ROMAN DOMINATION STABILITY IN GRAPHS

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Given a graph G, the minimum number of vertices whose removal changes the Roman domination number of G is called the Roman domination stability number of G. In this paper, we present various bounds and characterizations for the Roman domination stability number, and show that the decision problem for determining the Roman domination stability number is NP-hard even when restricted to bipartite graphs.

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1. INTRODUCTION

See [12] for notation and terminology not given here. Let $G = (V(G), E(G))$ be a simple graph of order n with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, let $N_G(v) = \{u|uv \in E(G)\}$ denotes the *open neighborhood* of v and $N_G[v] = N_G(v) \cup \{v\}$ denotes the *closed neighborhood* of v. For a set $S \subseteq V(G)$, we denote $N(S) = \cup_{v\in S} N(v)$ and $N[S] = \cup_{v\in S} N[v]$. The private neighborhood $pn(v, S)$ of a vertex $v \in S$ is defined by $pn(v, S) = N[v] - N[S - \{v\}]$. Equivalently, $pn(v, S) = \{u \in V(G)|N(u) \cap S = \{v\}\}$. Each vertex in $pn(v, S)$ is called a private neighbor of v. The *degree* of a vertex v, $\deg_G(v)$, or just $\deg(v)$, in a graph G denotes the number of neighbors of v in G. We refer to $\Delta(G)$ and $\delta(G)$ as the *maximum degree* and the *minimum degree* among the vertices of G, respectively. A *leaf* in a graph is a vertex of degree one, and a *support vertex* is one that is adjacent to a leaf. A *strong support vertex* is a support vertex adjacent to at least two leaves. A *double-star* is a tree with precisely two (central) vertices that are not leaves. If T is a rooted tree, then for each vertex v, we denote by T_v the sub-rooted tree rooted at v. If S is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of G induced by S. The *diameter* $diam(G)$ of a graph G is the maximum distance $d(x, y)$ among all pairs of distinct vertices x and y of G.

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A set $S \subseteq V(G)$ is a *dominating set* of G, if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set.

A *Roman dominating function* (or just RDF) is a function $f : V \longrightarrow \{0, 1, 2\}$ with the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a vertex $u \in N(v)$ with $f(u) = 2$. The *weight* of an RDF f is the sum $w(f) = f(V) = \sum_{v \in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. A Roman dominating function on G with weight $\gamma_R(G)$ is called a γ_R-function of G or just a $\gamma_R(G)$-function. If f is an RDF for G, $f(v) = 0$ and $f(u) = 2$ for a vertex $u \in N(v)$, then we say that v is Roman dominated by u, or just f Roman dominates v. For an RDF f we denote by V_i (or V_i^f to refer to f) the set of all vertices x with $f(x) = i$, for $i = 0, 1, 2$. Thus an RDF f can be represented by a triple (V_0^f, V_1^f, V_2^f). Roman domination, as originally defined and discussed by Stewart [23], and ReVelle and Rosing [19], and subsequently developed by Cockayne *et al.* [6].

A graph G is called *domination vertex critical* if removal of any vertex decreases the domination number. Similarly, a graph G is called *domination vertex super-critical* if removal of any vertex increases the domination number. A *domination-critical* vertex in a graph G is a vertex whose removal decreases the domination number. The concept of domination criticality has been introduced by Bauer *et al.* [1] and Sumner *et al.* [25], and has been further studied by several authors, see for example [2–5, 8, 10, 11, 13, 14, 21, 22, 24]. Jafari Rad *et al.* [15, 18] considered the concept of criticality on Roman domination number. A graph G is called *Roman domination vertex critical* if removal of any vertex decreases the Roman domination number. Similarly, a graph G is called *Roman domination vertex super-critical* if removal of any vertex increases the Roman domination number. A *Roman domination-critical* vertex in a graph G is a vertex whose removal decreases the Roman domination number. Similar concepts were studied by Samodivkin [20]. For references on critical concept on Roman domination, see also [17].

The concept of domination stability in graph has been introduced by Bauer *et al.* [1]. The *domination stability* of a graph G is the minimum number of vertices whose removal changes the domination number. We denote the domination stability of a graph G by $st_\gamma(G)$. Domination stability has been further studied in [16].

In this paper, we consider the stability for Roman domination. The *Roman domination stability* number of G, denoted by $st_{\gamma_R}(G)$, is the minimum number of vertices whose removal changes the Roman domination number of

G. Note that the Roman domination-stability number is defined for every graph G with $\gamma_R(G) > 1$, since the removal of $|V(G)| - 1$ vertices of G results a K_1 with Roman domination number one. We also define $st_{\gamma_r}(K_1) = 0$.

OBSERVATION 1. *For a graph G, $st_{\gamma_R}(G) = 0$ if and only if $G = K_1$.*

In Section 2, we show that the decision problem for determining the Roman domination stability number is NP-hard even when restricted to bipartite graphs. In Section 3, we present various bounds for the Roman domination stability number, and characterize graphs achieving equality for the bounds.

