INFINITELY MANY SOLUTIONS FOR ELLIPTIC EQUATIONS INVOLVING A GENERAL OPERATOR IN DIVERGENCE FORM

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The existence of infinitely many weak solutions for a class of elliptic equations involving a general operator in divergence form, subject to Dirichlet boundary conditions in a smooth bounded domain in $\mathbb{R}^N$ by a critical point result for differentiable functionals is discussed.

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1. INTRODUCTION

The aim of this paper is to investigate the existence of infinitely many weak solutions for the following elliptic Dirichlet problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = \lambda k(x) f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, $p > N$, $a : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a suitable continuous map of gradient type, and λ is a positive real parameter. Further, $f : \mathbb{R} \to \mathbb{R}$ and $k : \bar{\Omega} \to \mathbb{R}^+$ are two continuous functions.

The operator $-\mathrm{div}(a(x,\nabla u))$ arises, for example, from the expression of the p-Laplacian in curvilinear coordinates. We refer to the overview papers [11, 12, 18, 22, 23] for the investigation on Dirichlet problems involving a general operator in divergence form. For example, De Nápoli and Mariani in [11] studied the existence of solutions to equations of p-Laplacian type. They proved the existence of at least one solution, and under further assumptions, the existence of infinitely many solutions. In order to apply mountain pass results, they introduced a notion of uniformly convex functional that generalizes the notion of uniformly convex norm. Duc and Vu in [12] studied the non-uniform case. The authors in [23] established the existence and multiplicity of weak solutions of a problem involving a uniformly convex elliptic operator in divergence form. They discussed the existence of one nontrivial solution by the mountain pass lemma, when the nonlinearity has a $(p-1)$-superlinear growth at infinity, and two nontrivial solutions by minimization and mountain pass when the nonlinear term has a $(p-1)$-sublinear growth at infinity. Molica Bisci and Repovš in [18], exploiting variational methods, investigated the existence of three weak solutions for the problem (1.1). They analyzed several special cases. They presented a concrete example of an application by finding the existence of three nontrivial weak solutions for an uniformly elliptic second-order problem on a bounded Euclidean domain.

In [9], Colasuonno, Pucci and Varga studied different and very general classes of elliptic operators in divergence form looking at the existence of multiple weak solutions. Their contributions represent a nice improvement, in several directions, of the results obtained by Kristály *et al.* in [16] in which a uniform Dirichlet problem with parameter is investigated.

In the present paper, employing a smooth version of [5, Theorem 2.1] which is a more precise version of Ricceri's variational principle [20, Theorem 2.5], which we recall in the next section, we investigate the existence of infinitely many solutions for the problem (1.1). We shall present two types of results as follows: the existence of either an unbounded sequence of solutions (Theorem 3.1 and its consequences, *i.e.* Corollaries 3.3 and 3.6) and a sequence of pairwise distinct non-zero solutions which converges to zero (Theorem 3.1 and Remark 3.7), depending on whether the nonlinear term has a suitable oscillating behaviour, respectively, at infinity or at zero.

For a discussion about the existence of infinitely many solutions for differential equations, using Ricceri's Variational Principle [20], applying a smooth version of Theorem 2.1 of [5] which is a more precise version of Ricceri's Variational Principle [20] and employing a non-smooth version of Ricceri's Variational Principle [20], we refer the reader to the papers [1–3, 6, 8, 10, 13–15, 17].

The outline of the paper is organized as follows: in the forthcoming section, we shall recall our main tool (Theorem 2.1) and some basic notations which we need in the proofs. Whereas, Section 3 is devoted to the existence of infinitely many weak solutions for the system (1.1). To be precise, our main result (Theorem 3.1), some of its possible consequences, the proofs and some examples to illustrate the results are presented.

The following theorem is a special case of our main result.

THEOREM 1.1. *Assume that $f:\mathbb{R}\to\mathbb{R}$ is a non-negative continuous function such that*

$$\liminf_{\xi\to+\infty}\frac{\int_0^\xi f(t)dt}{\xi^p}=0 \qquad \textit{and} \qquad \limsup_{\xi\to+\infty}\frac{\int_0^\xi f(t)dt}{\xi^p}=+\infty.$$

Then, for each $\lambda > 0$, the problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = \lambda k(x) f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

admits a sequence of weak solutions which is unbounded in $W_0^{1,p}(\Omega)$.

