# DUAL PAIR AND APPROXIMATE DUAL FOR CONTINUOUS FRAMES IN HILBERT SPACES

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In this manuscript, the concept of dual and approximate dual for continuous frames in Hilbert spaces will be introduced. Some of its properties will be studied. Also, the relations between two continuous Riesz bases in Hilbert spaces will be clarified through examples.

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*Key words:* continuous frame, continuous Riesz basis, dual continuous frame, approximate dual continuous frame.

## 1. INTRODUCTION

Frames for Hilbert spaces have been first introduced by Duffin and Scheaffer in the study of some problems in nonharmonic Fourier series in 1952, [12]. A discrete frame is a countable family of elements in a separable Hilbert space which allows for a stable, not necessarily unique, decomposition of an arbitrary element into an expansion of the frame elements.

Recall that for a Hilbert space $\mathcal{H}$ and a countable index set $I$, a family of vectors $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ is called a discrete frame for $\mathcal{H}$, if there exist constants $0 < A \leq B < +\infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \qquad f \in \mathcal{H},$$

the constants $A$ and $B$ are called frame bounds. The frame $\{f_i\}_{i\in I}$ is called tight if $A = B$ and Parseval if $A = B = 1$. The frame decomposition is the most important frame results. It shows for the frame $\{f_i\}_{i\in I}$, every element in $\mathcal{H}$ has a representation as an infinite linear combination of the frame elements; *i.e.*, there exist coefficients $\{c_i(f)\}_{i\in I}$ such that $f = \sum_{i\in I} c_i(f) f_i$, where $f \in \mathcal{H}$ is arbitrary. Thus, it is natural to say that a frame is some kind of a generalized basis. Usually, we want to work with coefficients which depend continuously and linearly on $f$, by "Riesz representation theorem", this implies that the

$i$-th coefficient in the expansion of $f$ should have on the form $c_i(f) = \langle f, g_i \rangle$ for some $g_i \in \mathcal{H}$. The sequence $\{g_i\}_{i\in I}$ is called a dual frame of $\{f_i\}_{i\in I}$. Li [19] provided a characterization and construction of general frame decomposition. He showed that for generating all duals for a given frame, it is enough to find the left-inverses of a one-to-one mapping and found a general parametric and algebraic formula for all duals. It is usually complicated to calculate a dual frame explicitly. Hence Christensen and Laugesen seek methods for constructing approximate duals, see [9]. Note that the idea of approximate dual frames has appeared previously, especially for wavelets [17], Gabor systems [5,14], in the general context of coorbit theory [13] and the sensor modeling [20].

New applications of frames, especially in the last decade, motivated the researcher to find some generalizations of frames like continuous frames [1,24], fusion frames [7], $g$-frames [25], controlled and weighted frames [3,23], $p$-frames [22], $K$-frames [15] and etc.

The notion of continuous frames was introduced by Kaiser in [18] and independently by Ali, Antoine and Gazeau [1]. The windowed Fourier transform and the continuous wavelet transform are just two instances of continuous frames but at the same time the main motivation for its definition. Gabardo and Han in [16] defined the concept of dual frames for continuous frames.

## 2. PRELIMINARIES

In this section, we review some notations and definitions. Throughout this paper, $\mathcal{H}$ is a Hilbert space and $(\Omega, \mu)$ a measure space with positive measure $\mu$. The set of all bounded operators on $\mathcal{H}$ denoted by $L(\mathcal{H})$.

*Definition* 2.1. A weakly-measurable mapping $F : \Omega \to \mathcal{H}$ is called a continuous frame for $\mathcal{H}$ with respect to $(\Omega, \mu)$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 \mathrm{d}\mu(\omega) \leq B\|f\|^2, \qquad f \in \mathcal{H}.$$

The constants $A$ and $B$ are called continuous frame bounds. The mapping $F$ is called tight continuous frame if $A = B$ and if $A = B = 1$, it called a Parseval continuous frame. This mapping is called Bessel if the second inequality holds. In this case, $B$ is called Bessel constant.

Suppose $F : \Omega \to \mathcal{H}$ is a Bessel mapping with bound $B$. The operator $T_F : L^2(\Omega, \mu) \to \mathcal{H}$ weakly defined by

$$\langle T_F\varphi, f\rangle = \int_\Omega \varphi(\omega)\langle F(\omega), f\rangle \mathrm{d}\mu(\omega), \qquad \varphi \in L^2(\Omega, \mu), \qquad f \in \mathcal{H},$$

is well-defined, linear, bounded with bound $\sqrt{B}$ and its adjoint is given by

$$T_F^*: \mathcal{H} \to L^2(\Omega, \mu), \quad (T_F^* f)(\omega) = \langle f, F(\omega)\rangle, \quad \omega \in \Omega, \quad f \in \mathcal{H}.$$

The operator $T_F$ is called the synthesis operator and $T_F^*$ is called the analysis operator of $F$. For continuous frame $F$ with bounded $A$ and $B$, the operator $S_F = T_F T_F^*$ is called continuous frame operator and this is bounded, invertible, positive and $AI_{\mathcal{H}} \le S_F \le BI_{\mathcal{H}}$. In fact,

$$\langle S_F f, g\rangle = \int_\Omega \langle f, F(\omega)\rangle\langle F(\omega), g\rangle \mathrm{d}\mu(\omega), \quad f, g \in \mathcal{H}.$$

For all $f, g \in \mathcal{H}$, the reconstruction formulas are as follows

$$\langle f, g\rangle = \int_\Omega \langle f, F(\omega)\rangle\langle S_F^{-1} F(\omega), g\rangle \mathrm{d}\mu(\omega) = \int_\Omega \langle f, S_F^{-1} F(\omega)\rangle\langle F(\omega), g\rangle \mathrm{d}\mu(\omega).$$

We end this short section by a well known example named wavelet frames.

*Example* 2.2. Let $\Omega = (0, +\infty) \times \mathbb{R}$ be the affine group, with group law $(a, b)(a', b') = (aa', b + ab')$. An element $\psi \in L^2(\mathbb{R})$ is said to be admissible if $\|\psi\|_2 = 1$ and $C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\xi)|^2}{\xi} \mathrm{d}\xi < +\infty$. For such admissible function $\psi$, we have

$$\int_{-\infty}^{+\infty} \int_0^{+\infty} |\langle f, T_a D_b \psi\rangle|^2 \frac{\mathrm{d}a \mathrm{d}b}{a^2} = C_\psi \|f\|^2, \quad f \in L^2(\mathbb{R}),$$

where $T_a f(t) = f(t - a)$ and $D_b f(t) = \frac{1}{\sqrt{b}} f(\frac{t}{b})$, see [8]. That is, $\{T_a D_b \psi\}_{a \neq 0, 0 < b \in \mathbb{R}}$ is a tight continuous frame with respect to $(\Omega, \frac{\mathrm{d}a\mathrm{d}b}{a^2})$ with the frame operator $S = C_\psi I$.

