A NOTE ON p-EXTENSIONS

DRISS BENNIS and ABDULAZIZ SHAIEA

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This paper investigates two questions on p-extensions given in [9]. Precisely, we investigate the question of whether p-extensions and associate p-extensions coincide. We note by studying some particular cases that they tend to be the same notion as in the classical case of integral domains. The second question concerns the transfer of the unique factorization property for domains under associate p-extensions. We give a partial positive answer to this question. Moreover, we investigate the transfer of other factorization properties under p-extensions.

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1. INTRODUCTION

Throughout this paper all rings are commutative with identity; in particular, R and S denote such rings, and all modules are unitary. The set of all units of R will be denoted by $U(R)$.

Recall, from [8], that a ring extension $R \hookrightarrow S$ is said to be a *p-extension*, if the principal ideals of S are generated by elements of R; that is, for every $s \in S$, there is an $r \in R$ such that $sS = rS$. Also recall that a ring extension $R \hookrightarrow S$ is said to be an *associate p-extension* if, for every $s \in S$, there is an $r \in R$ and a unit $u \in S$ such that $r = su$.

Clearly, a ring extension $R \hookrightarrow S$ is a p-extension if and only if, for each $s \in S$, there are an $r \in R$ and $t_1, t_2 \in S$ such that $r = st_1$ and $s = rt_2$. Recall that two nonzeros, nonunits $e, f \in R$ are said to be *associates* (resp., *strong associates*), if $Re = Rf$ (resp., $e = uf$ for some $u \in U(R)$) (see [4, Definition 2.1]). Thus, a ring extension $R \hookrightarrow S$ is a p-extension (resp., an associate p-extension) if and only if, every element of S is an associate (resp., a strong associate) to an element of R. Also, clearly, an associate p-extension is a p-extension, while if S is an integral domain, a ring extension $R \hookrightarrow S$ is an associate p-extension if and only if it is a p-extension; and, in this case, they are already known by the well-centered concept (see [15]). The term well-centered is also used in [13] for rings with zero-divisors to mean an associate p-extension. In [8], the authors asked whether there is a p-extension which is not an associate p-extension. In the second section of this paper, we deal with this question. We show that in fact it seems like the two notions coincide rather than they are different. This is confirmed by analyzing some particular cases. We end the section with a study of p-extensions and associate p-extensions between n-trivial ring extensions which are recently introduced in [3] (see Theorem 2.6 and Corollaries 2.7, 2.8 and 2.10). It gives both new results and an extension of some results established in [8].

In Section 3, we investigate the transfer of some factorization properties under associate p-extensions. Namely, Theorem 3.8 gives a partial answer to the question posed at the end of the paper [8] concerning the transfer of the unique factorization property for domains under associate p-extensions.

2. p-EXTENSIONS AND ASSOCIATE p-EXTENSIONS

In this section, we investigate the question of whether p-extensions and associate p-extensions coincide. We show, following the study of some situations, that indeed they tend to be the same notions.

We start with extensions of Présimplifiable rings. Recall that a ring S is said to be *présimplifiable* if, for every a and b in S, $ab = a$ implies $a = 0$ or $b \in U(S)$. Présimplifiable rings were introduced and studied by Bouvier in a series of papers (see, for instance, [10] and [11]). Integral domains and quasi-local rings are examples of présimplifiable rings.

PROPOSITION 2.1. *Let $f : R \hookrightarrow S$ be a ring extension. If S is présimplifiable, then f is a p-extension if and only if it is an associate p-extension.*

Proof. Suppose that $f : R \hookrightarrow S$ is a p-extension, and consider an $s \in S$. The case $s = 0$ is trivial. Assume that $s \neq 0$. Then, there are an $r \in R$ and two elements $t_1, t_2 \in S$ such that $st_1 = r$ and $rt_2 = s$. Then, $s = t_1t_2s$ and so $t_1t_2 \in U(S)$ (since S is présimplifiable), as desired. $\square$