This article is organized as follows. In Section 2, we present some necessary preliminary facts that will be needed in the paper. In Section 3, we establish our main existence results.

2. AUXILIARY RESULTS

Assume that Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$. Further, denote by X the space $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p \,\mathrm{d}x \right)^{1/p},$$

the functional $I_\lambda : X \to \mathbb{R}$ associated with (1.1) is introduced as following:

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u),$$

for every $u \in X$, where

$$\Phi(u) := \int_\Omega A(x, \nabla u(x))\mathrm{d}x,$$

and

$$\Psi(u) := \int_\Omega k(x) F(u(x))\mathrm{d}x,$$

for every $u \in X$, where $k : \bar{\Omega} \to \mathbb{R}^+$ is a positive and continuous function, and

$$F(s) = \int_0^s f(t)\mathrm{d}t,$$

for every $s \in \mathbb{R}$. By standard arguments, Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) := \int_\Omega a(x, \nabla u(x))\nabla v(x)\mathrm{d}x,$$

for every $v \in X$. Moreover, ψ is a Gâteaux differentiable sequentially weakly upper continuous functional whose Gâteaux derivative is given by

$$\Psi'(u)(v) := \int_\Omega k(x) f(u(x))v(x)\mathrm{d}x,$$

for every $v \in X$. Fixing the real parameter λ, a function $u : \Omega \to \mathbb{R}$ is said to be a weak solution of (1.1) if $u \in X$ and

$$\int_\Omega a(x, \nabla u(x))\nabla v(x)\mathrm{d}x - \lambda \int_\Omega k(x) f(u(x))v(x)\mathrm{d}x = 0,$$

for every $v \in X$. Therefore, the critical points of I_λ are exactly the weak solutions of (1.1). Our main tool is the celebrated Ricceri's Variational Principle [20, Theorem 2.5] that we now recall as given by Bonanno and Molica Bisci in [5]:

THEOREM 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, put*

$$\varphi(r) := \inf_{\Phi(u)<r} \frac{\left(\sup_{\Phi(v)<r} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \text{and} \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in]0, 1/\varphi(r)[$, the restriction of the functional*

$$I_\lambda := \Phi - \lambda\Psi$$

to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) *If $\gamma < +\infty$, then for each $\lambda \in]0, 1/\gamma[$, the following alternative holds: either*

(b_1) *I_λ possesses a global minimum, or*

(b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$, then for each $\lambda \in]0, 1/\delta[$, the following alternative holds: either*

(c_1) *there is a global minimum of Φ which is a local minimum of I_λ, or*

(c_2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ, with*

$$\lim_{n\to+\infty} \Phi(u_n) = \inf_{u\in X} \Phi(u).$$

Put

$$k := \sup\left\{ \frac{\max_{x\in\overline{\Omega}}|u(x)|}{\|u\|} : u \in W_0^{1,p}(\Omega), u \neq 0 \right\}.$$

Since $p > N$, one has $k < \infty$. For our goal it is enough to know an explicit upper bound for the constant k. In this connection (see [21, formula (6b)] put

$$m := \frac{N^{-\frac{1}{p}}}{\sqrt{\pi}} \left[\Gamma\left(1 + \frac{N}{2}\right)\right]^{\frac{1}{N}} \left(\frac{p-1}{p-N}\right)^{1-\frac{1}{p}} (\mathrm{meas}(\Omega))^{\frac{1}{N}-\frac{1}{p}},$$

one has $k < m$. Hence

$$\|u\|_\infty = \max_{x\in\overline{\Omega}} |u(x)| \leq m\|u\|, \tag{2.1}$$

for every $u \in X$. Here Γ is the Gamma function defined by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1}e^{-z}\mathrm{d}z \qquad (\forall t > 0),$$

and "meas(Ω)" denotes the usual Lebesgue measure of Ω. Moreover, let

$$D := \sup_{x\in\Omega} \mathrm{dist}(x, \partial\Omega).$$

Simple calculations show that there is $x_0 \in \Omega$ such that $B(x_0, D) \subseteq \Omega$, where $B(x_0, D)$ is the open ball radius D centered at the point x_0. We also denote by