## 3. THE DUALITY OF CONTINUOUS FRAMES

Reconstruction of the original vector from frames, g-frames, fusion frames, continuous frames as well as their extensions, is typically achieved by using a so-called (alternate or standard) dual system.

*Definition* 3.1. Let $F$ and $G$ be two Bessel mappings with synthesis operators $T_F$ and $T_G$, respectively. We call $G$ a dual of $F$ if the following equality holds

$$\langle f, g\rangle = \int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), g\rangle \mathrm{d}\mu(\omega), \quad f, g \in \mathcal{H}.$$

In this case, $(F, G)$ is called a dual pair for $\mathcal{H}$.

This definition is equivalent to $T_G T_F^* = I_{\mathcal{H}}$. The condition

$$\langle f, g\rangle = \int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), g\rangle \mathrm{d}\mu(\omega), \quad f, g \in \mathcal{H}$$

is equivalent

$$\langle f, g\rangle = \int_{\Omega} \langle f, G(\omega)\rangle \langle F(\omega), g\rangle \mathrm{d}\mu(\omega), \qquad f, g \in \mathcal{H}$$

because $T_G T_F^* = I$ if and only if $T_F T_G^* = I$.

For the continuous frame $F$, the mapping $S_F^{-1}F$ is called standard dual of $F$. It is certainly possible for a continuous frame $F$ to have only one dual. In this case, the continuous frame $F$ is called a Riesz-type frame. Riesz-type frames are actually frames, for which the analysis operator is onto. Also, the continuous frame $F$ is Riesz-type frame if and only if $T_F^*$ is onto [2].

Similar to discrete case, by a simple calculation, it is easy to show that $G$ is a dual of $F$ if and only if $G = S_F^{-1}F + H$, where $H$ satisfies in the condition $\int_{\Omega} \langle f, F(\omega)\rangle \langle H(\omega), g\rangle \mathrm{d}\mu(\omega) = 0$, for all $f, g \in \mathcal{H}$.

It is easy verifiable that $F$ is a tight continuous frame for $\mathcal{H}$ with bound $C$ if and only if $(F, \frac{1}{C}F)$ is dual pair for it.

By using Example 2.2, the pair $(T_a D_b \psi, \frac{1}{C_\psi} T_a D_b \psi)$ is a dual pair in $L^2(\mathbb{R})$ and we have

$$f = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \langle f, T_a D_b \psi\rangle T_a D_b \psi \frac{\mathrm{d}a\mathrm{d}b}{a^2}, \qquad f \in L^2(\mathbb{R}).$$

It is not clear for which wavelet frames the standard dual frame consists of wavelet as well. More generally, there are some wavelet frames with no dual wavelet frames at all [10].

The following is another example of continuous frame with one of its duals.

*Example* 3.2. Consider $\mathcal{H} = \mathbb{R}^2$ with the standard basis $\{e_1, e_2\}$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Put $B_{\mathbb{R}^2} = \{x \in \mathbb{R}^2 : \ \|x\| \le 1\}$. Let $\Omega = B_{\mathbb{R}^2}$ and $\lambda$ be the Lebesgue measure. Define $F : B_{\mathbb{R}^2} \to \mathbb{R}^2$ and $G : B_{\mathbb{R}^2} \to \mathbb{R}^2$ such that

$$F(\omega) = \begin{cases} \frac{1}{\sqrt{\lambda(B_1)}} e_1, & \omega \in B_1, \\ \frac{1}{\sqrt{\lambda(B_2)}} e_2, & \omega \in B_2, \\ 0, & \omega \in B_3, \end{cases}$$

and

$$G(\omega) = \begin{cases} \frac{1}{\sqrt{\lambda(B_1)}} e_1, & \omega \in B_1, \\ \frac{1}{\sqrt{\lambda(B_2)}} e_2, & \omega \in B_2, \\ \frac{2}{\sqrt{\lambda(B_3)}} e_2, & \omega \in B_3, \end{cases}$$

where $\{B_1, B_2, B_3\}$ is a partition of $B_{\mathbb{R}^2}$. It is easy to check that $F$ and $G$ are continuous frames for $\mathbb{R}^2$ with respect to $(B_{\mathbb{R}^2}, \lambda)$. For each $x \in \mathbb{R}^2$, we have

$$\begin{aligned}
(T_G T_F^*)(x) &= \left(\int_{B_1} + \int_{B_2} + \int_{B_3}\right) (T_F^* x)(\omega) G(\omega) \mathrm{d}\lambda(\omega) \\
&= \int_{B_1} \langle x, \frac{1}{\sqrt{\lambda(B_1)}} e_1 \rangle \frac{1}{\sqrt{\lambda(B_1)}} e_1 \mathrm{d}\lambda(\omega) \\
&\quad + \int_{B_2} \langle x, \frac{1}{\sqrt{\lambda(B_2)}} e_2 \rangle \frac{1}{\sqrt{\lambda(B_2)}} e_2 \mathrm{d}\lambda(\omega) \\
&\quad + 0 \\
&= \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2 \\
&= x
\end{aligned}$$

*i.e.*, $(F, G)$ is a dual pair for $\mathbb{R}^2$.

Similar to discrete frames, for continuous frame we have the following assertion.

PROPOSITION 3.3. *The Bessel mapping $F : \Omega \to \mathcal{H}$ is a continuous frame for $\mathcal{H}$ with respect to $(\Omega, \mu)$ if and only if there exists a Bessel mapping $G : \Omega \to \mathcal{H}$ such that for each $f, g \in \mathcal{H}$,*

$$\langle f, g \rangle = \int_\Omega \langle f, F(\omega) \rangle \langle G(\omega), g \rangle \mathrm{d}\mu(\omega). \tag{3.1}$$

*Proof.* At first, assume that there exists a Bessel mapping $G$ with Bessel constant $B_G$, satisfying (3.1). Then for any $f \in \mathcal{H}$,

$$\begin{aligned}
\|f\|^4 &= \left| \int_\Omega \langle f, F(\omega) \rangle \langle G(\omega), f \rangle \mathrm{d}\mu(\omega) \right|^2 \\
&\leq \left( \int_\Omega |\langle f, F(\omega) \rangle \langle G(\omega), f \rangle| \mathrm{d}\mu(\omega) \right)^2 \\
&\leq \left( \int_\Omega |\langle f, F(\omega) \rangle|^2 \mathrm{d}\mu(\omega) \right) \left( \int_\Omega |\langle f, G(\omega) \rangle|^2 \mathrm{d}\mu(\omega) \right) \\
&\leq \left( \int_\Omega |\langle f, F(\omega) \rangle|^2 \mathrm{d}\mu(\omega) \right) . B_G \|f\|^2.
\end{aligned}$$

Thus, $F$ is a continuous frame for $\mathcal{H}$ with lower bound $B_G^{-1}$.