COROLLARY 2.2. *Let $f : R \hookrightarrow S$ be a ring extension. If S is quasi-local, then f is a p-extension if and only if it is an associate p-extension.*

It is clear that when $f : R \hookrightarrow S$ is a ring extension and S is an integral domain, then R is also an integral domain, and, in this case, f being a p-extension or an associate p-extension is the same. In the following result, we show that if $f : R \hookrightarrow S$ is a p-extension and R is an integral domain, then also S is an integral domain, and thus f is an associate p-extension. In fact, we establish this result in a more general context. Recall from [8] that an extension $f : R \hookrightarrow S$ is said to be *rigid* if, given an $s \in S$, there exists an $r \in R$ such that $Ann_S(s) = Ann_S(r)$. Clearly, every p-extension is rigid (see [8]).

PROPOSITION 2.3. *Let $R \hookrightarrow S$ be a rigid extension. If R is an integral domain, then S is an integral domain.*

Proof. Let $s_1, s_2 \in S$ such that $s_1 s_2 = 0$. By hypothesis, there are $r_1, r_2 \in R$ such that $Ann_S(s_1) = Ann_S(r_1)$ and $Ann_S(s_2) = Ann_S(r_2)$. Then, $s_1 \in Ann_S(s_2) = Ann_S(r_2)$, so $r_2 s_1 = 0$. Hence, $r_2 \in Ann_S(s_1) = Ann_S(r_1)$, which implies that $r_1 r_2 = 0$. Then, $r_1 = 0$ or $r_2 = 0$ (since R is an integral domain). Then, $s_1 = 0$ or $s_2 = 0$. Therefore, S is an integral domain. $\square$

As a simple consequence, we get the following result.

COROLLARY 2.4. *Let $f : R \hookrightarrow S$ be a ring extension. If R is an integral domain, then f is a p-extension if and only if it is an associate p-extension.*

Recall that a subset I of R is said to be *absorbing* if, $R \backslash I \neq \phi$ and for every $x \in I$ and nonzero $r \in R \backslash I$, $xr \in I$ (see [9]). From [9, Theorem 3.4], if, for an extension $R \hookrightarrow S$, $S \backslash R$ is an absorbing subset of S, then R is an integral domain. Then we get the following result.

COROLLARY 2.5. *Suppose that $f : R \hookrightarrow S$ is a p-extension. If $S \backslash R$ is an absorbing subset of S, then f is an associate p-extension.*

The results above show that if one would like to look for the existence of an example of a p-extension which is not an associate p-extension, it is necessary to consider only rings with zero-divisors. As an important class of rings with zero-divisors, the trivial extension of rings (also called idealization) has been used by many authors and in various contexts in order to produce examples of rings with zero-divisors satisfying preassigned conditions (see, for instance, [6] and [14]). Here we investigate p-extensions in the following generalization of the classical trivial extension which is recently introduced in [3] as follows[1]: Let $M = (M_i)_{i=1}^{n}$ be a family of R-modules and $\{\varphi_{i,j}\}_{\substack{i+j \leq n \\ 1 \leq i,j \leq n-1}}$ be a family of bilinear maps such that each $\varphi_{i,j}$ is written multiplicatively:

$$\begin{array}{rccl} \varphi_{i,j} : & M_i \times M_j & \longrightarrow & M_{i+j} \\ & (m_i, m_j) & \longmapsto & \varphi_{i,j}(m_i, m_j) := m_i m_j \end{array}$$

such that the following two assertions hold:

- $(m_i m_j) m_k = m_i (m_j m_k)$ for $m_i \in M_i$, $m_j \in M_j$ and $m_k \in M_k$ with $1 \leq i, j, k \leq n-2$ and $i + j + k \leq n$, and

[1] Note that a particular associate p-extension was introduced in [13, Definition 3.0]. The authors used the trivial extension to investigate questions raised by Heinzer and Roitman in [15].

- $m_i m_j = m_j m_i$ for every $m_i \in M_i$ and $m_j \in M_j$ with $1 \leq i, j \leq n-1$ and $i + j \leq n$.