$$\omega_s := s^N \frac{\pi^{\frac{N}{2}}}{\Gamma\left(1 + \frac{N}{2}\right)}$$

the measure of the N-dimensional ball of radius $s > 0$. At this point, for $d_n > 0$, let $w_n \in X$ be the following function

$$w_n(x) := \begin{cases} 0, & x \in \Omega \setminus B(x_0, D), \\ d_n, & x \in B(x_0, \frac{D}{2}), \\ \dfrac{2d_n}{D}(D - |x - x_0|), & x \in B(x_0, D) \setminus B(x_0, \frac{D}{2}), \end{cases} \tag{2.2}$$

that will be useful in the sequel in the proof of our theorems. One has that

$$\|w_n\|^p := \left(\int_\Omega |\nabla w_n(x)|^p \,\mathrm{d}x\right) = \frac{2^p d_n^p \omega_D}{D^p}\left(1 - \frac{1}{2^N}\right).$$

Indeed,

$$\int_\Omega |\nabla w_n(x)|^p \,\mathrm{d}x \;=\; d_n^p \int_{B(x_0,D)\setminus B(x_0,\frac{D}{2})} \frac{2^p}{D^p}\mathrm{d}x$$

$$= \frac{2^p d_n^p}{D^p}(\text{meas}(B(x_0, D)) - \text{meas}(B(x_0, \frac{D}{2})))$$
$$= \frac{2^p d_n^p \omega_D}{D^p}\left(1 - \frac{1}{2^N}\right),$$

where, from now on, "meas$(B(x_0, s))$" for $s > 0$ stands for the Lebesgue measure of the open ball $B(x_0, s)$.

3. MAIN RESULTS

In this section, we formulate our main results. Let $p \geq 1$ and let $\Omega \subseteq \mathbb{R}^N$ be a bounded Euclidean domain, where $N \geq 2$. Further, let $A : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ and let $A = A(x, \xi)$ be a continuous function in $\bar{\Omega} \times \mathbb{R}^N$, with continuous gradient $a(x, \xi) := \nabla_\xi A(x, \xi) : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$, and assume that the following conditions hold:
(α_1) $A(x, 0) = 0$ for all $x \in \Omega$;
(α_2) A satisfies $\Lambda_1|\xi|^p \leq A(x, \xi) \leq \Lambda_2|\xi|^p$ for all $x \in \bar{\Omega}$, $\xi \in \mathbb{R}^N$, where Λ_1 and Λ_2 are positive constants.
(α_3) a satisfies the growth condition $|a(x, \xi)| \leq c(1 + |\xi|^{p-1})$ for all $x \in \Omega$, $\xi \in \mathbb{R}^N$, $c > 0$;
(α_4) A is p-uniformly convex, that is

$$A(x, \frac{\xi + \eta}{2}) \leq \frac{1}{2}A(x, \xi) + \frac{1}{2}A(x, \eta) - k|\xi - \eta|^p,$$

for every $x \in \bar{\Omega}$, $\xi, \eta \in \mathbb{R}^N$ and some $k > 0$;

Put

$$B^\infty := \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^p},$$

Our main result is the following theorem.

Theorem 3.1. *Assume that there exist two sequences $\{d_n\}$ and $\{b_n\}$ in $]0, +\infty[$, with $\lim_{n \to +\infty} b_n = +\infty$, such that*

(i_0) $F(s) \geq 0$ *for every* $s \in \mathbb{R}^+$;

(i_1) $d_n^p < \dfrac{\Lambda_1 D^p 2^{N-p}}{\Lambda_2 m^p \omega_D (2^N - 1)} b_n^p$;

(i_2) $\mathcal{A}_\infty := \lim\limits_{n \to +\infty} \dfrac{\text{meas}(\Omega)\left(\max\limits_{|\xi| \leq b_n} F(\xi)\right) \max\limits_{x \in \bar{\Omega}} k(x) - \omega_{D/2} F(d_n) \min\limits_{x \in \bar{\Omega}} k(x)}{2^N (D b_n)^p \Lambda_1 - (2 m d_n)^p \Lambda_2 \omega_D (2^N - 1)}$
$< \dfrac{\min\limits_{x \in \bar{\Omega}} k(x)}{m^p \Lambda_2 2^{N+p} (2^N - 1)} B^\infty.$