We make use of the following.

PROPOSITION 2 (Jafari Rad, *et al.* [15]). *Let v be a vertex of a graph G. Then $\gamma_R(G - v) < \gamma_R(G)$ if and only if there is a $\gamma_R(G)$–function f with $f(v) = 1$.*

COROLLARY 3. *If $st_{\gamma_R}(G) > 1$ then $V_1^f = \emptyset$ for every $\gamma_R(G)$-function f.*

Note that, by Corollary 3, if $st_{\gamma_R}(G) > 1$ then $\gamma_R(G)$ is even.

2. NP-HARDNESS OF ROMAN DOMINATION STABILITY PROBLEM

The decision problem of Roman domination-stability problem is stated in this paper as follows:

Roman domination stability number problem (RDSNP)
Instance: Graph $G = (V, E)$ and the Roman domination number $\gamma_R(G)$.
Question: Is $st_{\gamma_R}(G) > 0$?

We use a transformation from 3-SAT, which was proven to be NP-complete in [9]. The problem 3-SAT is the problem of determining if there exists an interpretation that satisfies a given Boolean formula. The formula in 3-SAT is given in conjunctive normal form, where each clause contains three literals. We assume that the formula contains the instance of any literal u and its negation $\bar{u}$ (in the other case all clauses containing the literal u are satisfied by the true assignment of u).

THEOREM 4. *Roman domination-stability problem is NP-hard even for bipartite graphs.*

Proof. Given an instance, the set of literals $U = \{u_1, u_2, \ldots, u_n\}$ and the set of clauses $C = \{c_1, c_2, \ldots, c_m\}$ of 3-SAT, we construct the following graph G. For each literal u_i construct a graph G_i with vertex set $V(G_i) = \{u_i, a_i, a_i', \bar{u}_i, d_i, e_i, e_i', b_i\}$ and edge set

$$E(G_i) = \{u_i a_i, u_i a_i', u_i b_i, \bar{u}_i a_i, \bar{u}_i a_i', \bar{u}_i d_i, d_i e_i, d_i e_i', b_i e_i, b_i e_i'\}.$$

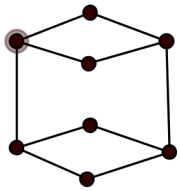

Fig. 1. The graph in G_i.

Figure 1 shows the graph G_i.

For each clause C_j we have a clause vertex c_j, where vertex c_j is adjacent to the literal vertices that correspond to the three literals it contains. For example, if $C_j = (u_1 \vee \bar{u}_2 \vee u_3)$, then the clause vertex c_j is adjacent to the literal vertices u_1, $\bar{u}_2$ and u_3. Then add path $P_3 = x_1xx_2$, and join x to every clause vertex c_j, for $i = 1, 2, ..., m$. Hence x is of degree $m+2$. Clearly, we can see that G is a bipartite graph and it can be built in polynomial time.

Let f be a $\gamma_R(G)$-function. Clearly, $\sum_{u\in V(G_i)} f(u) \geq 4$ for $i = 1, 2, ..., n$. Since $f(x) + f(x_1) + f(x_2) \geq 2$, we obtain that $\gamma_R(G) \geq 4n + 2$. On the other hand, assigning 2 to x, u_i and d_i for $i = 1, 2, ..., n$, and 0 to other vertices of G yields an RDF for G, implying that $\gamma_R(G) \leq 4n + 2$. We deduce that $\gamma_R(G) = 4n + 2$.

Assume that C has a satisfying truth assignment t. We show that $st_{\gamma_R}(G) > 1$. Let $v \in V(G)$, and g be a $\gamma_R(G - v)$-function. Assume that $v \in V(G_i)$ for some integer i. Clearly $\sum_{u\in V(G_i)-\{v\}} g(v) \geq 4$. Since $\sum_{u\in V(G_j)} g(u) \geq 4$ for each $j \in \{1, 2, ..., n\} - \{i\}$ and $g(x) + g(x_1) + g(x_2) \geq 2$, we obtain that $w(g) = \gamma_R(G - v) \geq 4n + 2$. Assume that $v = u_i$. Then assigning 2 to x, $\overline{u_i}$, b_i and u_j, d_j for each $j \in \{1, 2, ..., n\} - \{i\}$, and 0 to other vertices of $G - v$, yields an RDF for $G - v$, implying that $\gamma_R(G - v) \leq 4n + 2$. We deduce that $\gamma_R(G - v) = 4n + 2 = \gamma_R(G)$. Similarly, we have $\gamma_R(G - v) = \gamma_R(G)$ if $v \in \{\overline{u_i}, b_i, d_i\}$. Assume that $v = a_i$. Observe that $\sum_{u\in V(G_i)-\{v\}} g(v) \geq 4$, and we obtain that $w(g) \geq \gamma_R(G - v) \geq 4n + 2$. On the other hand assigning 2 to x, u_i and d_i for $i = 1, 2, ..., n$, and 0 to other vertices of G yields an RDF for G, implying that $\gamma_R(G) \leq 4n + 2$. We deduce that $\gamma_R(G - v) = 4n + 2 = \gamma_R(G)$. Similarly, we have $\gamma_R(G - v) = \gamma_R(G)$ if $v \in \{a'_i, e_i, e'_i\}$. If $v \in \{c_1, c_2, ..., c_m, x_1, x_2\}$, then similarly we can see that $\gamma_R(G - v) = \gamma_R(G)$. Now assume that $v = x$. Clearly $w(g) \geq 4n+2$, since $\sum_{u\in V(G_i)} g(u) \geq 4$ for $i = 1, 2, ..., n$, and $g(x_1)+g(x_2) = 2$. We form a set D as follows. For each $i = 1, 2, ..., n$ if $t(u_i) = T$ then $u_i, d_i \in D$, and if $t(u_i) = F$ then $\overline{u_i}, b_i \in D$. Clearly $|D| = 2n$. We define a function h on $V(G) - \{v\}$ by assigning 2 to every vertex of D, 1 to x_1 and x_2, and 0 to each other vertex of $G - v$. Since t is a truth assignment, h is an RDF for $G - v$,