The n-trivial extension of R by M is the ring, denoted by $R \ltimes_n M_1 \ltimes \cdots \ltimes M_n$ or simply $R \ltimes_n M$, whose underlying additive group is $R \oplus M_1 \oplus \cdots \oplus M_n$ with multiplication given by

$$(m_0, ..., m_n)(m'_0, ..., m'_n) = (\sum_{j+k=i} m_j m'_k)$$

for all (m_i), $(m'_i) \in R \ltimes_n M$.

As done in [13, Example 3.3], if we consider inclusions of rings $A \subseteq B \subsetneqq C$. Then, we can show that $A \ltimes B \subseteq B \ltimes C$ is not an associate p-extension. For this reason, we restrict our attention to extensions of trivial extensions with the same family of modules. Namely, for the remainder of this section we use the following notations:

Let $f : R \hookrightarrow S$ be a ring extension. Let $M = (M_i)_{i=1}^n$ be a family of S-modules and $\{\varphi_{i,j}\}_{\substack{i+j \leq n \\ 1 \leq i,j \leq n-1}}$ be a family of bilinear maps. Then, naturally we get the following ring extension $\widetilde{f} : R \ltimes_n M \hookrightarrow S \ltimes_n M$ such that $\widetilde{f}(m_i) := (f(m_0), m_1, ..., m_n)$.

Theorem 2.6. *The following assertions hold:*

1. *The ring extension $\widetilde{f}$ is an associate p-extension if and only if f is an associate p-extension.*
2. *If $\widetilde{f}$ is a p-extension, then f is also a p-extension.*

Furthermore, if S is a présimplifiable ring, then $\widetilde{f}$ is a p-extension if and only if $\widetilde{f}$ is an associate p-extension.

Proof. 1) Suppose that $\widetilde{f}$ is an associate p-extension. Let $0 \neq s \in S$. By hypothesis, there are $(e_i) \in R \ltimes_n M$ and $(x_i) \in U(S \ltimes_n M)$ such that $(s, 0, ..., 0)(x_i) = (e_i)$. So $sx_0 = e_0 \in R$. Therefore, f is an associate p-extension (since $U(S \ltimes_n M) = U(S) \ltimes_n M$ by [3, Proposition 4.9 (2)]).

Conversely, suppose that f is an associate p-extension. Let $(e_i) \in S \ltimes_n M$. We may suppose that $(e_i) \neq 0$ and $e_0 \neq 0$. Then, there are an $u \in U(S)$ and an $r \in R$ such that $e_0 u = r$. We have $(e_i)(u, 0, ..., 0) \in R \ltimes_n M$. Therefore, $\widetilde{f}$ is an associate p-extension.

2) Let $0 \neq s \in S$. We have $(s, 0, ..., 0) \in S \ltimes_n M$, so there are $(e_i) \in R \ltimes_n M$ and $(x_i), (y_i) \in S \ltimes_n M$ such that $(s, 0, ..., 0) = (e_i)(x_i)$ and $(e_i) = (s, 0, ..., 0)(y_i)$. Then, $s = e_0 x_0$ and $e_0 = s y_0$. Therefore, $sS = e_0 S$, as desired.

Suppose that S is présimplifiable and $\widetilde{f}$ is a p-extension. Then, by (2), f is a p-extension. Then, by Proposition 2.1, f is an associate p-extension. Hence, from (1), $\widetilde{f}$ is an associate p-extension. $\square$

As consequences we get the following results.

Corollary 2.7. *Suppose that S is a présimplifiable ring. Then the following assertions are equivalent:*

1. *$f : R \hookrightarrow S$ is a p-extension.*
2. *$f : R \hookrightarrow S$ is an associate p-extension.*
3. *$\widetilde{f} : R \ltimes_n M \hookrightarrow S \ltimes_n M$ is an associate p-extension.*
4. *$\widetilde{f} : R \ltimes_n M \hookrightarrow S \ltimes_n M$ is a p-extension.*

Corollary 2.8. *If $f : R \hookrightarrow S$ is an associate p-extension, then $\widetilde{f}$ is a p-extension if and only if $\widetilde{f}$ is an associate p-extension.*

We end this section with the case where S is a field.