Then, for every

$$\lambda \in \Lambda := \left] \frac{\Lambda_2 2^p (2^N - 1)}{D^p \left(\min_{x \in \bar{\Omega}} k(x) \right) B^\infty}, \frac{1}{2^N (mD)^p \mathcal{A}_\infty} \right[,$$

problem (1.1) *admits a sequence of weak solutions which is unbounded in* X.

Proof. Fix $\lambda \in \Lambda$. Our aim is to apply Theorem 2.1 part (b) with $X := W_0^{1,p}(\Omega)$ and where Φ and Ψ are the functionals introduced in Section 2. Clearly, Φ is coercive since, by condition (α_2), it follows that

$$\Phi(u) \geq \Lambda_1 \|u\|^p \to +\infty,$$

when $\|u\| \to \infty$. As seen before, the functionals Φ and Ψ satisfy the regularity assumptions requested in Theorem 2.1. Now, we look on the existence of critical points of the functional $I_\lambda := \Phi - \lambda\Psi$ in X. Therefore, our conclusion follows provided that $\gamma < \infty$ as well as I_λ turns out to be unbounded from below. To this end, set

$$r_n := \frac{\Lambda_1 {b_n}^p}{m^p},$$

for all $n \in \mathbb{N}$. Let $u \in X$ be such that $\Phi(u) < r_n$, that is

$$\int_\Omega A(x, \nabla u(x)) \mathrm{d}x < r_n.$$

Hence the above relation together with condition (α_2) implies that

$$\|u\| < \left(\frac{r_n}{\Lambda_1} \right)^{\frac{1}{p}}.$$

Owing to (2.1) we have $\|u\|_\infty \leq b_n$ for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$, we obtain

$$\varphi(r_n) \leq \inf_{\Phi(u) < r_n} \frac{\operatorname{meas}(\Omega) \left(\max_{|\xi| \leq b_n} F(\xi) \right) \max_{x \in \bar{\Omega}} k(x) - \int_\Omega k(x) F(u(x)) \mathrm{d}x}{\frac{\Lambda_1 {b_n}^p}{m^p} - \Lambda_2 \|u\|^p}.$$

Next, let w_n be the function defined in (2.2). Clearly, $w_n \in X$ and since (α_2) holds, we have

$$\Phi(w_n) = \int_\Omega A(x, \nabla w_n(x)) \mathrm{d}x \leq \frac{\Lambda_2 2^p d_n^p \omega_D (2^N - 1)}{2^N D^p}.$$

Hence, by (i_1), we have $\Phi(w_n) < r_n$. Moreover, from (i_0) and taking into account that the map k is positive and continuous in $\bar{\Omega}$, we have

$$\int_\Omega k(x) F(w_n(x)) \mathrm{d}x \geq \omega_{D/2} F(d_n) \min_{x \in \bar{\Omega}} k(x).$$

Therefore, one has

$$\varphi(r_n) \;\le\; 2^N(mD)^p \frac{\operatorname{meas}(\Omega)\left(\max\limits_{|\xi|\le b_n} F(\xi)\right)\max\limits_{x\in\bar\Omega} k(x) - \omega_{D/2}F(d_n)\min\limits_{x\in\bar\Omega} k(x)}{2^N(Db_n)^p\Lambda_1 - (2md_n)^p\Lambda_2\omega_D(2^N-1)}.$$

Hence, bearing in mind assumption (i_2), we can write

$$\gamma \le \liminf_{n\to+\infty}\varphi(r_n) \le 2^N(mD)^p\mathcal{A}_\infty < +\infty, \tag{3.1}$$

