In this paper, we derive some combinatorial properties of generalized Fibonacci quaternion $Q_n$ and generalized Lucas quaternion $K_n$, where the components of $Q_n$ and $K_n$ are generalized Fibonacci number $U_n$ and generalized Lucas number $V_n$, respectively.

Firstly, we obtain some basic identities and use them to prove Catalan identity for $Q_n$ and $K_n$. Then we find some sum formulas for $Q_n$ and $K_n$ in terms of $Q_n$ and $K_n$. Moreover, we derive the generating functions of these generalized quaternions.

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**Key words:** generalized Fibonacci numbers, generalized Lucas numbers, generalized quaternions, generating functions.

**1. PRELIMINARIES**

Let $p, q$ be arbitrary nonzero real numbers. The generalized Fibonacci sequence $(U_n) = (U_n(p, q))$ and the generalized Lucas sequence $(V_n) = (V_n(p, q))$ are defined by

\[(1.1)\quad U_0 = 0, U_1 = 1, U_n = pU_{n-1} + qU_{n-2}\]

and

\[(1.2)\quad V_0 = 2, V_1 = p, V_n = pV_{n-1} + qV_{n-2}\]

for $n \geq 2$. The terms $U_n$ and $V_n$ are called the $n$th generalized Fibonacci and Lucas numbers, respectively. Moreover, generalized Fibonacci and Lucas numbers can be extended to negative indices by $U_{-n} = -(-q)^{-n}U_n$ and $V_{-n} = (-q)^{-n}V_n$ for all $n \in \mathbb{N}$. Properties of these sequences are determined in [1,2,4,7].

Let $\alpha = \frac{p + \sqrt{\Delta}}{2}$ and $\beta = \frac{p - \sqrt{\Delta}}{2}$ be the roots of the characteristic
equation $x^2 - px - q = 0$ where $\Delta = p^2 + 4q$. Then it is easy to verify that $\alpha + \beta = p$, $\alpha - \beta = \sqrt{\Delta}$ and $\alpha \beta = -q$. Moreover, it can be seen that $\alpha^n = \alpha U_n + qU_{n-1}$ and $\beta^n = \beta V_n + qV_{n-1}$. If $\Delta \neq 0$, then Binet formulae for $U_n$ and $V_n$ can be given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n$$

for all $n \geq 0$.

With the same initial conditions given in (1.1) and (1.2), it is obvious that $(U_n(1,1))$ is the known Fibonacci sequence $(F_n)$ and $(V_n(1,1))$ is the known Fibonacci sequence $(L_n)$, respectively. Moreover, taking $k$ instead of $p$ and $1$ instead of $q$ in (1.1) and (1.2), we obtain $k$–Fibonacci and $k$–Lucas sequences, respectively. It is well known that Pell sequence $(P_n)$ is a special case of $(U_n(p,q))$ with $p = 2$ and $q = 1$.

### 2. SOME IDENTITIES CONCERNING THE GENERALIZED QUATERNIONS $Q_n$ AND $K_n$

A quaternion $q$ is a hyper complex number of the form

$$q = a + ib + jc + kd$$

with real components $a, b, c, d$ and basis $1, i, j, k$, where

$$i^2 = j^2 = k^2 = -1, \quad ij = k = -ji, \quad jk = i = -kj, \quad \text{and} \quad ki = j = -ik,$$

and the conjugate of quaternion $q$ is defined as $\bar{q} = a - ib - jc - kd$ in [3]. Quaternions are first observed by William R. Hamilton in 1843. Hamilton called the pure real term a scalar and the imaginary part a vector. The group of quaternions are denoted as $\mathbb{H}$. Addition is closed over the quaternion group $\mathbb{H}$ and addition is commutative. But quaternion multiplication is not commutative.

Generalized Fibonacci quaternion $Q_n$ is defined by

$$Q_n = U_n + iU_{n+1} + jU_{n+2} + kU_{n+3}, \quad n \geq 0$$

where $U_n$ is the $n$th generalized Fibonacci number and the generalized Lucas quaternion $K_n$ is defined by

$$K_n = V_n + iV_{n+1} + jV_{n+2} + kV_{n+3}, \quad n \geq 0$$

where $V_n$ is the $n$th generalized Lucas number in [4,6].