Conversely, let $F$ be a continuous frame for $\mathcal{H}$ with the frame operator $S_F$. Thus, for all $f, g \in \mathcal{H}$

$$\langle f, g \rangle = \int_\Omega \langle f, F(\omega) \rangle \langle S_F^{-1} F(\omega), g \rangle \mathrm{d}\mu(\omega).$$

Put $G = S_F^{-1}F$. $\square$

Like discrete frames [8], for Bessel mappings, we have the following corollary.

COROLLARY 3.4. *If $(F, G)$ is a dual pair for $\mathcal{H}$, then both $F$ and $G$ are continuous frames for $\mathcal{H}$.*

Improving and extending Theorem 3.6 of [24] for standard dual, we have the following theorem for any dual pairs. It shows that we can remove some elements from a continuous frame so that the remaining set is still a continuous frame.

THEOREM 3.5. *Let $(F, G)$ be a dual pair for $\mathcal{H}$ and there exists $\omega_0 \in \Omega$ such that $\mu(\{\omega_0\})\langle F(\omega_0), G(\omega_0)\rangle \neq 1$. Then $F : \Omega\backslash\{\omega_0\} \to \mathcal{H}$ is a continuous frame for $\mathcal{H}$.*

*Proof.* We assume that the Bessel constants $F$ and $G$ are $B_F$ and $B_G$, (respectively). If $f \in \mathcal{H}$, then

$$\begin{aligned}\langle F(\omega_0), f\rangle = &\int_{\Omega\backslash\{\omega_0\}} \langle F(\omega_0), G(\omega)\rangle\langle F(\omega), f\rangle \mathrm{d}\mu(\omega)\\ &+ \langle F(\omega_0), G(\omega_0)\rangle\langle F(\omega_0), f\rangle\mu(\{\omega_0\}).\end{aligned}$$

Therefore,

$$\begin{aligned}|\langle f, F(\omega_0)\rangle|^2 \leq &\frac{1}{|1-\mu(\{\omega_0\})\langle F(\omega_0), G(\omega_0)\rangle|^2}\left(\int_{\Omega\backslash\{\omega_0\}} |\langle f, F(\omega)\rangle|^2 \mathrm{d}\mu(\omega)\right)\\ &\left(\int_{\Omega\backslash\{\omega_0\}} |\langle F(\omega_0), G(\omega)\rangle|^2 \mathrm{d}\mu(\omega)\right)\\ \leq &\frac{B_G\|F(\omega_0)\|^2}{|1-\mu(\{\omega_0\})\langle F(\omega_0), G(\omega_0)\rangle|^2}\left(\int_{\Omega\backslash\{\omega_0\}} |\langle f, F(\omega)\rangle|^2 \mathrm{d}\mu(\omega)\right).\end{aligned}$$

Put $C = \frac{B_G\|F(\omega_0)\|^2}{|1-\mu(\{\omega_0\})\langle F(\omega_0), G(\omega_0)\rangle|^2}$. We have

$$B_G^{-1}\|f\|^2 \leq (1 + C\mu(\{\omega_0\}))\int_{\Omega\backslash\{\omega_0\}} |\langle f, F(\omega)\rangle|^2 \mathrm{d}\mu(\omega).$$

Hence $F : \Omega\backslash\{\omega_0\} \to \mathcal{H}$ is a continuous frame with lower bound $\frac{B_G^{-1}}{1+C\mu(\{\omega_0\})}$. $\square$

Concerning the above theorem, a question occurs. What happens in case $\mu(\{\omega_0\})\langle F(\omega_0), S_F^{-1}F(\omega_0)\rangle = 1$? To achieve this purpose, we need the following lemma.

LEMMA 3.6 ([24]). *The continuous frame coefficients* $\{\langle f, S_F^{-1}F(\omega)\rangle\}_{\omega\in\Omega}$ *have minimal* $L^2$*-norm among all coefficients* $\{\phi(\omega)\}_{\omega\in\Omega}$ *for which*

$$f = \int_\Omega \phi(\omega)F(\omega)\mathrm{d}\mu(\omega)$$

*for some* $\phi \in L^2(\Omega,\mu)$*; i.e.,*

$$\int_\Omega |\phi(\omega)|^2\mathrm{d}\mu(\omega) = \int_\Omega |\langle f, S_F^{-1}F(\omega)\rangle|^2\mathrm{d}\mu(\omega) + \int_\Omega |\phi(\omega) - \langle f, S_F^{-1}F(\omega)\rangle|^2\mathrm{d}\mu(\omega).$$

THEOREM 3.7. *Let* $F$ *be a continuous frame for* $\mathcal{H}$ *with respect to* $(\Omega,\mu)$ *with frame operator* $S_F$ *and* $\omega_0 \in \Omega$*. If* $\mu(\{\omega_0\})\langle F(\omega_0), S_F^{-1}F(\omega_0)\rangle = 1$*, then there exists a non-empty measurable set* $\Omega_0 \subset \Omega$ *such that* $\omega_0 \in \Omega_0$*,* $\mu(\Omega_0) = 0$ *and* $\{F(\omega)\}_{\omega\in\Omega\setminus\Omega_0}$ *is incomplete.*

*Proof.* We have

$$F(\omega_0) = \int_\Omega \langle F(\omega_0), S_F^{-1}F(\omega)\rangle F(\omega)\mathrm{d}\mu(\omega) = \int_\Omega \frac{\chi_{\{\omega_0\}}}{\mu(\{\omega_0\})}F(\omega)\mathrm{d}\mu(\omega),$$

such that $\chi_{\{\omega_0\}}$ is the characteristic function of a set $\{\omega_0\} \subset \Omega$. So Lemma 3.6 yields the following relation between $\{\frac{\chi_{\{\omega_0\}}(\omega)}{\mu(\{\omega_0\})}\}$ and $\{\langle F(\omega_0), S_F^{-1}F(\omega)\rangle\}$

$$\begin{aligned}
\int_\Omega |\frac{\chi_{\{\omega_0\}}(\omega)}{\mu(\{\omega_0\})}|^2\mathrm{d}\mu(\omega) &= \int_\Omega |\langle F(\omega_0), S_F^{-1}F(\omega)\rangle|^2\mathrm{d}\mu(\omega)\\
&\quad + \int_\Omega |\frac{\chi_{\{\omega_0\}}(\omega)}{\mu(\{\omega_0\})} - \langle F(\omega_0), S_F^{-1}F(\omega)\rangle|^2\mathrm{d}\mu(\omega)\\
&= \int_{\Omega\setminus\{\omega_0\}} |\langle F(\omega_0), S_F^{-1}F(\omega)\rangle|^2\mathrm{d}\mu(\omega)\\
&\quad + |\langle F(\omega_0), S_F^{-1}F(\omega_0)\rangle|^2\mu(\{\omega_0\})\\
&\quad + \int_{\Omega\setminus\{\omega_0\}} |\frac{\chi_{\{\omega_0\}}(\omega)}{\mu(\{\omega_0\})} - \langle F(\omega_0), S_F^{-1}F(\omega)\rangle|^2\mathrm{d}\mu(\omega)\\
&\quad + |\frac{1}{\mu(\{\omega_0\})} - \langle F(\omega_0), S_F^{-1}F(\omega_0)\rangle|^2\mu(\omega_0).
\end{aligned}$$