implying that $\gamma_R(G-v) \leq 4n+2 = \gamma_R(G)$, and so $\gamma_R(G-v) = \gamma_R(G)$. We conclude that $\gamma_R(G-v) = \gamma_R(G)$. Consequently, $st_{\gamma_R}(G) > 1$.

Assume now that C does not have a satisfying truth assignment. We consider the graph $G-x$. Let h_1 be a $\gamma_R(G-x)$-function. Clearly $h_1(x_1) = h_1(x_2) = 1$, and $\sum_{u \in V(G_i)} h_1(u) \geq 4$ for each $i = 1, 2, ..., n$, and thus $w(h_1) \geq 4n+2$. Assume that $w(h_1) = 4n+2$. Clearly there is no integer $i \in \{1, 2, ..., n\}$ such that $h_1(u_i) = h_1(\overline{u_i}) = 2$, since then e_i and e_i' are not Roman dominated by h_1. Let $A = \{u_i : h_1(u_i) = 2 \text{ or } h_1(\overline{u_i}) = 2, i = 1, 2, ..., n\}$. Since h_1 is an RDF for $G-v$, any vertex of $\{c_1, c_2, ..., c_m\}$ is dominated by a vertex of A. Now we define an assignment $t_1 : U \to \{T, F\}$ by $t(u_i) = T$ if $u_i \in A$ and $t(u_i) = F$ if $\overline{u_i} \in A$. Then t_1 is a truth assignment for C, a contradiction. Thus $w(h_1) > 4n+2$. We conclude that $st_{\gamma_R}(G) = 1$, as desired. $\square$

A vertex v in a graph G is called a *Roman domination-critical* vertex if $\gamma_R(G-v) < \gamma_R(G)$. Jafari Rad and Volkmann [17] studied some graphs with no Roman domination-critical vertex. Since the class of graphs G with $st_{\gamma_R}(G) > 1$ is a subclass of graphs with no Roman domination-critical vertex, we have the following result.

THEOREM 5. *The decision problem for determining graphs with no Roman domination-critical vertex is NP-hard even for bipartite graphs.*

3. BOUNDS ON THE ROMAN DOMINATION STABILITY NUMBER

In this section, we present various bounds for the Roman domination stability number of a graph. We begin with the following.

THEOREM 6. *For any graph G, $st_{\gamma_R}(G) \leq \delta(G) + 1$.*

Proof. Let G be a graph. Suppose that $st_{\gamma_R}(G) > \delta(G) + 1$. Let x be a vertex of G with $\deg(x) = \delta(G)$. Then $\gamma_R(G - N(x)) = \gamma_R(G)$, and $\gamma_R(G - N[x]) = \gamma_R(G)$. Since x is an isolated vertex in $G - N(x)$, any $\gamma_R(G - N(x))$-function assigns 1 to x, and by Proposition 2, $\gamma_R(G - N[x]) < \gamma_R(G - N(x))$, a contradiction. Thus $st_{\gamma_R}(G) \leq \delta(G) + 1$. $\square$

PROPOSITION 7. *For a graph G of order $n \geq 2$, $st_{\gamma_R}(G) \leq n-1$, with equality if and only if $G = K_n$.*

Proof. Let G be a graph of order $n \geq 2$. The result follows by Theorem 6 if $\delta(G) < n-1$. Thus assume that $\delta(G) = n-1$. Then $G = K_n$. Since $\gamma_R(K_n) = 2$, we have $st_{\gamma_R}(G) = st_{\gamma_R}(K_n) = n-1$. The equality part is now obvious. $\square$

THEOREM 8. *For a graph G of order $n \geq 2$, $st_{\gamma_R}(G) \leq 2n/\gamma_R(G) - 1$. This bound is sharp.*