In [8], Ramirez considered $k$–Fibonacci quaternions and $k$–Lucas quaternions with $k$–Fibonacci and $k$–Lucas number components. He proved Cassini’s identity for $k$–Fibonacci quaternions. Then he gave a conjecture related to Catalan identity of $k$–Fibonacci quaternions. In [9], the authors proved
Ramirez’s conjecture. In this paper, we prove Catalan identity for the generalized Fibonacci and Lucas quaternions. If someone takes \( p = k \) and \( q = 1 \) in the components of generalized Fibonacci quaternion \( Q_n \) and Lucas quaternion \( K_n \), then he/she gets \( k \)--Fibonacci quaternions and \( k \)--Lucas quaternions. Therefore \( k \)--Fibonacci and \( k \)--Lucas quaternions are the special cases of the generalized Fibonacci quaternion \( Q_n \) and Lucas quaternion \( K_n \). Thus the following theorems are more general forms of the theorems given in [8] by Ramirez and in [9, 10] by Polatlı and Kesim.

The Binet formulae for the generalized Fibonacci quaternions \( Q_n \) and the generalized Lucas quaternions \( K_n \) were discovered by Iakin in [5]. We will prove these formulae in the following theorem.

**Theorem 1.** For all \( n \in \mathbb{Z} \),

\[
Q_n = \frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha - \beta}
\]

and

\[
K_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n
\]

where \( \hat{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3 \) and \( \hat{\beta} = 1 + i\beta + j\beta^2 + k\beta^3 \).

**Proof.** If we consider Binet formulae for generalized Fibonacci number \( U_n \) and generalized Lucas number \( V_n \), then we obtain

\[
Q_n = U_n + iU_{n+1} + jU_{n+2} + kU_{n+3}
\]

\[
= \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + j\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} + k\frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta}
\]

\[
= \frac{(1 + i\alpha + j\alpha^2 + k\alpha^3) \alpha^n}{\alpha - \beta} - \frac{(1 + i\beta + j\beta^2 + k\beta^3) \beta^n}{\alpha - \beta}
\]

\[
= \frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha - \beta}
\]

and

\[
K_n = V_n + iV_{n+1} + jV_{n+2} + kV_{n+3}
\]

\[
= (\alpha^n + \beta^n) + i(\alpha^{n+1} + \beta^{n+1}) + j(\alpha^{n+2} + \beta^{n+2}) + k(\alpha^{n+3} + \beta^{n+3})
\]

\[
= (1 + i\alpha + j\alpha^2 + k\alpha^3) \alpha^n + (1 + i\beta + j\beta^2 + k\beta^3) \beta^n
\]

\[
= \hat{\alpha}\alpha^n + \hat{\beta}\beta^n. \quad \Box
\]

In the proofs of the following theorems we will need the formulas \( \hat{\alpha}, \hat{\beta}, \alpha \hat{\alpha}, \alpha \hat{\beta} - \hat{\beta} \alpha, \alpha \hat{\beta} + \hat{\beta} \alpha, \hat{\alpha} \alpha, \hat{\beta} \beta, \hat{\alpha} \beta \beta r - \hat{\beta} \alpha r \hat{\beta} \beta \hat{\alpha}, \beta \alpha - \beta \alpha, \hat{\beta} \alpha + \hat{\alpha} \beta, \beta \alpha^m - \hat{\alpha} \beta^m \).
and \( \hat{\beta} \alpha^m + \hat{\alpha} \beta^m \) for all \( m, r \in \mathbb{Z} \). So, we list these elementary results without proof by using the definitions of \( \hat{\alpha} \) and \( \hat{\beta} \) as follows:

