + \ldots + e_1 g_1 (h^{m+1} - 1) \\ + e_0(h^m - 1) \equiv 0, \end{aligned}$$

which implies that $h^s = 1$, where $s = GCD(n+m, \ldots, n+m-i, \ldots, m)$, $e_{m_1-i} \neq 0$ for some $i \in \{0, 1, \ldots, m_1\}$. Thus $f - c \equiv t(g - c)$ for a constant t such that $t^s = 1$, where $s = GCD(n+m, \ldots, n+m-i, \ldots, m)$, $e_{m_1-i} \neq 0$ for some $i \in \{0, 1, \ldots, m_1\}$.

If h is not constant, then we must have $P(f)(f^{(k)})^m \equiv P(g)(g^{(k)})^m$. This completes the proof. □

Lemma 3.15 ([2]). *Let f and g be two non-constant meromorphic functions sharing $(1, k_1)$, where $2 \leq k_1 \leq \infty$. Then*

$$\begin{aligned} \overline{N}(r,1;f|=2) + 2\,\overline{N}(r,1;f|=3) + \ldots + (k_1-1)\,\overline{N}(r,1;f|=k_1) + k_1\,\overline{N}_L(r,1;f) \\ + (k_1+1)\,\overline{N}_L(r,1;g) + k_1\,\overline{N}_E^{(k_1+1}(r,1;g) \leq N(r,1;g) - \overline{N}(r,1;g). \end{aligned}$$

4. PROOFS OF THE THEOREMS

Proof of the Theorem 2.1. Let us take $F = P(f)(f^{(k)})^m$. In view of

Lemma 3.13 and by the second theorem for small functions ([18]) we get

$$\begin{aligned}(n-m)T(r,f) &\leq T(r,F)-m\,N(r,\infty;f)-N(r,0;(f^{(k)})^m)+S(r,f)\\ &\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,a;F)-m\,N(r,\infty;f)\\ &\quad -N(r,0;(f^{(k)})^m)+(\varepsilon+o(1))T(r,f)\\ &\leq \overline{N}(r,0;P_1(f))+\overline{N}(r,0;f-c)+\overline{N}(r,a;F)\\ &\quad +(\varepsilon+o(1))T(r,f)\\ &\leq (\Gamma_1+\frac{1}{k^*})T(r,f)+\overline{N}(r,a;F)+(\varepsilon+o(1))T(r,f),\end{aligned}$$

for all $\varepsilon>0$. Take $\varepsilon<1$. Since $n>m+\Gamma_1+\frac{1}{k^*}$, one can easily say that $F-a$ has infinitely many zeros. This completes the proof. □

Proof of the Theorem 2.3. Let $F=\frac{P(f)(f^{(k)})^m}{p}$ and $G=\frac{P(g)(g^{(k)})^m}{p}$. Note that F and G share $(1,k_1)$ except for the zeros of p and f, g share $(\infty,0)$.

Case 1. Let $H\not\equiv 0$. From (3.1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) those poles of F and G whose multiplicities are different, (iv) zeros of $F'(G')$ which are not the zeros of $F(F-1)(G(G-1))$.

Since H has only simple poles we get

$$\begin{aligned}&N(r,\infty;H)\\ \leq\;& \overline{N}_*(r,\infty;f,g)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)\\ &+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g),\end{aligned}\tag{4.1}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Let z_0 be a simple zero of $F-1$ but $p(z_0)\neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$N(r,1;F|=1)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,f)+S(r,g).\tag{4.2}$$

Using (4.1) and (4.2) we get

$$\begin{aligned}&\overline{N}(r,1;F)\\ \leq\;& N(r,1;F|=1)+\overline{N}(r,1;F|\geq 2)\\ \leq\;& \overline{N}_*(r,\infty;f,g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G)\\ &+\overline{N}(r,1;F|\geq 2)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g)\\ \leq\;& \overline{N}(r,\infty;f)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G)\\ &+\overline{N}(r,1;F|\geq 2)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g).\end{aligned}\tag{4.3}$$

Now in view of Lemmas 3.2 and 3.15 we get

$$
\begin{aligned}
(4.4)\qquad & \overline{N}_0(r,0;G') + \overline{N}(r,1;F\mid\geq 2) + \overline{N}_*(r,1;F,G)\\
\leq\; & \overline{N}_0(r,0;G') + \overline{N}(r,1;F|=2) + \overline{N}(r,1;F|=3)\\
& + \ldots + \overline{N}(r,1;F|=k_1)\\
& + \overline{N}_E^{(k_1+1}(r,1;F) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_*(r,1;F,G)\\
\leq\; & \overline{N}_0(r,0;G') - \overline{N}(r,1;F|=3) - \ldots - (k_1-2)\overline{N}(r,1;F|=k_1)\\
& -(k_1-1)\overline{N}_L(r,1;F)\\
& -k_1\overline{N}_L(r,1;G) - (k_1-1)\overline{N}_E^{(k_1+1}(r,1;F)\\
& +N(r,1;G) - \overline{N}(r,1;G) + \overline{N}_*(r,1;F,G)\\
\leq\; & \overline{N}_0(r,0;G') + N(r,1;G) - \overline{N}(r,1;G) - (k_1-2)\overline{N}_L(r,1;F)\\
& -(k_1-1)\overline{N}_L(r,1;G)\\
\leq\; & N(r,0;G' \mid G\neq 0) - (k_1-2)\overline{N}_L(r,1;F) - (k_1-1)\overline{N}_L(r,1;G)\\
\leq\; & \overline{N}(r,0;G) + \overline{N}(r,\infty;g) - (k_1-2)\overline{N}_L(r,1;F) - (k_1-1)\overline{N}_L(r,1;G)\\
=\; & \overline{N}(r,0;G) + \overline{N}(r,\infty;g) - (k_1-2)\overline{N}_*(r,1;F,G) - \overline{N}_L(r,1;G).
\end{aligned}
$$