Proof. Let G be a graph of order $n \geq 2$. Note that $\gamma_R(G) \geq 2$. Clearly the result holds if $st_{\gamma_R}(G) = 1$, since $\gamma_R(G) \leq n$. Thus assume that $st_{\gamma_R}(G) > 1$. Let f be a $\gamma_R(G)$-function. By Corollary 3, $V_1^f = \emptyset$. Let $v \in V_2^f$ be a vertex with minimum number of private neighbors in V_0^f. Then Removal of its private neighbors in V_0^f produces a graph with Roman domination number less than $\gamma_R(G)$. We conclude that $st_{\gamma_R}(G) \leq \deg(v) \leq n/|V_2^f| - 1 = 2n/\gamma_R(G) - 1$. To see the sharpness, consider a cycle C_n, where $n \equiv 0 \pmod 3$, and observe that $\gamma_R(C_n) = 2n/3$ and $st_{\gamma_R}(C_n) = 2$. $\square$

PROPOSITION 9. *For a graph G of order $n \geq 2$ with $\gamma_R(G) > 2$, $st_{\gamma_R}(G) \leq n - \delta(G) - 1$, with equality if and only if G is a $(n-2)$-regular graph.*

Proof. Let G be a graph of order $n \geq 2$ with $\gamma_R(G) > 2$. Clearly $G \neq K_n$ and so $\delta(G) \leq n - 2$. If $st_{\gamma_R}(G) = 1$ then $st_{\gamma_R}(G) = 1 \leq n - \delta(G) - 1$, since $\delta(G) \leq n - 2$. Thus assume that $st_{\gamma_R}(G) > 1$. Let f be a $\gamma_R(G)$-function. By Corollary 3, we have $V_1^f = \emptyset$. Then $\gamma_R(G) \geq 4$. Let $x, y \in V_2^f$. Let G' be obtained from G by removal of $V(G) - (N[x] \cup \{y\})$. Then $\gamma_R(G') = 3$, and so $st_{\gamma_R}(G) \leq n - (1 + \deg(x) + 1) \leq n - \delta(G) - 2 < n - \delta(G) - 1$.

Next we prove the equality part. Let G be a graph with $\gamma_R(G) > 2$ and $st_{\gamma_R}(G) = n - \delta(G) - 1$. From the above argument we have $st_{\gamma_R}(G) = 1$ and so $1 = st_{\gamma_R}(G) = n - \delta(G) - 1$, implying that $\delta(G) = n - 2$. Since $G \neq K_n$, we obtain that $\Delta(G) = n - 2$. Consequently, G is a $(n-2)$-regular graph. Conversely, assume that G is a $(n-2)$-regular graph. Then $\gamma_R(G) = 3$, and evidently $st_{\gamma_R}(G) = 1 = n - \delta(G) - 1$. $\square$

We next give Nordhaus Gaddum type inequalities for the sum of the Roman domination stabilities of a graph and its complement.

PROPOSITION 10. *For a graph G of order $n \geq 3$, $st_{\gamma_R}(G) + st_{\gamma_R}(\overline{G}) \leq n$, with equality if and only if G is K_n or $\overline{K_n}$.*

Proof. Let G be a graph of order $n \geq 3$. Clearly $\gamma_R(G) \geq 2$ and $\gamma_R(\overline{G}) \geq 2$. Assume that $\gamma_R(G) = 2$. Then $\delta(G) > 0$, $\Delta(G) = n - 1$, and so $\overline{G}$ has an isolated vertex. Clearly $st_{\gamma_R}(\overline{G}) = 1$. By Theorem 8, $st_{\gamma_R}(G) + st_{\gamma_R}(\overline{G}) \leq 2n/2 - 1 + 1 = n$. Thus assume that $\gamma_R(G) \geq 3$ and by symmetry, $\gamma_R(\overline{G}) \geq 3$. If $\gamma_R(G) = 3$ then we observe that $st_{\gamma_R(G)} = 1$, and by Theorem 8, $st_{\gamma_R}(G) + st_{\gamma_R}(\overline{G}) \leq 1 + 2n/3 - 1 < n$. Thus assume that $\gamma_R(G) \geq 4$ and by symmetry, $\gamma_R(\overline{G}) \geq 4$. Now Proposition 8 implies that

$$\begin{aligned} st_{\gamma_R}(G) + st_{\gamma_R}(\overline{G}) &\leq 2n/\gamma_R(G) - 1 + 2n/\gamma_R(\overline{G}) - 1 \\ &\leq n/2 + n/2 - 2 < n. \end{aligned}$$

Next, we prove the equality part. Assume that $st_{\gamma_R}(G) + st_{\gamma_R}(\overline{G}) = n$. From the above argument, we may assume, without loss of generality, that

$\gamma_R(G) = 2$. Then $\Delta(G) = n-1$, and so $\overline{G}$ has an isolated vertex. Thus $st_{\gamma_R}(\overline{G}) = 1$. Now we obtain that $st_{\gamma_R}(G) = n-1$. By Proposition 7, G is a complete graph. The converse is obvious. $\square$

Thus from Proposition 10 we obtain that $st_{\gamma_R}(G) + st_{\gamma_R}(\overline{G}) \leq n-1$ if $\gamma_R(G) \geq 3$ and $\gamma_R(\overline{G}) \geq 3$. We next improve this bound.