\begin{align}
\hat{\alpha} \hat{\beta} &= A + q\sqrt{\Delta B}, \\
\hat{\beta} \hat{\alpha} &= A - q\sqrt{\Delta B}, \\
\hat{\alpha} \hat{\beta} + \hat{\beta} \hat{\alpha} &= 2A, \\
\hat{\alpha} \hat{\beta} - \hat{\beta} \hat{\alpha} &= 2q\sqrt{\Delta B}, \\
\hat{\alpha} \hat{\alpha} &= 2 \hat{\alpha} - (1 + \alpha^2) (1 + \alpha^4), \\
\hat{\beta} \hat{\beta} &= 2 \hat{\beta} - (1 + \beta^2) (1 + \beta^4), \\
\hat{\alpha} \hat{\beta}^r - \hat{\beta} \hat{\alpha}^r &= \sqrt{\Delta} (qB \nu_r - A \omega_r), \\
\hat{\alpha} \hat{\alpha} - \hat{\beta} \hat{\beta} &= q\sqrt{\Delta Q_{-1}}, \\
\hat{\beta} \hat{\alpha} + \hat{\alpha} \hat{\beta} &= -qK_{-1}, \\
\hat{\beta} \alpha^m - \hat{\alpha} \beta^m &= -(-q)^m \sqrt{\Delta Q_{-m}}, \\
\hat{\beta} \alpha^m + \hat{\alpha} \beta^m &= (-q)^m K_{-m}
\end{align}

where \( A = K_0 - (1 - q)(1 + q^2) \) and \( B = i(-q) + j(-p) + k \).

**Theorem 2.** Let \( n \geq r \geq 1 \) be integers. Then

\[ Q_{n-r} Q_{n+r} - Q_n^2 = (-q)^{n-r} U_r (qB \nu_r - A \omega_r) \]

and

\[ K_{n-r} K_{n+r} - K_n^2 = (-q)^{n-r} \Delta U_r (A \omega_r - qB \nu_r). \]

**Proof.** If we apply Binet formula for generalized Fibonacci quaternion \( Q_n \), then we get

\[
Q_{n-r} Q_{n+r} - Q_n^2 = \left( \frac{\hat{\alpha} \alpha^{n-r} \beta^{n-r} - \hat{\beta} \beta^{n-r}}{\alpha - \beta} \right) \left( \frac{\hat{\alpha} \alpha^{n+r} \beta^{n+r} - \hat{\beta} \beta^{n+r}}{\alpha - \beta} \right) - \left( \frac{\hat{\alpha} \alpha^n - \hat{\beta} \beta^n}{\alpha - \beta} \right)^2
\]

\[
= \frac{(\alpha \beta)^{n-r}}{(\alpha - \beta)^2} \left[ \hat{\alpha} \hat{\beta} \left( 1 - \frac{\beta^r}{\alpha^r} \right) + \hat{\beta} \hat{\alpha} \left( 1 - \frac{\alpha^r}{\beta^r} \right) \right]
\]

\[
= \frac{(\alpha \beta)^{n-r}}{(\alpha - \beta)} \left( \frac{\hat{\alpha} \beta \beta^r - \hat{\beta} \alpha \alpha^r}{\alpha - \beta} \right)
\]

\[
= (-q)^{n-r} U_r (qB \nu_r - A \omega_r),
\]

using the facts that \( \alpha \beta = -q, \alpha - \beta = \sqrt{\Delta} \) and (2.7).

Similarly, applying Binet formula for generalized Lucas quaternion \( K_n \) and using the facts \( \alpha \beta = -q, \alpha - \beta = \sqrt{\Delta} \) and (2.7), we obtain
\[ K_{n-r}K_{n+r} - K_n^2 \]
\[ = \left( \hat{\alpha}^{n-r} + \hat{\beta}\beta_n^{n-r} \right) \left( \hat{\alpha}^{n+r} + \hat{\beta}\beta_n^{n+r} \right) - \left( \hat{\alpha}^n + \hat{\beta}\beta^n \right)^2 \]
\[ = (\alpha\beta)^n \left( \frac{\hat{\beta}\alpha^r - \hat{\alpha}\hat{\beta}\beta^r}{(\alpha\beta)^r} \right) \]
\[ = - (\alpha\beta)^{n-r} (\alpha - \beta)^2 \left( \frac{\hat{\alpha}\beta\beta^r - \hat{\beta}\hat{\alpha}\beta^r}{\alpha - \beta} \right) \]
\[ = (-q)^{n-r} \Delta U_r (AU_r - qBV_r). \quad \square \]

Since the following corollary can be proved by Theorem 2, we omit its proof.