Lemma 2.9. *If S is a field, then $R \hookrightarrow S$ is an associate p-extension.*

Proof. Obvious. $\square$

Corollary 2.10. *If S is a field, then $R \ltimes_n M \hookrightarrow S \ltimes_n M$ is an associate p-extension.*

Proof. Use Lemma 2.9 and Theorem 2.6. $\square$

3. TRANSFER RESULTS OF SOME FACTORIZATION PROPERTIES

Let S be a commutative ring. A nonunit $a \in S$ is said to be *irreducible* or an *atom* if $a = bc$ implies a and b are associates or a and c are associates. Recall that a ring S is called *atomic* if every (nonzero) nonunit of S is a product of irreducible elements (atoms) of S. Also recall that an integral domain S is said to be a *unique factorization domain* (UFD) if S is atomic and if $0 \neq r_1 \cdots r_m = s_1 \cdots s_n$ are two factorizations into atoms, then $n = m$ and, after a suitable reordering, r_i and s_i are associates for each $i = 1, ..., n$.

In [8, Unresolved Questions (2)] the authors asked: Is it the case that if S is a p-extension of R and R is a UFD, then so is S?

At the end of this section, we will show that the desired transfer holds under the condition that $S \backslash R$ is an absorbing subset of S (see Theorem 3.8). Before that, we present some results that study the transfer results for various kind of factorization properties. For a background on factorization in commutative rings see, for instance, [1, 4] and [5].

We start with the transfer result of the atomicity.

Lemma 3.1. *Suppose that $R \hookrightarrow S$ is a p-extension, and $S \backslash R$ is an absorbing subset of S. Then every irreducible element in R is irreducible in S.*

Proof. Let $r \in R$ be a nonzero nonunit. Suppose that r is irreducible in R but it is not irreducible in S. Let then $r = s_1 s_2$ for some $s_1, s_2 \in S \backslash U(S)$. Since $R \hookrightarrow S$ is an associate p-extension (by Corollary 2.5), there are $u_1, u_2 \in U(S)$ and $r_1, r_2 \in R$ such that $s_1 u_1 = r_1$ and $s_2 u_2 = r_2$. Then, $r = u_1^{-1} u_2^{-1} r_1 r_2$. We have $u_1^{-1} u_2^{-1} \in R$. Otherwise $u_1^{-1} u_2^{-1} \in S \backslash R$. Then, since $S \backslash R$ is an absorbing subset of S and $r_1 r_2 \in R$, $u_1^{-1} u_2^{-1} r_1 r_2 \in S \backslash R$; that is $r \in S \backslash R$ which is absurd. Hence $u_1^{-1} u_2^{-1} \in R$. Since r is irreducible in R, either $u_1^{-1} u_2^{-1} r_1 \in U(R)$ or $r_2 \in U(R)$. If $r_2 \in U(R)$, then $r_2 \in U(S)$; absurd since r is not irreducible in S. Then, $u_1^{-1} u_2^{-1} r_1 \in U(R)$ and so $r_1 \in U(R)$. But this implies that $r_1 \in U(S)$ which is also absurd. Therefore, r is irreducible in S. $\square$

THEOREM 3.2. *Suppose that $R \hookrightarrow S$ is a p-extension, and $S \backslash R$ is an absorbing subset of S. If R is atomic, then S is atomic.*

Proof. Let $s \in S$. Since $R \hookrightarrow S$ is an associate p-extension, there are $u \in U(S)$ and $r \in R$ such that $su = r$ (by Corollary 2.5), so $s = u^{-1} r$. Since R is atomic, r is a product of irreducible elements of R, so, by Lemma 3.1, r is a product of irreducible elements of S. Then, s is a product of irreducible elements of S. This shows that S is atomic. $\square$

We say that R satisfies the ascending chain condition on principal ideals (ACCP) if there does not exist an infinite strictly ascending chain of principal ideals of R. It is well-known that this property guarantees that the ring is atomic but the converse does not hold (see [12]). In [12, Proposition 2.1], it was observed that if $R \subseteq S$ is an extension of integral domains with $U(S) \cap R = U(R)$, then R satisfies ACCP if S satisfies ACCP. Next result studies the ascent of ACCP property.