PROPOSITION 11. *For a graph G of order $n \geq 3$, with $\gamma_R(G) \geq 3$ and $\gamma_R(\overline{G}) \geq 3$, $st_{\gamma_R}(G) + st_{\gamma_R}(\overline{G}) \leq n-2$. This bound is sharp.*

Proof. Let G be a graph of order $n \geq 3$ with $\gamma_R(G) \geq 3$ and $\gamma_R(\overline{G}) \geq 3$. By Proposition 10, $st_{\gamma_R}(G) + st_{\gamma_R}(\overline{G}) \leq n-1$. Suppose that $st_{\gamma_R}(G) + st_{\gamma_R}(\overline{G}) = n-1$. If $\gamma_R(G) \geq 4$ and $\gamma_R(\overline{G}) \geq 4$, then by Proposition 8,

$$\begin{aligned} st_{\gamma_R}(G) + st_{\gamma_R}(\overline{G}) &\leq 2n/\gamma_R(G) - 1 + 2n/\gamma_R(\overline{G}) - 1 \\ &\leq n/2 + n/2 - 2 < n-1. \end{aligned}$$

This is a contradiction. Thus we may assume, without loss of generality, that $\gamma_R(G) = 3$. Then $st_{\gamma_R}(G) = 1$ and so $st_{\gamma_R}(\overline{G}) = n-2$. Furthermore, $\Delta(G) = n-2$ and $\delta(\overline{G}) = 1$. By Proposition 7, $\overline{G}$ is not a $(n-2)$ -regular graph. Then by Proposition 9, $st_{\gamma_R}(\overline{G}) \leq n - \delta(\overline{G}) - 2 < n-2$, a contradiction. We conclude that $st_{\gamma_R}(G) + st_{\gamma_R}(\overline{G}) \leq n-2$. To see the sharpness, consider a path P_4. $\square$

We note that $st_{\gamma_R(G)}$ is not bounded above or bellow by $\gamma_R(G)$. In fact, the following proposition shows that the differences $\gamma_R(G) - st_{\gamma_R}(G)$ and $st_{\gamma_R}(G) - \gamma_R(G)$ can be arbitrarily large.

PROPOSITION 12. *For any even integer b and any integer a, there is a graph G with $\gamma_R(G) = b$ and $st_{\gamma_R}(G) = a$.*

Proof. Let b be an even integer. Let $V(K_{b/2}) = \{v_1, ..., v_{b/2}\}$. For $i = 1, 2, ..., b$ identify v_i by a vertex of a complete graph K_{a+1}. Clearly $\gamma_R(G) = b$, and it is straightforward to see that $st_{\gamma_R}(G) = a$. $\square$

3.1. TREES

Theorem 6 indicates that $1 \leq st_{\gamma_R}(T) \leq 2$ for any tree T. In this section, we give a constructive characterization for all trees T with $st_{\gamma_R}(T) = 2$. A vertex v of a graph T is a *special* vertex if $f(v) = 0$ for every $\gamma_R(G)$-function f. Note that from Proposition 2, we obtain that any leaf of a tree T with $st_{\gamma_R}(T) > 1$ is a special vertex. Let $\mathcal{T}$ be the family of trees T that can be obtained from a sequence $T_1, T_2, ...T_j$ $(j \geq 2)$ such that $T_1 = P_3$, and if $j \geq 2$,

then T_{i+1} (for $i = 1, 2, ..., j-1$) can be obtained recursively from T_i by the following operation:

Operation $\mathcal{O}_1$. Assume that $v \in V(T_i)$ is a special vertex of T_i. Then T_{i+1} is obtained from T_i by joining v to a vertex of path P_3.

THEOREM 13. *For a tree T of order $n \geq 2$, $st_{\gamma_R}(T) \leq 2$, with equality if and only if $T \in \mathcal{T}$.*