**Corollary 1.** Let \( n, r \in \mathbb{Z} \) and \( n \geq r \geq 1 \). Then
\[ K_{n-r}K_{n+r} - K_n^2 = -\Delta \left[ Q_{n-r}Q_{n+r} - Q_n^2 \right]. \]

**Theorem 3.** Let \( n \in \mathbb{N} \) and \( m, r \in \mathbb{Z} \). Then
\[ \sum_{i=0}^{n} Q_{mi+r} = \frac{(-q)^m [Q_{mn+r} - Q_{r-m}] - Q_{mn+m+r} + Q_r}{1 - v_m + (-q)^m} \]

and
\[ \sum_{i=0}^{n} K_{mi+r} = \frac{(-q)^m [K_{mn+r} - K_{r-m}] - K_{mn+m+r} + K_r}{1 - v_m + (-q)^m}. \]

**Proof.** If we use Binet formulae for the quaternions \( Q_n \) and \( K_n \), then we get
\[ \sum_{i=0}^{n} Q_{mi+r} = \frac{1}{\alpha - \beta} \left[ \hat{\alpha}^r \sum_{i=0}^{n} \alpha^{mi} - \hat{\beta}^r \sum_{i=0}^{n} \beta^{mi} \right] \]
\[ = \frac{1}{\alpha - \beta} \left[ \hat{\alpha}^r \left( \frac{\alpha^{mn+m} - 1}{\alpha^m - 1} \right) - \hat{\beta}^r \left( \frac{\beta^{mn+m} - 1}{\beta^m - 1} \right) \right] \]
\[ = \frac{1}{\alpha - \beta} \left[ \frac{(\alpha\beta)^m \left( \hat{\alpha}^{mn+r} - \hat{\beta}\beta^{mn+r} \right) - \left( \hat{\alpha}^{mn+m+r} - \hat{\beta}\beta^{mn+m+r} \right)}{(\alpha\beta)^m - (\alpha^m + \beta^m) + 1} \right. \]
\[ - \left. \frac{(\alpha\beta)^k \left( \hat{\alpha}^{r-m} - \hat{\beta}\beta^{r-m} \right) + \left( \hat{\alpha}^r - \hat{\beta}\beta^r \right)}{(\alpha\beta)^m - (\alpha^m + \beta^m) + 1} \right] \]
\[ = \frac{1}{1 - v_m + (-q)^m} \left[ (\alpha\beta)^m \frac{\hat{\alpha}^{mn+r} - \hat{\beta}\beta^{mn+r}}{\alpha - \beta} - (\alpha\beta)^m \frac{\hat{\alpha}^{r-m} - \hat{\beta}\beta^{r-m}}{\alpha - \beta} \right] \]
\[
-\frac{\hat{\alpha}a^{mn+m+r} - \hat{\beta}b^{mn+m+r}}{\alpha - \beta} + \frac{\hat{\alpha}r - \hat{\beta}r}{\alpha - \beta}
\]

\[
= (\alpha\beta)^m Q_{mn+r} - (\alpha\beta)^m Q_{r-m} - Q_{mn+m+r} + Q_r
\]

\[
= \frac{(\alpha\beta)^m [Q_{mn+r} - Q_{r-m}] - Q_{mn+m+r} + Q_r}{1 - V_m + (-q)^m}
\]