From the above formula,

$$\int_{\Omega\setminus\{\omega_0\}} |\langle F(\omega_0), S_F^{-1}F(\omega)\rangle|^2\mathrm{d}\mu(\omega) = 0,$$

so that $\langle F(\omega_0), S_F^{-1}F(\omega)\rangle = 0$ a.e. on $\Omega\setminus\{\omega_0\}$. Put

$$\Omega_0 = \{\omega \in \Omega : \quad \langle S_F^{-1}F(\omega_0), F(\omega)\rangle \neq 0\}.$$

It is clear that $\Omega_0$ is a measurable set with zero measure and $\omega_0 \in \Omega_0$. Thus we have the non-zero element $S_F^{-1}F(\omega_0)$ which is orthogonal to $\{F(\omega)\}_{\omega\in\Omega\setminus\Omega_0}$, *i.e.*, $\{F(\omega)\}_{\omega\in\Omega\setminus\Omega_0}$ is incomplete. $\square$

Now we are going to give simple ways for construction of many dual pairs of a given dual pair.

THEOREM 3.8. *Let $(F, G)$ be a dual pair for $\mathcal{H}$ and let $U$ and $V$ be two bounded operators on $\mathcal{H}$ such that $VU^* = I_{\mathcal{H}}$. Then, $(UF, VG)$ is a dual pair for $\mathcal{H}$.*

*Proof.* It is clear that if $F$ is a Bessel mapping with synthesis operator $T_F$ and $U$ is a bounded operator on $\mathcal{H}$, then $UF$ is a Bessel mapping with synthesis operator $T_{UF} = UT_F$. Hence $UF$ and $VG$ are Bessel mappings and

$$T_{VG}T_{UF}^* = VT_GT_F^*U^* = VIU^* = I_{\mathcal{H}}. \quad \square$$

COROLLARY 3.9. *If $(F, G)$ is a dual pair for $\mathcal{H}$ and $U$ is a unitary operator, then $(UF, UG)$ is a dual pair for $\mathcal{H}$.*

THEOREM 3.10. *Let $(F, G)$ be a dual pair for $\mathcal{H}$ and there exists a bounded operator $U \in L(\mathcal{H})$ such that $(F, UG)$ is a dual pair for $\mathcal{H}$. Then $U = I_{\mathcal{H}}$.*

*Proof.* For all $f, g \in \mathcal{H}$,

$$\langle f, U^*g\rangle = \int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), U^*g\rangle \mathrm{d}\mu = \int_\Omega \langle f, F(\omega)\rangle\langle UG(\omega), g\rangle \mathrm{d}\mu = \langle f, g\rangle.$$

Therefore, $U^* = I_{\mathcal{H}}$. Hence $U = I_{\mathcal{H}}$. $\square$

THEOREM 3.11. *Assume that $(F, G)$ and $(F, K)$ are dual pairs for $\mathcal{H}$. Then for all $\alpha \in \mathbb{C}$, $(F, \alpha G + (1-\alpha)K)$ is a dual pair for $\mathcal{H}$.*

*Proof.* Put $F_1 = \alpha G + (1-\alpha)K$. For all $f, g \in \mathcal{H}$, we have

$$\begin{aligned}\int_\Omega \langle f, F(\omega)\rangle\langle F_1(\omega), g\rangle \mathrm{d}\mu(\omega) &= \int_\Omega \langle f, F(\omega)\rangle\langle \alpha G(\omega) + (1-\alpha)K(\omega), g\rangle \mathrm{d}\mu(\omega)\\ &= \alpha \int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), g\rangle \mathrm{d}\mu(\omega)\\ &\quad + (1-\alpha)\int_\Omega \langle f, F(\omega)\rangle\langle K(\omega), g\rangle \mathrm{d}\mu(\omega)\\ &= \alpha\langle f, g\rangle + (1-\alpha)\langle f, g\rangle\\ &= \langle f, g\rangle. \quad \square\end{aligned}$$

Now, we want to find a relationship between the arbitrary continuous Riesz basis of $\mathcal{H}$. For this purpose, we need to following definitions and propositions from [2]. Denote by $L^2(\Omega, \mathcal{H})$ the set of all mapping $F : \Omega \to \mathcal{H}$ such that for all $f \in \mathcal{H}$, the functions $\omega \mapsto \langle f, F(\omega)\rangle$ defined almost everywhere on $\Omega$, belong to $L^2(\Omega)$.

*Definition* 3.12 ([2]). A Bessel mapping $F : \Omega \to \mathcal{H}$ is called $\mu$-complete, if

$$cspan\{F(\omega)\}_{\omega\in\Omega} = \left\{\int_\Omega \varphi(\omega)F(\omega)\mathrm{d}\mu(\omega); \quad \varphi \in L^2(\Omega)\right\}$$

is dense in $\mathcal{H}$.

It is worthwhile to mention that if $F$ is $\mu$-complete, then $\{F(\omega)\}_{\omega\in\Omega}$ is a complete subset of $\mathcal{H}$. The converse is also true when $0 < \mu(\{\omega\}) < +\infty$ for all $\omega \in \Omega$, since $span\{F(\omega)\}_{\omega\in\Omega} \subseteq cspan\{F(\omega)\}_{\omega\in\Omega}$.

PROPOSITION 3.13 ([2]). *Let $F$ be a Bessel mapping. The following are equivalent*

(1) *$F$ is $\mu$-complete;*
(2) *If $f \in \mathcal{H}$ so that $\langle f, F(\omega)\rangle = 0$ for almost all $\omega \in \Omega$, then $f = 0$.*

*Definition* 3.14 ([2]). A mapping $F \in L^2(\Omega, \mathcal{H})$ is called a continuous Riesz base for $\mathcal{H}$ with respect to $(\Omega, \mu)$, if $\{F(\omega)\}_{\omega\in\Omega}$ is $\mu$-complete and there are two positive numbers $A$ and $B$ such that

$$A\left(\int_{\Omega_1} |\phi(\omega)|^2 \mathrm{d}\mu(\omega)\right)^{\frac{1}{2}} \le \left\|\int_{\Omega_1} \phi(\omega)F(\omega)\mathrm{d}\mu(\omega)\right\| \le B\left(\int_{\Omega_1} |\phi(\omega)|^2 \mathrm{d}\mu(\omega)\right)^{\frac{1}{2}},$$

for every $\phi \in L^2(\Omega)$ and any measurable subset $\Omega_1$ of $\Omega$ with $\mu(\Omega_1) < +\infty$. The integral is taken in the weak sense and the constant $A$ and $B$ are called continuous Riesz base bounds. It is obvious that any continuous Riesz basis is a continuous frame.