Proof. By Theorem 6, $st_{\gamma_R}(T) \leq 2$. We establish the equality part. Let T be a tree of order $n \geq 2$ with $st_{\gamma_R}(T) = 2$. We prove by induction on n to show that $T \in \mathcal{T}$. From $st_{\gamma_R}(T) = 2$ and Proposition 2, we obtain that for any $\gamma_R(T)$-function g, $V_1^g = \emptyset$. Clearly, $n \geq 3$. For the base step, if $n = 3$, then $T = P_3 \in \mathcal{T}$. Thus assume that $n \geq 4$. Let $d = diam(T)$ and $x_0x_1...x_d$ be a diametral path in T, and f be any $\gamma_R(T)$-function. Note that $V_1^f = \emptyset$. If $d = 2$ then T is a star with $\gamma_R(T) = 2$, and removal of the central vertex produces a $\overline{K_{n-1}}$ with Roman domination number greater than 2, a contradiction. Thus $d \geq 3$. Suppose that $d = 3$. Then T is a double star with $\gamma_R(T) = 4$. Clearly, $\deg(x_1) \geq 3$ and $\deg(x_2) \geq 3$. If $\deg(x_2) = 3$, then removal of x_3 produces a double star with Roman domination number $3 < \gamma_R(G)$, a contradiction. Thus assume that $\deg(x_2) \geq 4$ and similarly $\deg(x_1) \geq 4$. Now removal of x_2 produces a graph with at least three isolated vertices and Roman domination number greater than $\gamma_R(G)$, a contradiction. Thus $d \geq 4$. Observe that $f(x_{d-1}) = 2$, since $V_1^g = \emptyset$ for any $\gamma_R(T)$-function g.

It can be easily seen that if $\deg(x_{d-1}) \geq 4$, then $\gamma_R(T - x_{d-1}) > \gamma_R(T)$, contradicting $st_{\gamma_R}(T) = 2$. Thus $\deg(x_{d-1}) \leq 3$. Assume that $\deg(x_{d-1}) = 2$. Clearly $f(x_{d-2}) = 0$, and there is no vertex $u \in N(x_{d-2}) - \{x_{d-1}\}$ with $f(u) = 2$, since $V_1^g = \emptyset$ for any $\gamma_R(T)$-function g. Thus $\deg(x_{d-2}) = 2$ and $f(x_{d-3}) = 0$. Let $T' = T - T_{x_{d-2}}$. Then $f|_{V(T')}$ is an RDF for T', implying that $\gamma_R(T') \leq \gamma_R(T) - 2$. On the other hand any $\gamma_R(T')$-function can be extended to an RDF for T by assigning 2 to x_{d-1} and 0 to x_{d-2} and x_d, and so $\gamma_R(T) \leq \gamma_R(T') + 2$. Thus $\gamma_R(T) = \gamma_R(T') + 2$. Assume that $st_{\gamma_R}(T') = 1$. Then there is a vertex $v \in V(T')$ such that $\gamma_R(T' - v) \neq \gamma_R(T')$. An argument similar to that used to show $\gamma_R(T) = \gamma_R(T') + 2$, shows that $\gamma_R(T - v) = \gamma_R(T' - v) + 2$. Then $\gamma_R(T - v) = \gamma_R(T' - v) + 2 \neq \gamma_R(T') + 2 = \gamma_R(T)$, a contradiction. Thus $st_{\gamma_R}(T') = 2$. Note that $|V(T')| \geq 3$. By the inductive hypothesis, $T' \in \mathcal{T}$. If there is a $\gamma_R(T')$-function g with $g(x_{d-3}) = 2$, then we extend g to an RDF g_1 for T by assigning 0 to x_{d-2} and 1 to x_{d-1} and x_d. Since $\gamma_R(T) = \gamma_R(T') + 2$, g_1 is a $\gamma_R(T)$-function with $V_1^{g_1} \neq \emptyset$, a contradiction. Similarly, there is no $\gamma_R(T')$-function g with $g(x_{d-3}) = 1$. Thus x_{d-3} is a special vertex of T'. Hence T is obtained from T_1 by Operation $\mathcal{O}_1$.