and

\[
\sum_{i=0}^{n} K_{mi+r} = \sum_{i=0}^{n} \left( \hat{\alpha}a^{mi+r} + \hat{\beta}b^{mi+r} \right) = \hat{\alpha}a^r \sum_{i=0}^{n} a^{mi} + \hat{\beta}b^r \sum_{i=0}^{n} b^{mi}
\]

\[
= \hat{\alpha}a^r \left( \frac{a^{mn+m} - 1}{\alpha^m - 1} \right) + \hat{\beta}b^r \left( \frac{b^{mn+m} - 1}{\beta^m - 1} \right)
\]

\[
= \frac{\hat{\alpha}a^r (\alpha^{mn+m} - 1) (\beta^m - 1) + \hat{\beta}b^r (\beta^{mn+m} - 1) (\alpha^m - 1)}{\alpha^m - 1} (\beta^m - 1)
\]

\[
= \frac{(\alpha\beta)^m \left( \hat{\alpha}a^{mn+r} + \hat{\beta}b^{mn+r} \right) - \left( \hat{\alpha}a^{mn+m+r} + \hat{\beta}b^{mn+m+r} \right)}{1 - V_m + (-q)^m}
\]

\[
- (\alpha\beta)^k \left( \hat{\alpha}a^{r-m} + \hat{\beta}b^{r-m} \right) + \left( \hat{\alpha}a^r + \hat{\beta}b^r \right)
\]

\[
= \frac{(\alpha\beta)^m \left[ \left( \hat{\alpha}a^{mn+r} + \hat{\beta}b^{mn+r} \right) - \left( \hat{\alpha}a^{r-m} + \hat{\beta}b^{r-m} \right) \right]}{1 - V_m + (-q)^m}
\]

\[
- \left( \hat{\alpha}a^{mn+m+r} + \hat{\beta}b^{mn+m+r} \right) + \left( \hat{\alpha}a^r + \hat{\beta}b^r \right)
\]

\[
= \frac{(\alpha\beta)^m \left[ K_{mn+r} - K_{r-m} \right] - K_{mn+m+r} + K_r}{1 - V_m + (-q)^m}
\]

\[\square\]

**Theorem 4.** Let \( n \in \mathbb{N} \). Then

\[
\sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} Q_i = Q_{2n}
\]

and

\[
\sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} K_i = K_{2n}.
\]

**Proof.** If we use Binet formulae for the quaternions \( Q_n \) and \( K_n \), then we
get
\[
\sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} Q_i = \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} \left( \frac{\hat{\alpha}^i - \hat{\beta}^i}{\alpha - \beta} \right)
\]
\[
= \frac{\hat{\alpha}}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} (p\alpha)^i q^{n-i} - \frac{\hat{\beta}}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} (p\beta)^i q^{n-i}
\]
\[
= \frac{\hat{\alpha}}{\alpha - \beta} (p\alpha + q)^n - \frac{\hat{\beta}}{\alpha - \beta} (p\beta + q)^n
\]
\[
= \frac{\hat{\alpha}\alpha^2 - \hat{\beta}\beta^2 n}{\alpha - \beta} = Q_{2n}
\]

and
\[
\sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} K_i = \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} \left( \hat{\alpha}^i + \hat{\beta}^i \right)
\]
\[
= \hat{\alpha} \sum_{i=0}^{n} \binom{n}{i} (p\alpha)^i q^{n-i} + \hat{\beta} \sum_{i=0}^{n} \binom{n}{i} (p\beta)^i q^{n-i}
\]
\[
= \hat{\alpha} (p\alpha + q)^n + \hat{\beta} (p\beta + q)^n
\]
\[
= \hat{\alpha}\alpha^2 + \hat{\beta}\beta^2 n = K_{2n}. \quad \square
\]

**Theorem 5.** The generating functions of the generalized Fibonacci and Lucas quaternions are given by

\[
G(x) = \frac{x + i + j [p + xq] + k [(p^2 + q) + qpx]}{1 - px - qx^2}
\]

and

\[
(2.12)
J(x) = \frac{[2 - px] + i [p + 2q] + j [(p^2 + 2q) + pqx] + k [(p^3 + 3pq) + (p^2 + 2q) qx]}{1 - px - qx^2},
\]

respectively.