Hence using (4.3), (4.4), Lemma 3.1 we get from second fundamental theorem that

$$
\begin{aligned}
(4.5)\qquad & T(r,F)\\
\leq\; & \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) - N_0(r,0;F')\\
\leq\; & 2\,\overline{N}(r,\infty,f) + N_2(r,0;F) + \overline{N}(r,0;G|\geq 2) + \overline{N}(r,1;F|\geq 2)\\
& +\overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;G') + S(r,f) + S(r,g)\\
\leq\; & 3\,\overline{N}(r,\infty;f) + N_2(r,0;F) + N_2(r,0;G) - (k_1-2)\,\overline{N}_*(r,1;F,G)\\
& +S(r,f) + S(r,g)\\
\leq\; & 3\,\overline{N}(r,\infty;f) + 2\overline{N}(r,0;f-c) + N_2(r,0;P_1(f)) + N_2(r,0;(f^{(k)})^m)\\
& +2\overline{N}(r,0;g-c) + N_2(r,0;P(g)) + N_2(r,0;(g^{(k)})^m)\\
& -(k_1-2)\,\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
\leq\; & 3\overline{N}(r,\infty;f) + (\Gamma_2 + \frac{2}{k^*})\,T(r,f) + N(r,0;(f^{(k)})^m)\\
& +(\Gamma_2 + \frac{2}{k^*})\,T(r,g) + m\,N_2(r,0;g^{(k)})\\
& -(k_1-2)\,\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
\leq\; & (3+mk)\,\overline{N}(r,\infty;f) + (\Gamma_2 + \frac{2}{k^*})T(r,f)\\
& +(\Gamma_2 + \frac{2}{k^*} + m)T(r,g) + N(r,0;(f^{(k)})^m)
\end{aligned}
$$

$$-(k_1-2)\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).$$

Now using Lemmas 3.12 and 3.13, we get from (4.5) that

$$\begin{aligned}
(4.6)\qquad & (n-m)\,T(r,f)\\
\le\ & T(r,F)-m\,N(r,\infty;f)-N(r,0;(f^{(k)})^m)+S(r,f)\\
\le\ & [m(k-1)+3]\,\overline{N}(r,\infty;f)+(\Gamma_2+\frac{2}{k^*})T(r,f)\\
& +(\Gamma_2+\frac{2}{k^*}+m)\,T(r,g)\\
& -(k_1-2)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
\le\ & \frac{(k^*+k^*\Gamma_1+1)[m(k-1)+3]}{k^*(m+n+(m-2)k-1)}\,[T(r,f)+T(r,g)]\\
& +(\Gamma_2+\frac{2}{k^*})T(r,f)+(\Gamma_2+\frac{2}{k^*}+m)T(r,g)\\
& +\frac{m(k-1)+3}{n+m+(m-2)k-1}\overline{N}_*(r,1;F,G)\\
& -(k_1-2)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
\le\ & \left[2\Gamma_2+m+\frac{4}{k^*}+2\,\frac{(k^*+k^*\Gamma_1+1)[m(k-1)+3]}{k^*(m+n+(m-2)k-1)}\right]T(r)+S(r).
\end{aligned}$$

In a similar way we can obtain

$$\begin{aligned}
(4.7)\qquad & (n-m)\,T(r,g)\\
\le\ & \left[2\Gamma_2+\frac{4}{k^*}+m+2\,\frac{(k^*+k^*\Gamma_1+1)[m(k-1)+3]}{k^*(m+n+(m-2)k-1)}\right]T(r)+S(r).
\end{aligned}$$