Next assume that $\deg(x_{d-1}) = 3$. Clearly $f(x_{d-2}) = 0$. We show that we may assume that there is a vertex $w \in N(x_{d-2}) - \{x_{d-1}\}$ with $f(w) = 2$. To see this, suppose to the contrary, that there is no vertex $w \in N(x_{d-2}) - \{x_{d-1}\}$ with value 2 under any $\gamma_R(T)$-function. In particular, $\deg(x_{d-2}) = 2$ and $f(x_{d-3}) = 0$. Let $a \neq x_d$ be the second child of x_{d-1}. Since a and x_d are isolate vertices of $T - x_{d-1}$, we can see that $\gamma_R(T) \leq \gamma_R(T - x_{d-1})$. Suppose that $\gamma_R(T) = \gamma_R(T - x_{d-1})$. Let h be a $\gamma_R(T - x_{d-1})$-function. It is obvious that $h(a) = h(x_d) = 1$. If $h(x_{d-2}) = 2$ then h_1 defined on $V(T)$ by $h_1(x_{d-1}) = 0$, $h_1(a) = h_1(x_d) = 1$, and $h_1(u) = h(u)$ otherwise, is a $\gamma_R(T)$-function with $V_1^{h_1} \neq \emptyset$, a contradiction. If $h(x_{d-2}) = 1$ then h_2 defined on $V(T)$ by $h_2(x_{d-1}) = 2$, $h_2(a) = h_2(x_d) = h_2(x_{d-2}) = 1$, is an RDF for T of weight less than $\gamma_R(T)$, a contradiction. Thus $h(x_{d-2}) = 0$, and so $h(x_{d-3}) = 2$. Then h_3 defined on $V(T)$ by $h_3(x_{d-1}) = 2$, $h_3(a) = h_3(x_d) = 0$ is a $\gamma_R(T)$-function with $h_3(x_{d-1}) = h_3(x_{d-3}) = 2$, a contradiction with the assumption that there is no vertex $w \in N(x_{d-2}) - \{x_{d-1})$ with value 2 under any $\gamma_R(T)$-function. Thus $\gamma_R(T) < \gamma_R(T - x_{d-1})$, contradicting $st_{\gamma_R}(T) = 2$. We conclude that there is a vertex $w \in N(x_{d-2}) - \{x_{d-1}\}$ with $f(w) = 2$. Let $T' = T - T_{x_{d-1}}$. Then $f|_{V(T')}$ is an RDF for T', implying that $\gamma_R(T') \leq \gamma_R(T) - 2$. On the other hand any $\gamma_R(T')$-function can be extended to an RDF for T by assigning 2 to x_{d-1} and 0 to both children of x_{d-1}, and so $\gamma_R(T) \leq \gamma_R(T') + 2$. Thus $\gamma_R(T) = \gamma_R(T') + 2$. Assume that $st_{\gamma_R}(T') = 1$. Then there is a vertex $v \in V(T')$ such that $\gamma_R(T' - v) \neq \gamma_R(T')$. As before, $\gamma_R(T - v) = \gamma_R(T' - v) + 2$, and we find that $\gamma_R(T - v) \neq \gamma_R(T)$, a contradiction. Thus $st_{\gamma_R}(T') = 2$. By the inductive hypothesis, $T' \in \mathcal{T}$. If there is a $\gamma_R(T')$-function g with $g(x_{d-3}) = 2$, then we extend g to an RDF g_1 for T by assigning 0 to x_{d-1} and 1 to the children of x_{d-1}. Since $\gamma_R(T) = \gamma_R(T') + 2$, g_1 is a $\gamma_R(T)$-function with $V_1^{g_1} \neq \emptyset$, a contradiction. Similarly, there is no $\gamma_R(T')$-function g with $g(x_{d-3}) = 1$. Thus x_{d-3} is a special vertex of T'. Hence T is obtained from T_1 by Operation $\mathcal{O}_1$.

Conversely, assume that $T \in \mathcal{T}$. We use an induction on the number that the Operation $\mathcal{O}_1$ performed to form T to show that $st_{\gamma_R}(T) = 2$. Clearly $st_{\gamma_R}(T) = 2$ if $T = T_1 = P_3$. Assume the result holds for T_i, and T_{i+1} is obtained from T_i by Operation $\mathcal{O}_1$. Thus $st_{\gamma_R}(T_i) = 2$, and for any $\gamma_R(T)$-function g, $V_1^g = \emptyset$. Assume that $v \in V(T_i)$ is a special vertex of T_i, and T_{i+1} is obtained from T_i by joining v to a vertex of path $P_3 : xyz$. Assume first that v is adjacent to x. Any $\gamma_R(T_i)$-function can be extended to an RDF for T_{i+1} by assigning 2 to y and 0 to x and z, and thus $\gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 2$. Let f be a $\gamma_R(T_{i+1})$-function. Clearly $f(x) + f(y) + f(z) \geq 2$. If $f(v) \neq \emptyset$ then $f|V(T_i)$ is an RDF for T_i, implying that $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 2$. Thus assume that $f(v) = 0$. If $f(x) = 2$ then $f(x) + f(y) + f(z) = 3$, and g defined

on $V(T_i)$ by $g(v) = 1$, and $g(u) = f(u)$ otherwise, is an RDF for T_i, implying that $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 2$. Thus assume that $f(x) \neq 2$, and so $f|V(T_i)$ is an RDF for T_i, implying that $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 2$. We deduce that $\gamma_R(T_{i+1}) = \gamma_R(T_i) + 2$. Suppose that $st_{\gamma_R}(T_{i+1}) = 1$. Let $w \in V(T_{i+1})$ be a vertex with $\gamma_R(T_{i+1} - w) \neq \gamma_R(T_{i+1})$. If $w \in V(T_i)$ then similarly to the proof of $\gamma_R(T_{i+1}) = \gamma_R(T_i) + 2$, we can easily see that $\gamma_R(T_{i+1} - w) = \gamma_R(T_i - w) + 2$. Then we obtain $\gamma_R(T_i - w) \neq \gamma_R(T_i)$, contradicting $st_{\gamma_R}(T_i) = 2$. Thus $w \in \{x, y, z\}$. It is straightforward to see that $\gamma_R(T_{i+1} - x) = \gamma_R(T_{i+1})$, and so $w \in \{y, z\}$. Assume that $w = y$. Observe that $\gamma_R(T_{i+1} - y) \leq \gamma_R(T_{i+1})$, since we can extend any $\gamma_R(T_i)$-function to an RDF for $T_{i+1} - y$ by assigning 1 to x and z. Thus $\gamma_R(T_{i+1} - y) < \gamma_R(T_{i+1})$. Then by Proposition 2, there is a $\gamma_R(T_{i+1})$-function f_1 with $f_1(y) = 1$. Then $f_1(z) = 1$, and we may assume that $f_1(x) = 0$ and $f_1(v) = 2$. Then $f_1|_{V(T_i)}$ is a $\gamma_R(T_i)$-function, since $\gamma_R(T_{i+1}) = \gamma_R(T_i) + 2$. This is a contradiction, since v is a special vertex of T_i. Next assume that $w = z$.