Assumption (i_2) together with (3.1), implies

$$\Lambda \subseteq]0, \frac{1}{\gamma}[.$$

Fix $\lambda \in \Lambda$. We conclude that condition (b) of Theorem 2.1 can be applied, and either I_λ has a global minimum or there exists a sequence $\{u_n\} \subset X$ of weak solutions of the system (1.1) such that $\lim_{n\to\infty}\|u_n\| = +\infty$. The other step is to show that the functional I_λ has no global minimum. For fixed λ, we claim that the functional $\Phi - \lambda\Psi$ is unbounded from below, since condition (i_2) yields $\mathcal{A}_\infty < +\infty$. Now, we claim that the functional I_λ is unbounded from below. Since

$$\frac{1}{\lambda} < \frac{D^p\left(\min\limits_{x\in\bar\Omega} k(x)\right)B^\infty}{\Lambda_2 2^p(2^N-1)},$$

we can consider a real sequence $\{c_n\}$ and a positive constant τ such that $\{c_n\} \to \infty$ as $n \to \infty$ and

$$\frac{1}{\lambda} < \tau < \frac{D^p\left(\min\limits_{x\in\bar\Omega} k(x)\right)\left(\limsup_{n\to+\infty} F(c_n)\right)}{\Lambda_2 2^p(2^N-1){c_n}^p},$$

First, assume that $B^\infty = +\infty$. Accordingly, fix M such that

$$M > \frac{\Lambda_2 2^p(2^N-1)\omega_D}{2^N D^p\omega_{D/2}\left(\min\limits_{x\in\bar\Omega} k(x)\right)\lambda}$$

and let $\{c_n\}$ be a real sequence such that $\lim_{n\to+\infty} c_n = +\infty$, and

$$F(c_n) > Mc_n^p \qquad (\forall n \in N).$$

Further, for each $n \ge 1$, define $s_n \in X$ given by

$$s_n(x) := \begin{cases} 0, & x \in \Omega \setminus B(x_0, D), \\ c_n, & x \in B(x_0, \frac{D}{2}), \\ \dfrac{2c_n}{D}(D - |x - x_0|), & x \in B(x_0, D) \setminus B(x_0, \frac{D}{2}). \end{cases} \tag{3.2}$$

By using condition (i_0), we infer

$$\Psi(s_n) = \int_\Omega k(x)F(s_n(x))\mathrm{d}x \geq \int_{B(x_0,\frac{D}{2})} k(x)F(c_n)\mathrm{d}x,$$

for every $n \in \mathbb{N}$. Then, we have

$$I_\lambda(s_n) \leq \Phi(s_n) - \lambda \int_{B(x_0,\frac{D}{2})} k(x)F(c_n)\mathrm{d}x.$$

Consequently, one has

$$\begin{aligned} I_\lambda(s_n) &< \Lambda_2\|s_n\|^p - \lambda M\omega_{D/2} \min_{x\in\bar{\Omega}} k(x)c_n^p \\ &= \left(\frac{\Lambda_2 2^p\omega_D}{D^p}(1-\frac{1}{2^N}) - \lambda M\omega_{D/2}\min_{x\in\bar{\Omega}} k(x)\right)c_n^p \end{aligned}$$

for every $n \in \mathbb{N}$. Then, it follows that

$$\lim_{n\to+\infty} I_\lambda(s_n) = -\infty.$$

Next, assume that $B^\infty < +\infty$. Since $\lambda > \frac{\Lambda_2 2^p(2^N-1)}{D^p\left(\min_{x\in\bar{\Omega}} k(x)\right)B^\infty}$, we can fix $\varepsilon > 0$ such that $\varepsilon < B^\infty - \frac{\Lambda_2 2^p(2^N-1)}{D^p\left(\min_{x\in\bar{\Omega}} k(x)\right)\lambda}$. Therefore, also calling $\{c_n\}$ a sequence of positive numbers such that $\lim_{n\to+\infty} c_n = +\infty$ and

$$\frac{\omega_D}{2^N}(B^\infty-\varepsilon)c_n^p < F(c_n)\left(\omega_{D/2}\min_{x\in\bar{\Omega}} k(x)\right) < \frac{\omega_D}{2^N}(B^\infty+\varepsilon)c_n^p \qquad (\forall n \in N),$$

arguing as before and by choosing $\{s_n\}$ in X as above, one has

$$I_\lambda(s_n) < \left(\frac{\Lambda_2 2^p\omega_D(2^N-1)}{2^N D^p} - \lambda\frac{\omega_D}{2^N}(B^\infty-\varepsilon)\right)c_n^p.$$