*Definition* 3.15 ([2]). A Bessel mapping $F$ is said to be $L^2$-independent if $\int_\Omega \varphi(\omega)F(\omega)\mathrm{d}\mu(\omega) = 0$ for $\varphi \in L^2(\Omega, \mu)$, implies that $\varphi = 0$ a. e.

PROPOSITION 3.16 ([2]). *Let $F \in L^2(\Omega, \mathcal{H})$ be a continuous frame for $\mathcal{H}$. Then $F$ is a continuous Riesz base for $\mathcal{H}$ if and only if $F$ is $\mu$-complete and $L^2$-independent.*

PROPOSITION 3.17 ([2]). *Let $F \in L^2(\Omega, \mathcal{H})$ be a continuous frame. The following are equivalent*

(1) *$F$ is a continuous Riesz base for $\mathcal{H}$;*
(2) *$F$ is a Riesz-type continuous frame for $\mathcal{H}$;*
(3) *$T_F^*$ is onto.*

THEOREM 3.18. *Let $F$ and $G$ be two continuous Riesz bases for $\mathcal{H}$. Then there exists an invertible operator $\Theta \in L(\mathcal{H})$ such that $G = S_G\Theta^* F$.*

*Proof.* Assume $f \in \mathcal{H}$ such that $(T_G T_F^*)f = 0$. We have for all $\omega \in \Omega$, $T_G(T_F^* f(\omega)) = 0$. Hence $\int_\Omega \langle f, F(\omega)\rangle G(\omega)\mathrm{d}\mu(\omega) = 0$. Since $G$ is $L^2$-independent, then $\langle f, F(\omega)\rangle = 0$ for almost all $\omega \in \Omega$. The $\mu$-completeness of $F$ implies $f = 0$. Therefore $T_G T_F^*$ is one to one. According to Proposition 3.17, $T_F^*$ is onto, because $F$ is a continuous Riesz base. The synthesis $T_G$ is onto because $G$ is a continuous frame [24]. Hence, $T_G T_F^*$ is onto. Putting $\Theta = (T_G T_F^*)^{-1}$, for any $f, g \in \mathcal{H}$,

$$\begin{aligned}
\langle f, g\rangle &= \langle \Theta^{-1}\Theta f, g\rangle \\
&= \langle T_F^*\Theta f, T_G^* g\rangle \\
&= \int_\Omega \langle \Theta f, F(\omega)\rangle\langle G(\omega), g\rangle \mathrm{d}\mu(\omega) \\
&= \int_\Omega \langle f, \Theta^* F(\omega)\rangle\langle G(\omega), g\rangle \mathrm{d}\mu(\omega).
\end{aligned}$$

It follows that $\Theta^* F$ is a dual of $G$, but $G$ has only one dual. Hence $S_G^{-1}G = \Theta^* F$, or $G = S_G \Theta^* F$. □

*Definition* 3.19. [2] A continuous orthonormal basis for $\mathcal{H}$ with respect to $(\Omega, \mu)$ is a continuous Parseval frame $F$ for which

$$\left\| \int_\Omega \phi(\omega) F(\omega)\mathrm{d}\mu(\omega) \right\| = \|\phi\|_2, \quad \phi \in L^2(\Omega).$$

One may easily see that if $F$ is a continuous orthonormal base for $\mathcal{H}$, then it is a continuous Riesz base.

Corollary 3.20. *If $F$ and $G$ are two continuous orthonormal basis for $\mathcal{H}$, then there exists an invertible operator $\Theta \in L(\mathcal{H})$ such that $G = \Theta^* F$.*

## 4. APPROXIMATE DUALITY OF CONTINUOUS FRAMES

In Section 3, by definition of the dual frame, we saw that if the Bessel mapping $G$ is a dual of Bessel mapping $F$, then for all arbitrary elements $f, g \in \mathcal{H}$, we have the dual frame expansion $\langle f, g\rangle = \int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), g\rangle \mathrm{d}\mu(\omega)$. Unfortunately, it might be difficult, or even impossible, to calculate a dual frame explicitly. This limitation leads us to seek continuous frames that are "close to dual". For solving this problem in discrete frames, Christensen and Laugesen in [9] introduced the concept of *approximately dual frames.* By using their ideas in this section, we investigate and improve this notion for continuous frames. Here, we are generalizing the definition 3.1. of [9] to continuous cases and then we will obtain our results. For clarifying, some examples will be presented.

Throughout this section, we assume that $F$ and $G$ are Bessel mappings with synthesis operators $T_F$ and $T_G$, respectively.

*Definition* 4.1. Two Bessel mappings $F$ and $G$ are called approximately dual continuous frames for $\mathcal{H}$ if $\|I_{\mathcal{H}} - T_G T_F^*\| < 1$ or $\|I_{\mathcal{H}} - T_F T_G^*\| < 1$.

It is clear that in this case, $T_G T_F^*$ is an invertible operator.

*Example* 4.2. Similar to Example 3.2, let $\mathcal{H} = \mathbb{R}^2$ and $\{e_1, e_2\}$ be standard base for it. Also, let $0 < \varepsilon < 1$ be arbitrary. Put $\Omega = B_{\mathbb{R}^2}$ and $\lambda$ is the Lebesgue measure. Define the continuous frames $F$ and $G$ for $\mathbb{R}^2$ with respect to $(B_{\mathbb{R}^2}, \lambda)$ by

$$F(\omega) = \begin{cases} \frac{1}{\sqrt{\lambda(B_1)}} e_1, & \omega \in B_1, \\ \frac{1}{\sqrt{\lambda(B_2)}} e_2, & \omega \in B_2, \\ 0, & \omega \in B_3, \end{cases}$$

and

$$G(\omega) = \begin{cases} \frac{\varepsilon}{\sqrt{\lambda(B_1)}} e_1, & \omega \in B_1, \\ 0, & \omega \in B_2, \\ \frac{1}{\sqrt{\lambda(B_3)}} e_2, & \omega \in B_3, \end{cases}$$

where $\{B_1, B_2, B_3\}$ is a partition of $B_{\mathbb{R}^2}$. For all $x \in \mathbb{R}^2$

$$\begin{aligned} (T_G T_F^*)(x) &= \int_{B_{\mathbb{R}^2}} (T_F^* x)(\omega) G(\omega) \mathrm{d}\lambda(\omega) \\ &= \left( \int_{B_1} + \int_{B_2} + \int_{B_3} \right) (T_F^* x)(\omega) G(\omega) \mathrm{d}\lambda(\omega) \\ &= \int_{B_1} \langle x, \frac{\varepsilon}{\sqrt{\lambda(B_1)}} e_1 \rangle \frac{1}{\sqrt{\lambda(B_1)}} e_1 \mathrm{d}\lambda(\omega) + 0 + 0 \\ &= \varepsilon \langle x, e_1 \rangle e_1. \end{aligned}$$

Thus

$$\|x - (T_G T_F^*)(x)\|^2 = (1-\varepsilon)^2 |\langle x, e_1 \rangle|^2 + |\langle x, e_2 \rangle|^2 < \|x\|^2.$$

Therefore, $F$ and $G$ are approximately dual continuous frames for $\mathbb{R}^2$.