Observe that $\gamma_R(T_{i+1} - z) \leq \gamma_R(T_{i+1})$, since we can extend any $\gamma_R(T_i)$-function to an RDF for $T_{i+1} - z$ by assigning 1 to x and y. Thus $\gamma_R(T_{i+1} - z) < \gamma_R(T_{i+1})$. Then by Proposition 2, there is a $\gamma_R(T_{i+1})$-function f_1 with $f_1(z) = 1$. Clearly $f_1(y) \neq 2$. If $f_1(x) = 0$, then $f_1(v) = 2$, and $f_1|_{V(T_i)}$ is a $\gamma_R(T_i)$-function, a contradiction, since v is a special vertex of T_i. Thus $f_1(x) \neq 0$, and so we obtain that $f_1(x) = 2$ and $f_1(y) = f_1(v) = 0$. Since $f_1(x) + f_1(y) + f_1(z) = 3$ and $\gamma_R(T_{i+1}) = \gamma_R(T_i) + 2$, we obtain that $f_1|_{V(T_i)}$ is not an RDF for T_i. Then f_2 defined on $V(T_i)$ by $f_2(v) = 1$ and $f_2(u) = f_1(u)$ otherwise, is a $\gamma_R(T_i)$-function with $V_1^{f_2} \neq \emptyset$, a contradiction. We conclude that $st_{\gamma_R}(T_{i+1}) = 2$.

Assume next that v is adjacent to y. Any $\gamma_R(T_i)$-function can be extended to an RDF for T_{i+1} by assigning 2 to y and 0 to x and z, and thus $\gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 2$. Let f be a $\gamma_R(T_{i+1})$-function. Clearly $f(x) + f(y) + f(z) \geq 2$. If $f(v) \neq \emptyset$ then $f|V(T_i)$ is an RDF for T_i, implying that $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 2$. Thus assume that $f(v) = 0$. Clearly we may assume that $f(y) = 2$. If there is a vertex $u \in N(v) - \{y\}$ with $f(u) = 2$ then $f|V(T_i)$ is an RDF for T_i, implying that $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 2$. Thus assume that there is no vertex $u \in N(v) - \{y\}$ with $f(u) = 2$. Let f_1 be defined on $V(T_i)$ by $f_1(v) = 1$, and $f_1(u) = f(u)$ otherwise. Then f_1 is an RDF for T_i, and so $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 1$. If $\gamma_R(T_i) = \gamma_R(T_{i+1}) - 1$, then f_1 is a $\gamma_R(T_i)$-function with $V_1^{g_1} \neq \emptyset$, a contradiction. Thus $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 2$. Consequently, $\gamma_R(T_{i+1}) = \gamma_R(T_i) + 2$. Suppose that $st_{\gamma_R}(T_{i+1}) = 1$. Let $w \in V(T_{i+1})$ be a vertex with $\gamma_R(T_{i+1} - w) \neq \gamma_R(T_{i+1})$. As before, we can assume that $w \in \{x, z\}$. Without loss of generality, assume that $w = x$. Observe that $\gamma_R(T_{i+1} - x) \leq \gamma_R(T_{i+1})$, since we can extend any $\gamma_R(T_i)$-function to an RDF for $T_{i+1} - x$ by assigning 1 to y and z. Thus

$\gamma_R(T_{i+1} - x) < \gamma_R(T_{i+1})$. Then by Proposition 2, there is a $\gamma_R(T_{i+1})$-function f_1 with $f_1(x) = 1$. Then we may assume that $f_1(z) = 1$, and thus $f_1(y) = 0$. Then $f_1(v) = 2$. Then $f_1|_{V(T_i)}$ is a $\gamma_R(T_i)$-function, since $\gamma_R(T_{i+1}) = \gamma_R(T_i)+2$. This is a contradiction, since v is a special vertex of T_i. We conclude that $st_{\gamma_R}(T_{i+1}) = 2$. $\square$

COROLLARY 14. *For any path P_n of order $n \geq 3$, $st_{\gamma_R}(P_n) = 2$ if and only if $n \equiv 0 \pmod 3$ and $st_{\gamma_R}(P_n) = 1$ if $n \not\equiv 0 \pmod 3$.*

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