So, $\lim_{n\to+\infty} I_\lambda(s_n) = -\infty$. Hence, our claim holds true; it follows that I_λ has no global minimum. Therefore, Theorem 2.1 assures that there is a sequence $\{u_n\} \subset X$ of critical points of I_λ such that $lim_{n\to\infty}\|u_n\| = +\infty$, and we have the conclusion. Therefore, owing to Theorem 2.1(b), the functional I_λ admits an unbounded sequence $\{u_n\} \subset X$ of critical points. Then, problem (1.1) admits a sequence of weak solutions which is unbounded in X. □

Now, put

$$B^0 := \limsup_{\xi\to0^+} \frac{F(\xi)}{\xi^p}.$$

Arguing as in the proof of Theorem 3.1 and applying part(c) of Theorem 2.1, we get the following theorem.

THEOREM 3.2. *Assume that* (i_0) *holds and there exist two sequences* $\{c_n\}$ *and* $\{a_n\}$ *in* $]0,+\infty[$, *with* $\lim_{n\to+\infty} a_n = 0$, *such that*

(i_3) $c_n^p < \frac{\Lambda_1 D^p 2^{N-p}}{\Lambda_2 m^p \omega_D (2^N-1)} a_n^p$;

(i_4) $\mathcal{A}_0 := \lim_{n\to+\infty} \frac{\operatorname{meas}(\Omega)\left(\max_{|\xi|\le a_n} F(\xi)\right)\max_{x\in\bar{\Omega}} k(x) - \omega_{D/2}F(c_n)\min_{x\in\bar{\Omega}} k(x)}{2^N(Da_n)^p\Lambda_1-(2mc_n)^p\Lambda_2\omega_D(2^N-1)}$
$< \frac{\min_{x\in\bar{\Omega}} k(x)}{m^p\Lambda_2 2^{N+p}(2^N-1)} B^0.$

Then, for every

$$\lambda \in \Lambda' := \left] \frac{\Lambda_2 2^p(2^N-1)}{D^p\left(\min_{x\in\bar{\Omega}} k(x)\right)B^0}, \frac{1}{2^N(mD)^p\mathcal{A}_0}\right[,$$

problem (1.1) *admits a sequence of non-zero solutions which converges to zero.*

Proof. We take X,Φ and Ψ as in the proof of Theorem 3.1. In a similar way, we prove that $\delta<\infty$. Put

$$r_n := \frac{\Lambda_1 {b_n}^p}{m^p},$$

for all $n\in\mathbb{N}$. We claim that the functional I_λ does not have a local minimum at zero. Now, we fixed λ such that

$$\frac{1}{\lambda} < \frac{D^p\left(\min_{x\in\bar{\Omega}} k(x)\right)B^0}{\Lambda_2 2^p(2^N-1)},$$

we can consider a real sequence $\{c_n\}$ and a positive constant τ such that $\{c_n\}\to 0$ as $n\to\infty$ and

$$\frac{1}{\lambda} < \tau < \frac{D^p\left(\min_{x\in\bar{\Omega}} k(x)\right)\left(\limsup_{n\to+\infty} F(c_n)\right)}{\Lambda_2 2^p(2^N-1){c_n}^p},$$

If we take s_n as in the proof of Theorem 3.1, of course the sequence $\{s_n\}$ strongly converges to 0 in X and $I_\lambda(s_n)<0$ for each $n\in N$. Since $I_\lambda(0)=0$, that means that 0 is not a local minimum of I_λ. The part (c) of Theorem 2.1 ensures that there exists a sequence $\{u_n\}$ in X of critical points of I_λ such that $lim_{n\to\infty}\|u_n\|=0$, and the proof is complete. □

Now, we point out some consequences of Theorem 3.1. First, by setting

$$A_\infty := \liminf_{\xi \to +\infty} \frac{\max_{|t| \leq \xi} F(t)}{\xi^p},$$

we get the following result.