The following theorem shows that a Bessel mapping $F$ is a continuous frame for $\mathcal{H}$ with respect to $(\Omega, \mu)$, if there exists a Bessel mapping $G$ such that $F$ and $G$ are approximately dual frames.

Theorem 4.3. *If $F$ and $G$ are approximately dual continuous frames, then both $F$ and $G$ are continuous frames for $\mathcal{H}$ with respect to $(\Omega, \mu)$.*

*Proof.* Since $\|I_{\mathcal{H}} - T_G T_F^*\| < 1$, the operator $T_G T_F^*$ is an invertible operator on $\mathcal{H}$ and

$$\|(T_G T_F^*)^{-1}\| \le \frac{1}{1 - \|I_{\mathcal{H}} - T_G T_F^*\|}.$$

Denote by $B_F$ and $B_G$ the Bessel constants of $F$ and $G$, respectively. For all $f \in \mathcal{H}$

$$\begin{aligned}
\|f\| &\leq \|(T_G T_F^*)^{-1}\| \|T_G T_F^* f\| \\
&\leq \frac{1}{1-\|I_{\mathcal{H}}-T_G T_F^*\|} \sup_{\|g\|=1} |\langle T_G T_F^* f, g\rangle| \\
&\leq \frac{1}{1-\|I_{\mathcal{H}}-T_G T_F^*\|} \sup_{\|g\|=1} \int_\Omega |\langle f, F(\omega)\rangle \langle G(\omega), g\rangle| \mathrm{d}\mu(\omega) \\
&\leq \frac{1}{1-\|I_{\mathcal{H}}-T_G T_F^*\|} \sup_{\|g\|=1} \left(\int_\Omega |\langle f, F(\omega)\rangle|^2 \mathrm{d}\mu(\omega)\right)^{\frac{1}{2}} \left(\int_\Omega |\langle G(\omega), g\rangle|^2 \mathrm{d}\mu(\omega)\right)^{\frac{1}{2}} \\
&\leq \frac{1}{1-\|I_{\mathcal{H}}-T_G T_F^*\|} \sup_{\|g\|=1} \left(\int_\Omega |\langle f, F(\omega)\rangle|^2 \mathrm{d}\mu(\omega)\right)^{\frac{1}{2}} \sqrt{B_G}\, \|g\| \\
&= \frac{\sqrt{B_G}}{1-\|I_{\mathcal{H}}-T_G T_F^*\|} \left(\int_\Omega |\langle f, F(\omega)\rangle|^2 \mathrm{d}\mu(\omega)\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence $F$ is a continuous frame with lower bound $B_G^{-1}(1-\|I_{\mathcal{H}}-T_G T_F^*\|)^2$. Similarly, $G$ is a continuous frame with lower bound $B_F^{-1}(1-\|I_{\mathcal{H}}-T_G T_F^*\|)^2$. □

The following theorem shows that the sum of two approximately dual continuous frames is a continuous frame.

THEOREM 4.4. *If $F$ and $G$ are approximately dual continuous frames, then $F+G$ is a continuous frame for $\mathcal{H}$ with respect to $(\Omega, \mu)$.*

*Proof.* Suppose $T_F$ and $T_G$ be synthesis operators of $F$ and $G$ (respectively) and $A_F, B_F, A_G$ and $B_G$ be lower and upper bounds of $F$ and $G$, respectively. It is clear that $F+G$ is a Bessel mapping with Bessel constant $B_F+B_G$. Since $\|I_{\mathcal{H}}-T_G T_F^*\| < 1$ or $\|I_{\mathcal{H}}-T_F T_G^*\| < 1$, then $\|2I_{\mathcal{H}}-(T_G T_F^*+T_F T_G^*)\| < 2$, and as $T_G T_F^*+T_F T_G^*$ is self-adjoint, then by Lemma 2.2.2 in [21], $T_G T_F^*+T_F T_G^*$ is a positive operator. For each $f \in \mathcal{H}$, we have

$$\begin{aligned}
\int_\Omega |\langle f, (F+G)(\omega)\rangle|^2 \mathrm{d}\mu(\omega) &= \int_\Omega |\langle f, F(\omega)\rangle|^2 \mathrm{d}\mu(\omega) + \langle (T_G T_F^* + T_F T_G^*) f, f\rangle \\
&\quad + \int_\Omega |\langle f, G(\omega)\rangle|^2 \mathrm{d}\mu(\omega) \\
&\geq \int_\Omega |\langle f, F(\omega)\rangle|^2 \mathrm{d}\mu(\omega) + \int_\Omega |\langle f, G(\omega)\rangle|^2 \mathrm{d}\mu(\omega) \\
&\geq (A_F + A_G)\|f\|^2,
\end{aligned}$$

*i.e.*, the lower bound condition holds. □

It is clear that if $(F, G)$ is a dual pair for $\mathcal{H}$, then $F$ and $G$ are approximately dual continuous frames, but its converse isn't true in general. The following theorem shows that any approximate dual continuous frames are a kind of dual pair (as special sense).

THEOREM 4.5. *If $F$ and $G$ are approximately dual continuous frames, then there exists an invertible operator $\Theta : \mathcal{H} \to \mathcal{H}$ such that $(\Theta^* F, G)$ is a dual pair for $\mathcal{H}$.*

*Proof.* Since $\|I_{\mathcal{H}} - T_G T_F^*\| < 1$, then $T_G T_F^*$ is an invertible operator on $\mathcal{H}$. For each $f, g \in \mathcal{H}$

$$\begin{aligned}\langle f, g\rangle &= \langle (T_G T_F^*)(T_G T_F^*)^{-1} f, g\rangle \\ &= \int_\Omega \langle f, ((T_G T_F^*)^{-1})^* F(\omega)\rangle \langle G(\omega), g\rangle \mathrm{d}\mu(\omega).\end{aligned}$$

The result follows by putting $\Theta = (T_G T_F^*)^{-1}$. □

THEOREM 4.6. *Let $(F, G)$ be a dual pair for $\mathcal{H}$ and let $U$ and $V$ be two bounded operators on $\mathcal{H}$ such that $\|I_{\mathcal{H}} - VU^*\| < 1$. Then $UF$ and $VG$ are approximately dual continuous frames.*

*Proof.* Since $UF$ and $VG$ are Bessel mappings with synthesis operators $T_{UF} = UT_F$ and $T_{VG} = VT_G$ ( resp.), so we have

$$\|I_{\mathcal{H}} - T_{VG} T_{UF}^*\| = \|I_{\mathcal{H}} - VT_G T_F^* U^*\| = \|I_{\mathcal{H}} - VU^*\| < 1. \quad \square$$

COROLLARY 4.7. *If $(F, G)$ is a dual pair for $\mathcal{H}$ and $U$ is a unitary operator on $\mathcal{H}$, then $(UF, UG)$ is approximately dual continuous frames.*

We proceed this section with the following result which gives a sufficient and necessary condition for two continuous frames $F$ and $G$ under which they are approximately continuous frames. To this end, recall that every bounded and positive operator $U : \mathcal{H} \to \mathcal{H}$ has a unique bounded and positive square root $U^{\frac{1}{2}}$. Moreover, if the operator $U$ is self-adjoint (resp. invertible), then $U^{\frac{1}{2}}$ is also self-adjoint (resp. invertible), see A.6.7 of [8].