In the proof of the following result we use the fact that, for an extension $R \hookrightarrow S$, if $S \backslash R$ is an absorbing subset of S, then $rS \cap R = rR$ for every $r \in R$ (see [9, Theorem 3.4]).

PROPOSITION 3.3. *Suppose that $R \hookrightarrow S$ is a p-extension, and $S \backslash R$ is an absorbing subset of S. If R satisfies ACCP, then S satisfies ACCP.*

Proof. Suppose that R satisfies ACCP and S does not. Consider an ascending chain of principal ideals of S:

$$s_1 S \subseteq s_2 S \subseteq \cdots \subseteq s_m S \subseteq \cdots$$

Then, by hypothesis, there is an $r_i \in R$, such that $s_i S = r_i S$ for every $i \in \mathbb{N}$. Then

$$r_1 S \cap R \subseteq r_2 S \cap R \subseteq \cdots \subseteq r_m S \cap R \subseteq \cdots$$

Since $S \backslash R$ is an absorbing subset of S, we get using [9, Theorem 3.4],

$$r_1 R \subseteq r_2 R \subseteq \cdots \subseteq r_m R \subseteq \cdots$$

Since R satisfies ACCP, there is an $n \in \mathbb{N}$ such that $r_nR = r_iR$ for every $i \geq n$. Let us prove that $s_nS = s_iS$ for every $i \geq n$. Let $i \geq n$. We prove that $s_iS \subseteq s_nS$. We have $s_i \in s_iS = r_iS$, so $s_i = r_ix$ for some $x \in S$. But $r_i \in r_iR = r_nR$, so $r_i = r_nr'$ for some $r' \in R$, then $s_i = r_nr'x = r_n(r'x) \in r_nS = s_nS$, then $s_iS \subseteq s_nS$ for every $i \geq n$. □

Recall that a ring R is said to be a *bounded factorial ring* (BFR) if, for each nonzero nonunit $x \in R$, there is a natural number $N(x)$ so that for any factorization $x = x_1 \cdots x_n$, where each x_i is irreducible, we have $n \leq N(x)$. For domains we say BFD instead of BFR. In [2, p. 16], it is proved that if $U(S) \cap R = U(R)$, then R is a BFD whenever S is a BFD. Next result studies the ascent of this property.

THEOREM 3.4. *Suppose that $R \hookrightarrow S$ is a p-extension with $S \backslash R$ is an absorbing subset of S. Then if R is a BFD, then S is also a BFD.*

Proof. Since $S \backslash R$ is an absorbing subset of S, $R \hookrightarrow S$ is an associate p-extension (by Corollary 2.5). Let $s \in S$ be a nonzero nonunit. Suppose that s has a factorization $s = s_1 \cdots s_n$ where $n \in \mathbb{N}$ and s_i is a nonunit. Since $R \hookrightarrow S$ is an associate p-extension, there are $u_i \in U(S)$ and $r_i \in R$, for $i = 1, ..., n$, such that $s_iu_i = r_i$, so $s = ur_1 \cdots r_n$, where $u = u_1^{-1} \cdots u_n^{-1}$, so $n \leq N(r)$ (since R is a BFD) where $r = r_1 \cdots r_n$. Hence, S is a BFD. □

Recall that an integral domain R is said to be a *finite factorization domain* (FFD) if each nonzero nonunit of R has only a finite number of nonassociate divisors and hence, only a finite number of factorizations up to order and associates.

Unlike the previous factorization properties, it was observed in [7, p. 8] that for an extension $R \subseteq S$ of integral domain, we may have $U(S) \cap R = U(R)$ and S is an FFD but R is not an FFD. However, in [2, p. 17], it was mentioned if S is an FFD and $U(S) \cap \mathrm{qf}(R) = U(R)$, where $\mathrm{qf}(R)$ is the quotient field of R, then R is an FFD.