COROLLARY 3.3. *Assume that* (i_0) *holds and*

(i_5) $$A_\infty < \frac{\Lambda_1 D^p \min_{x\in\bar{\Omega}} k(x)}{\operatorname{meas}(\Omega)\left(\max_{x\in\bar{\Omega}} k(x)\right) m^p \Lambda_2 2^p (2^N - 1)} B^\infty.$$

Then, for each

$$\lambda \in \left] \frac{\Lambda_2 2^p(2^N-1)}{D^p \left(\min_{x\in\bar{\Omega}} k(x)\right) B^\infty}, \frac{\Lambda_1}{\operatorname{meas}(\Omega)\left(\max_{x\in\bar{\Omega}} k(x)\right) m^p A_\infty} \right[,$$

problem (1.1) *admits an unbounded sequence of weak solutions in* X.

Proof. Let $\{b_n\}$ be a sequence of positive numbers which goes to infinity such that

$$\lim_{n\to+\infty} \frac{\max_{|\xi|\leq b_n} F(\xi)}{b_n^p} = A_\infty.$$

Taking $d_n = 0$ for every $n \in \mathbb{N}$, by Theorem 3.1 the conclusion follows. $\square$

A special case of Corollary 3.3 is the following.

COROLLARY 3.4. *Assume that* (i_0) *holds and*

(i_6) $$A_\infty < \frac{\Lambda_1}{\operatorname{meas}(\Omega)\left(\max_{x\in\bar{\Omega}} k(x)\right) m^p} \quad \text{and} \quad B^\infty > \frac{\Lambda_2 2^p(2^N-1)}{D^p \min_{x\in\bar{\Omega}} k(x)}.$$

Then, the following problem

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) = k(x) f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

admits a sequence of weak solutions which is unbounded in X.

Remark 3.5. When f is a nonnegative function, assumption (i_0) holds and condition (i_5) becomes

(i_5') $$A'_\infty := \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} < \frac{\Lambda_1 D^p \min_{x\in\bar{\Omega}} k(x)}{\operatorname{meas}(\Omega)\left(\max_{x\in\bar{\Omega}} k(x)\right) m^p \Lambda_2 2^p (2^N-1)} B^\infty.$$

In this case, (i_5') ensures that for each

$$\lambda \in \left] \frac{\Lambda_2 2^p(2^N-1)}{D^p\left(\min_{x\in\bar\Omega} k(x)\right) B^\infty}, \frac{\Lambda_1}{\operatorname{meas}(\Omega)\left(\max_{x\in\bar\Omega} k(x)\right) m^p A'_\infty} \right[,$$

problem (1.1) admits an unbounded sequence of weak solutions in X.

Proof. Obviously, from (i_5') we obtain (i_5). Taking $d_n = 0$ for every $n \in \mathbb{N}$, by Theorem 3.1 the conclusion follows. □

The next result is a consequence of Theorem 3.1 and guarantees the existence of infinitely many weak solutions to (1.1) for each λ which lies in a precise half-line.

Corollary 3.6. *Assume that (i_0) holds and let there exist two sequences $\{d_n\}$ and $\{b_n\}$ in $]0, +\infty[$, with $\lim_{n\to+\infty} b_n = +\infty$, such that (i_1) and*

(i_5) $\operatorname{meas}(\Omega)\left(\max_{|\xi|\le b_n} F(\xi)\right)\max_{x\in\bar\Omega} k(x) = \omega_{D/2} F(d_n)\min_{x\in\bar\Omega} k(x)$ *for each* $n \in \mathbb{N}$,

are satisfied. If $B^\infty > 0$, Then for each $\lambda > \frac{\Lambda_2 2^p(2^N-1)}{D^p\left(\min_{x\in\bar\Omega} k(x)\right)B^\infty}$, problem (1.1) admits an unbounded sequence of solutions.

Proof. From (i_5) we obtain $\mathcal{A}_\infty = 0$. Hence, since $B^\infty > 0$, (i_2) of Theorem 3.1 holds and the proof is complete. □

Remark 3.7. From Theorem 3.2 we obtain the same consequences of Theorem 3.1. Namely, substituting $\xi \to +\infty$ with $\xi \to 0^+$, statements such as Corollaries 3.3, 3.4 and 3.6 can be established.