THEOREM 4.8. *Let $F$ be continuous frame and $G$ a Bessel mapping for $\mathcal{H}$ with upper bounds $B_F$ and $B_G$, respectively. Then $F$ and $G$ are approximately dual continuous frames if and only if there exists a bounded operator $D \in \mathcal{B}(\mathcal{H})$ such that*

$$T_F T_G^* = S_F^{\frac{1}{2}} D, \quad DD^* \leq B_G I_{\mathcal{H}}, \quad \|I_{\mathcal{H}} - S_F^{\frac{1}{2}} D\| < 1.$$

*Proof.* Since $F$ is a continuous frame for $\mathcal{H}$, then $S_F$ is a bounded and positive operator. Hence it has a unique bounded square root $S_F^{\frac{1}{2}}$. The proof of the "only if" part is trivial. To prove the "if" part, suppose that $F$ and $G$ are approximately dual continuous frames. For each $f \in \mathcal{H}$, we have

$$\begin{aligned}
\|T_G T_F^* f\| &= \sup_{\|g\|=1} |\langle T_G T_F^* f, g\rangle| \\
&= \sup_{\|g\|=1} |\int_\Omega \langle f, F(\omega)\rangle \langle G(\omega), g\rangle \mathrm{d}\mu(\omega)| \\
&\leq \sup_{\|g\|=1} \left(\int_\Omega |\langle f, F(\omega)\rangle|^2 \mathrm{d}\mu(\omega)\right)^{\frac{1}{2}} \left(\int_\Omega |\langle g, G(\omega)\rangle|^2 \mathrm{d}\mu(\omega)\right)^{\frac{1}{2}} \\
&\leq \sqrt{B_G} \left(\int_\Omega |\langle f, F(\omega)\rangle|^2 \mathrm{d}\mu(\omega)\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, we have

$$\langle (T_F T_G^*)(T_F T_G^*)^* f, f\rangle \leq B_G \langle S_F f, f\rangle, \quad f \in \mathcal{H}.$$

Thus for all $f \in \mathcal{H}$

$$(T_F T_G^*)(T_F T_G^*)^* \leq B_G S_F^{\frac{1}{2}} S_F^{\frac{1}{2}}.$$

The above inequality results:

(1) According to Theorem 1 of [11], there exists a bounded operator $D \in B(\mathcal{H})$ such that $T_F T_G^* = S_F^{\frac{1}{2}} D$.
(2) Theorem 2.2.5 in [21] implies that $DD^* \leq B_G I_{\mathcal{H}}$. $\square$

Now, we can prove the following theorems.

THEOREM 4.9. *Let $F$ be a continuous frame for $\mathcal{H}$ with respect to $(\Omega, \mu)$. Then $F$ and the Bessel mapping $G$ are approximately dual continuous frames if and only if*

$$G = D^* S_F^{-\frac{1}{2}} F + K,$$

*where $D$ is a bounded operator on $\mathcal{H}$ for which $\|I_h - S_F^{\frac{1}{2}} D\| < 1$ and $K$ is a Bessel mapping with property $T_F T_K^* = 0$.*

*Proof.* First, we put $G = D^* S_F^{-\frac{1}{2}} F + K$, then for each $f \in \mathcal{H}$, the $\omega \mapsto \langle f, G(\omega)\rangle$ is a measurable function. Also, for each $f \in \mathcal{H}$, we have

$$\int_\Omega |\langle f, G(\omega)\rangle|^2 \mathrm{d}\mu(\omega) \leq (B_K + \|D\|^2)\|f\|^2.$$

Therefore, $G$ is a Bessel mapping with Bessel bound $B_G = B_K + \|D\|^2$.

Now, for all $f \in \mathcal{H}$ and $\omega \in \Omega$, we have

$$\begin{aligned}(T_G^* f)(\omega) &= \langle f, D^* S_F^{-\frac{1}{2}} F(\omega)\rangle + \langle f, K(\omega)\rangle \\ &= \langle S_F^{-\frac{1}{2}} Df, F(\omega)\rangle + \langle f, K(\omega)\rangle \\ &= T_F^*(S_F^{-\frac{1}{2}} Df)(\omega) + (T_K^* f)(\omega)\end{aligned}$$

and

$$\begin{aligned}T_F T_G^* f &= T_F T_F^*(S_F^{-\frac{1}{2}} Df) + T_F T_K^* f \\ &= S_F^{\frac{1}{2}} Df.\end{aligned}$$

Hence $T_F T_G^* = S_F^{\frac{1}{2}} D$. Moreover,

$$\langle DD^* f, f\rangle = \langle D^* f, D^* f\rangle = \|D^* f\|^2 \leq \|D\|^2 \|f\|^2 \leq B_G \|f\|^2 = B_G \langle f, f\rangle,$$

for all $f \in \mathcal{H}$. Now, we use previous theorem to conclude that $F$ and $G$ are approximately dual continuous frames.

Conversely, let $F$ and $G$ be approximately dual continuous frames. According to Theorem 4.8, there exists an invertible operator $D$ in $\mathcal{B}(\mathcal{H})$ such that $T_F T_G^* = S_F^{\frac{1}{2}} D$ and $\|I_{\mathcal{H}} - S_F^{\frac{1}{2}} D\| < 1$. Put $K = G - D^* S_F^{-\frac{1}{2}} F$. It is clear that $K$ is a Bessel mapping with bound $B_K = B_G + \|D\|^2$. We have for all $f \in \mathcal{H}$ and all $\omega \in \Omega$,

$$\begin{aligned}(T_G^* - T_F^* S_F^{-\frac{1}{2}} Df)(\omega) &= \langle f, G(\omega)\rangle - \langle S_F^{-\frac{1}{2}} Df, F(\omega)\rangle \\ &= \langle f, G(\omega)\rangle - \langle f, D^* S_F^{-\frac{1}{2}} F(\omega)\rangle \\ &= \langle f, G(\omega) - D^* S_F^{-\frac{1}{2}} F(\omega)\rangle \\ &= \langle f, K(\omega)\rangle \\ &= (T_K^* f)(\omega).\end{aligned}$$