THEOREM 3.5. *Suppose that $R \hookrightarrow S$ is a p-extension with $S \backslash R$ is an absorbing subset of S. If R is an FFD, then S is also an FFD.*

Proof. Let $0 \neq s \in S$ be a nonunit. Since $S \backslash R$ is an absorbing subset of S and by Corollary 2.5, $R \hookrightarrow S$ is an associate p-extension, so that there are $u \in U(S)$ and $r \in R$ such that $s = u^{-1}r$. Since R is an FFD, r has only a finite number of factorizations up to order and associates, hence s has only a finite number of factorizations up to order and associates, then S is FFD, as desired. □

Now we study the transfer result of the UFD property. We need the following lemmas.

LEMMA 3.6. *Assume that $S\backslash R$ is an absorbing subset of S, then $U(S)\cap R=U(R)$.*

Proof. Let $s\in U(S)\cap R$, then there is $s'\in S$ such that $ss'=1$. Then, $s'\in R$, otherwise, $s'\in S\backslash R$, since $s\in R$. Thus, $1=ss'\in S\backslash R$, absurd. Therefore, $U(S)\cap R=U(R)$. $\square$

LEMMA 3.7. *Suppose that $R\hookrightarrow S$ is an associate p-extension. If $U(S)\cap R=U(R)$, then every irreducible element of S is an associate to an irreducible element of R.*

Proof. Let $s\in S$ be a nonzero nonunit. Suppose that s is irreducible in S. Since $s\in S$ and by hypothesis, there are $u\in U(S)$ and $r\in R$ such that $su=r$. We show that r is irreducible in R. Assume that $r=r_1r_2$ for some $r_1,r_2\in R$, then $s=u^{-1}r_1r_2$. Since s is irreducible in S, we get either $u^{-1}r_1\in U(S)$ or $r_2\in U(S)$. If $r_2\in U(S)$, then $r_2\in U(S)\cap R=U(R)$. If $u^{-1}r_1\in U(S)$, then $r_1\in U(S)\cap R=U(R)$. Hence, r is irreducible in R, as desired. $\square$

THEOREM 3.8. *Suppose that $R\hookrightarrow S$ is a p-extension with $S\backslash R$ is an absorbing subset of S. Then if R is a UFD, then S is also a UFD.*

Proof. Since R is a UFD, R is atomic. Then, by Theorem 3.2, S is also atomic.

Now, suppose that $p_1\cdots p_n=q_1\cdots q_m$ where both p_i,q_j are irreducibles in S for $i=1,...,n$ and $j=1,...,m$. By Corollary 2.5, $R\hookrightarrow S$ is an associate p-extension, then there are $u_i,v_j\in U(S)$ and $p'_i,q'_j\in R$ such that $p_i=u_i^{-1}p'_i$ and $q_j=v_jq'_j$. By Lemmas 3.6 and 3.7, p'_i and q'_j are irreducibles in R. Hence we have $u'p'_1\cdots p'_n=v'q'_1\cdots q'_m$, where $u'=u_1^{-1}\cdots u_n^{-1}$ and $v'=v_1^{-1}\cdots v_m^{-1}$. Then, $u',v'\in R$, otherwise $p'_1\cdots p'_n=u'^{-1}v'q'_1\cdots q'_m\in S\backslash R$ (since $S\backslash R$ is an absorbing subset of S), absurd. Then, using the unicity of factorization in R, it follows that $n=m$ and p'_i and q'_i are associates, so p_i and q_i are associates for every $i=1,...,n$, as desired. $\square$

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Mohammed V University in Rabat,
Faculty of Sciences,
Centre de Recherche de Mathématiques
et Applications de Rabat (CeReMAR),
Rabat, Morocco
d.bennis@fsr.ac.ma; driss_bennis@hotmail.com
abdulazizsha@yahoo.com