Example 3.8. Set

$$a_n := \frac{2n!(n+2)!-1}{4(n+1)!}, \qquad b_n := \frac{2n!(n+2)!+1}{4(n+1)!},$$

for every $n \in \mathbb{N}$. Let $\{g_n\}$ be a sequence of non-negative functions such that

(i) $g_n \in C^0([a_n, b_n])$ such that $g(a_n) = g(b_n) = 0$ for every $n \in \mathbb{N}$;

(ii) $\int_{a_n}^{b_n} g_n(t)\mathrm{d}t \neq 0$ for every $n \in \mathbb{N}$.

For instance, we can choose the sequence $\{g_n\}$ as follows:

$$g_n(\xi) := \sqrt{\frac{1}{16(n+1)!^2} - \left(\xi - \frac{n!(n+2)}{2}\right)^2}$$

for all $n \in \mathbb{N}$. Define the function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(\xi) := \begin{cases} [(n+1)!^p - n!^p] \dfrac{g_n(\xi)}{\int_{a_n}^{b_n} g_n(t)\mathrm{d}t}, & \text{if } \xi \in \bigcup_{n=1}^{+\infty} [a_n, b_n], \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\int_{n!}^{(n+1)!} f(t)\mathrm{d}t = \int_{a_n}^{b_n} f(t)\mathrm{d}t = (n+1)!^p - n!^p$$

and

$$F(a_n) = n!^p - 1, \qquad F(b_n) = (n+1)!^p - 1$$

for every $n \in \mathbb{N}$. Hence,

$$\lim_{n\to+\infty} \frac{F(b_n)}{b_n^p} = 2^p, \qquad \lim_{n\to+\infty} \frac{F(a_n)}{a_n^p} = 0.$$

Therefore, we can prove that

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} = 0, \qquad \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} = 2^p.$$

Then,

$$0 = \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} < \frac{\min_{x\in\bar{\Omega}} k(x)}{m^p \Lambda_2 2^{N+p}(2^N - 1)} \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p}.$$

By Theorem 3.1, for each

$$\lambda > \frac{\Lambda_2(2^N - 1)}{D^p \min_{x\in\bar{\Omega}} k(x)},$$

the problem

$$\begin{cases} -\mathrm{div}(a(x, \nabla u)) = \lambda k(x) f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

possesses a sequence of weak solutions which is unbounded in X.

Following Omari and Zanolin in [19], we give a concrete example of positive continuous function $f : \mathbb{R} \to \mathbb{R}$ such that its potential F satisfies our growth conditions at zero. Precisely, let $\{s_n\}, \{t_n\}$ and $\{\delta_n\}$ be real sequences defined by

$$s_n := 2^{-\frac{n!}{2}}, \qquad t_n := 2^{-2n!}, \qquad \delta_n := 2^{-(n!)^2}.$$

Observe that, definitively, one has

$$s_{n+1} < t_n < s_n - \delta_n.$$

Example 3.9. With the above notations, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function such that $f(t) = 0$ in $]-\infty, 0]$, $f(t) > 0$ for every $t > 0$ and

$$f(t) := p^{-n!}, \qquad \forall t \in [s_{n+1}, s_n - \delta_n],$$

for n sufficiently large. Define $F : \mathbb{R} \to \mathbb{R}$ given by $F(\xi) = \int_0^\xi f(t)\mathrm{d}t$, for every $\xi \in \mathbb{R}$. Then

$$\frac{F(s_n)}{{s_n}^p} \leq \frac{q(s_{n+1})s_n + q(s_n)\delta_n}{{s_n}^p} \to 0,$$

and

$$\frac{F(t_n)}{{t_n}^p} \geq \frac{q(s_{n+1})t_n - s_{n+1}}{{t_n}^p} \to +\infty.$$

Hence

$$\liminf_{\xi \to 0^+} \frac{\max\limits_{|t| \leq \xi} F(t)}{\xi^p} = 0, \qquad \limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^p} = +\infty.$$

Thus, the following Dirichlet problem

$$\begin{cases} -\mathrm{div}(a(x, \nabla u)) = \lambda k(x) f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

for every $\lambda > 0$, admits a sequence $\{u_n\}$ of pairwise distinct weak solutions such that

$$\lim_{n \to \infty} \|u_n\| = 0.$$

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