**Proof.** Since the power series representation of the generating function for \((Q_n)\) is

\[
(2.13)
G(x) = \sum_{n=0}^{\infty} Q_n x^n = Q_0 + xQ_1 + x^2 Q_2 + x^3 Q_3 + x^4 Q_4 + ...
\]

we can write

\[
(2.14)
-pxG(x) = -pxQ_0 - px^2 Q_1 - px^3 Q_2 - px^4 Q_3 - px^5 Q_4 - ...
\]
and

\[(2.15) \quad -qx^2G(x) = -qx^2Q_0 - qx^3Q_1 - qx^4Q_2 - qx^5Q_3 - qx^6Q_4 - \ldots.\]

If we take the sum of (2.13), (2.14) and (2.15), then we obtain

\[(1 - px - qx^2) G(x) = Q_0 + xQ_1 - pxQ_0.\]

Hence it follows that

\[G(x) = \frac{Q_0 + x(Q_1 - pQ_0)}{1 - px - qx^2} = \frac{Q_0 + xQ_{-1}}{1 - px - qx^2} = \frac{x + i + j[p + qx] + k[(p^2 + q) + px]}{1 - px - qx^2}.\]

The proof of (2.12) follows by using the similar steps. □

**Theorem 6.** Let \(m, n, r \in \mathbb{Z}\). Then the generating function of the quaternions \(Q_{mn+r}\) and \(K_{mn+r}\) are given by

\[(2.16) \quad \sum_{n=0}^{\infty} Q_{mn+r}x^n = \frac{Q_r - (-q)^m Q_{r-m}x}{1 - V_mx + (-q)^m x^2}\]

and

\[(2.17) \quad \sum_{n=0}^{\infty} K_{mn+r}x^n = \frac{K_r - (-q)^m V_{r-m}x}{1 - V_mx + (-q)^m x^2},\]

respectively.

**Proof.** If we use Binet formula for \(Q_{mn+r}\), then we get

\[
\sum_{n=0}^{\infty} Q_{mn+r}x^n = \sum_{n=0}^{\infty} \frac{\hat{\alpha}^{mn+r} - \hat{\beta}^{mn+r}}{\alpha - \beta} x^n
\]

\[
= \frac{1}{\alpha - \beta} \left[ \hat{\alpha}^r \sum_{n=0}^{\infty} (\alpha^m x)^n - \hat{\beta}^r \sum_{n=0}^{\infty} (\beta^m x)^n \right]
\]

\[
= \frac{1}{\alpha - \beta} \left[ \frac{\hat{\alpha}^r}{1 - \alpha^m x} - \frac{\hat{\beta}^r}{1 - \beta^m x} \right]
\]

\[
= \frac{1}{\alpha - \beta} \left[ \frac{(\hat{\alpha}^r - \hat{\beta}^r) - (\hat{\alpha}^r \beta^m x - \hat{\beta} \alpha^m \beta^r x)}{1 - (\alpha^m + \beta^m)x + \alpha^m \beta^m x^2} \right]
\]

\[
= \frac{1}{\alpha - \beta} \left[ \frac{(\hat{\alpha}^r - \hat{\beta}^r) - (\alpha \beta)^m (\hat{\alpha}^{r-m} - \hat{\beta} \beta^{r-m}) x}{1 - V_mx + (-q)^m x^2} \right]
\]
\[
\frac{1}{1 - V_m x + (-q)^m x^2} \left[ \left( \hat{\alpha} \alpha^r - \hat{\beta} \beta^r \right) \alpha - \beta \right] - \left( \alpha \beta \right)^m \left( \hat{\alpha} \alpha^{r-m} - \hat{\beta} \beta^{r-m} \right) \frac{x}{\alpha - \beta} \right]
= \frac{Q_r - (-q)^m Q_{r-m} x}{1 - V_m x + (-q)^m x^2}.
\]

The proof of (2.17) is almost the same as the proof of (2.16). □

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