From (4.6) and (4.7) we see that

$$\begin{aligned}
& (n-m)\,T(r)\\
\le\ & \left[2\Gamma_2+\frac{4}{k^*}+m+2\,\frac{(k^*+k^*\Gamma_1+1)[m(k-1)+3]}{k^*(m+n+(m-2)k-1)}\right]T(r)+S(r),
\end{aligned}$$

i.e.,

$$\left[k^*n^2-(2k^*\Gamma_2+k^*m+2k^*k+k^*+4-mkk^*)n+A\right]T(r)\le S(r),$$

where

$$\begin{aligned}
A=\ & 2mkk^*+4mk^*+8k-2m+2m\Gamma_1k^*+4\Gamma_2kk^*+2\Gamma_2k^*-2m^2k^*-2m^2kk^*-2\\
& -6k^*-6\Gamma_1k^*-2m\Gamma_1kk^*-6mk-2m\Gamma_2k^*-2m\Gamma_2kk^*,
\end{aligned}$$

i.e.,

$$(4.8)\qquad [k^*(n-K_1)(n-K_2)]\ \ T(r)\le S(r),$$

where

$$K_1=\frac{2k^*\Gamma_2+k^*m+2k^*k+k^*+4-mkk^*+\sqrt{L}}{2k^*}$$

and

$$K_2=\frac{2k^*\Gamma_2+k^*m+2k^*k+k^*+4-mkk^*-\sqrt{L}}{2k^*}$$

with

$$L=(2k^*\Gamma_2+k^*m+2k^*k+k^*+4-mkk^*)^2-4k^*A.$$

Note that

$$\begin{aligned}
L &= (2k^*\Gamma_2+k^*m+2k^*k+k^*+4-mkk^*)^2-4k^*A\\
&= 4(k^*)^2\Gamma_2^2+9(k^*)^2m^2+(k^*)^2+16+(k^*)^2m^2k^2+6(k^*)^2m^2k\\
&\quad +8(k^*)^2mk+16k^*+16k^*mk+16mk^*+4(k^*)^2mk\Gamma_2+16k^*\Gamma_2\\
&\quad +12(k^*)^2m\Gamma_2+8(k^*)^2mk\Gamma_1-2(k^*)^2mk-2(k^*)^2m-16k^*k\\
&\quad -8(k^*)^2k\Gamma_2-4(k^*)^2\Gamma_2+(4(k^*)^2k^2-4(k^*)^2k^2m)\\
&\quad +(4(k^*)^2k-12(k^*)^2mk)+(24(k^*)^2-12(k^*)^2m)\\
&\quad +(24(k^*)^2\Gamma_1-8(k^*)^2m\Gamma_1)\\
&\leq 4(k^*)^2\Gamma_2^2+9(k^*)^2m^2+(k^*)^2+16+(k^*)^2m^2k^2+6(k^*)^2m^2k\\
&\quad +8(k^*)^2mk+16k^*+16k^*mk+16mk^*\\
&\quad +4(k^*)^2mk\Gamma_2+16k^*\Gamma_2+8(k^*)^2m\Gamma_2+8(k^*)^2mk\Gamma_2\\
&\quad -2(k^*)^2mk-2(k^*)^2m-16k^*k-8(k^*)^2k\Gamma_2-4(k^*)^2\Gamma_2-8(k^*)^2mk\\
&\quad +12(k^*)^2+16(k^*)^2\Gamma_2\\
&\leq m^2k^2(k^*)^2+9(k^*)^2m^2+13(k^*)^2+16+6(k^*)^2m^2k+16k^*\\
&\quad +16k^*mk+16mk^*+12(k^*)^2mk\Gamma_2+16k^*\Gamma_2+8(k^*)^2m\Gamma_2+16\Gamma_2^2(k^*)^2\\
&\quad -2(k^*)^2m-16k^*k-8(k^*)^2k\Gamma_2-2(k^*)^2mk\\
&\leq m^2k^2(k^*)^2+9m^2(k^*)^2+4(k^*)^2+36\Gamma_2^2(k^*)^2+9+6m^2k(k^*)^2\\
&\quad +4mk(k^*)^2+24m\Gamma_2k(k^*)^2+6mkk^*\\
&\quad +12m(k^*)^2+36m\Gamma_2(k^*)^2+18mk^*+24\Gamma_2(k^*)^2+12k^*+36\Gamma_2k^*\\
&\quad +\big(7+9(k^*)^2+10mkk^*-14m(k^*)^2-6mk(k^*)^2-16kk^*-2mk^*\big)\\
&< \Big[mkk^*+3mk^*+2k^*+6\Gamma_2k^*+3\Big]^2.
\end{aligned}$$

Therefore

$$K_1=\frac{2k^*\Gamma_2+k^*m+2k^*k+k^*+4-k^*mk+\sqrt{L}}{2k^*}$$

$$< \frac{2k^*\Gamma_2 + k^*m + 2k^*k + k^* + 4 - k^*mk + mkk^* + 3mk^* + 2k^* + 6\Gamma_2 k^* + 3}{2k^*}$$
$$= k + 2(m + 2\Gamma_2) + \frac{3k^* + 7}{2k^*}.$$

Since $n \geq k + 2(m + 2\Gamma_2) + \frac{3k^*+7}{2k^*}$, (4.8) leads to a contradiction.

Case 2. Let $H \equiv 0$. Note $n \geq k+2(m+2\Gamma_2)+\frac{3k^*+7}{2k^*} > 2\Gamma_1+\frac{2}{k^*}+m+k+1$. Remaining part of the theorem follows from Lemmas 3.14, 3.9 and 3.11. This completes the proof. □

Proof of the Theorem 2.2. When $H \not\equiv 0$ we follow the proof of Theorem 2.3 while for $H \equiv 0$ we follow Lemmas 3.14 and 3.9. So we omit the detail proof. □

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