Thus

$$T_F T_K^* = T_F T_G^* - T_F T_F^* S_F^{-\frac{1}{2}} D = T_F T_G^* - S_F^{\frac{1}{2}} D = 0,$$

and this completes the proof. □

THEOREM 4.10. *Let $F$ be a continuous frame for $\mathcal{H}$ with respect to $(\Omega, \mu)$. Then $F$ and the Bessel mapping $G$ are approximately dual continuous frames if and only if*

$$G = D^* S_F^{-\frac{1}{2}} F - F + S_F K,$$

*where $D$ is a bounded operator on $\mathcal{H}$ for which $\|I_{\mathcal{H}} - S_F^{\frac{1}{2}} D\| < 1$ and $K$ is a*

*Bessel mapping such that $(F, K)$ is a dual pair for $\mathcal{H}$.*

*Proof.* Put $G = D^* S_F^{-\frac{1}{2}} F - F + S_F K$ is a Bessel mapping with Bessel constant $B_G = \|D\|^2 + B_F + B_G$. Since $T_G^* = T_F^* S_F^{-\frac{1}{2}} D - T_F^* + T_K^* S_F$, then $T_F T_G^* = S_F^{\frac{1}{2}} D$. Moreover,

$$DD^* \leq B_G I_{\mathcal{H}}.$$

Conversely, let $F$ and $G$ be approximately dual continuous frames. According to Theorem 4.8, there exists an invertible operator $D$ in $\mathcal{B}(\mathcal{H})$ such that $T_F T_G^* = S_F^{\frac{1}{2}} D$ and $\|I_{\mathcal{H}} - S_F^{\frac{1}{2}} D\| < 1$. Put $K = S_F^{-1} G - S_F^{-1} D^* S_F^{-\frac{1}{2}} F - S_F^{-1} F$. It is clear that $K$ is a Bessel mapping with bound $B_K = (B_G + \|D\|^2 + B_F)\|S_F^{-1}\|^2$. Since

$$T_K^* = T_G^* S_F^{-1} - T_F^* S_F^{-\frac{1}{2}} D S_F^{-1} + T_F^* S_F^{-1},$$

then $T_F T_K^* = I_{\mathcal{H}}$ and this completes the proof. $\square$

## 4.1. ON PERTURBATION OF CONTINUOUS FRAMES

Perturbation theory is a very important concept in several areas of mathematics. It went back to classical perturbation results by Paley and Wienr in 1934. The perturbations of discrete frames have been discussed in [6]. For continuous frames, it was studied in [4, 24]. In this subsection, we are giving some results on perturbation of continuous frames from the point of view of the duality notion.

THEOREM 4.11. *Let $F$ be a Parseval continuous frame and $G$ be a Bessel mapping. Assume that there exist constants $\lambda, \gamma \geq 0$ such that*

$$\begin{aligned}\| \int_\Omega \phi(\omega)(F(\omega) - G(\omega)) \mathrm{d}\mu(\omega)\| \leq & \lambda \| \int_\Omega \phi(\omega) F(\omega) \mathrm{d}\mu(\omega)\| \\ & + \gamma \left( \int_\Omega |\phi(\omega)|^2 \mathrm{d}\mu(\omega) \right)^{\frac{1}{2}},\end{aligned}$$

*for all $\phi \in L^2(\Omega)$. If $\lambda + \gamma < 1$, then $(F, G)$ is an approximately dual continuous frames.*

*Proof.* For all $f \in \mathcal{H}$

$$\begin{aligned}\|f - (T_G T_F^*)(f)\| &= \| \int_\Omega \langle f, F(\omega)\rangle (F(\omega) - G(\omega)) \mathrm{d}\mu(\omega)\| \\ &\leq \lambda \| \int_\Omega \langle f, F(\omega)\rangle F(\omega) \mathrm{d}\mu(\omega)\|\end{aligned}$$

$$
\begin{aligned}
&+\gamma\left(\int_\Omega |\langle f, F(\omega)\rangle|^2 \mathrm{d}\mu(\omega)\right)^{\frac{1}{2}}\\
&< \|f\|.
\end{aligned}
$$

Thus $\|I_{\mathcal{H}} - T_G T_F^*\| < 1$. Consequently, $F$ and $G$ are approximately dual continuous frames. $\square$

THEOREM 4.12. *Let $F$, $G$ and $K$ be Bessel mappings and $B_G$ be the Bessel constant of $G$. Assume that there exists $\lambda > 0$ such that $\lambda B_G < 1$ and for all $f \in \mathcal{H}$*

$$\int_\Omega |\langle f, F(\omega) - K(\omega)\rangle|^2 \mathrm{d}\mu(\omega) \leq \lambda \|f\|^2.$$

*If $(F, G)$ is a dual pair for $\mathcal{H}$, then $G$ and $K$ are approximately dual continuous frames.*

*Proof.* Since for any $f \in \mathcal{H}$, $(T_F^* f - T_K^* f)(\omega) = \langle f, F(\omega) - K(\omega)\rangle$, then

$$\|(T_F^* - T_K^*) f\|^2 = \int_\Omega |\langle f, F(\omega) - K(\omega)\rangle|^2 \leq \lambda \|f\|^2$$

and consequently, $\|T_F^* - T_H^*\| \leq \sqrt{\lambda}$. Now we have

$$\|I_{\mathcal{H}} - T_G T_K^*\| = \|T_G(T_F^* - T_K^*)\| \leq \|T_G\| \|T_F^* - T_K^*\| \leq \sqrt{\lambda B_G} < 1. \quad \square$$

THEOREM 4.13. *Let $(F, G)$ be a dual pair for $\mathcal{H}$ and $K : \Omega \to \mathcal{H}$ be a Bessel mapping. Assume that there exist constants $\lambda, \gamma \geq 0$ such that*

$$
\begin{aligned}
\|\int_\Omega \phi(\omega)(F(\omega) - K(\omega)) \mathrm{d}\mu(\omega)\| \leq & \lambda \|\int_\Omega \phi(\omega) F(\omega) \mathrm{d}\mu(\omega)\| \\
&+ \gamma\left(\int_\Omega |\phi(\omega)|^2 \mathrm{d}\mu(\omega)\right)^{\frac{1}{2}},
\end{aligned}
$$

*for all $\phi \in L^2(\Omega)$. If $\lambda + \gamma\sqrt{B_G} < 1$, Then $G$ and $K$ are approximately dual continuous frames, where $B_G$ is Bessel constant of $G$.*

*Proof.* For all $f \in \mathcal{H}$

$$
\begin{aligned}
\|f - (T_K T_G^*)(f)\| &= \|\int_\Omega \langle f, G(\omega)\rangle (F(\omega) - K(\omega)) \mathrm{d}\mu(\omega)\| \\
&\leq \lambda \|\int_\Omega \langle f, G(\omega)\rangle F(\omega) \mathrm{d}\mu(\omega)\| \\
&\quad + \gamma\left(\int_\Omega |\langle f, G(\omega)\rangle|^2 \mathrm{d}\mu(\omega)\right)^{\frac{1}{2}} \\
&\leq (\lambda + \gamma\sqrt{B_G}) \|f\|.
\end{aligned}
$$

Thus $\|I_{\mathcal{H}} - T_K T_G^*\| < 1$. $\square$

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