QUATERNIONS *VIA* GENERALIZED FIBONACCI AND LUCAS NUMBER COMPONENTS

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In this paper, we derive some combinatorial properties of generalized Fibonacci quaternion Q_n and generalized Lucas quaternion K_n, where the components of Q_n and K_n are generalized Fibonacci number U_n and generalized Lucas number V_n, respectively.

Firstly, we obtain some basic identities and use them to prove Catalan identity for Q_n and K_n. Then we find some sum formulas $\sum_{i=0}^{n} Q_{mi+r}$, $\sum_{i=0}^{n} K_{mi+r}$, $\sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} Q_i$ and $\sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} K_i$ in terms of Q_n and K_n. Moreover, we derive the generating functions of these generalized quaternions.

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1. PRELIMINARIES

Let p, q be arbitrary nonzero real numbers. The generalized Fibonacci sequence $(U_n) = (U_n(p,q))$ and the generalized Lucas sequence $(V_n) = (V_n(p,q))$ are defined by

$$U_0 = 0, U_1 = 1, U_n = pU_{n-1} + qU_{n-2} \tag{1.1}$$

and

$$V_0 = 2, V_1 = p, V_n = pV_{n-1} + qV_{n-2} \tag{1.2}$$

for $n \geq 2$. The terms U_n and V_n are called the nth generalized Fibonacci and Lucas numbers, respectively. Moreover, generalized Fibonacci and Lucas numbers can be extended to negative indices by $U_{-n} = -(-q)^{-n} U_n$ and $V_{-n} = (-q)^{-n} V_n$ for all $n \in \mathbb{N}$. Properties of these sequences are determined in [1, 2, 4, 7].

Let $\alpha = \dfrac{p + \sqrt{\Delta}}{2}$ and $\beta = \dfrac{p - \sqrt{\Delta}}{2}$ be the roots of the characteristic

equation $x^2 - px - q = 0$ where $\Delta = p^2 + 4q$. Then it is easy to verify that $\alpha + \beta = p$, $\alpha - \beta = \sqrt{\Delta}$ and $\alpha\beta = -q$. Moreover, it can be seen that $\alpha^n = \alpha U_n + qU_{n-1}$ and $\beta^n = \beta V_n + qV_{n-1}$. If $\Delta \neq 0$, then Binet formulae for U_n and V_n can be given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n$$

for all $n \geq 0$.

With the same initial conditions given in (1.1) and (1.2), it is obvious that $(U_n(1,1))$ is the known Fibonacci sequence (F_n) and $(V_n(1,1))$ is the known Fibonacci sequence (L_n), respectively. Moreover, taking k instead of p and 1 instead of q in (1.1) and (1.2), we obtain $k-$Fibonacci and $k-$Lucas sequences, respectively. It is well known that Pell sequence (P_n) is a special case of $(U_n(p,q))$ with $p = 2$ and $q = 1$.

2. SOME IDENTITIES CONCERNING THE GENERALIZED QUATERNIONS Q_n AND K_n

A quaternion q is a hyper complex number of the form

$$q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$$

with real components a, b, c, d and basis $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$, where

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1},\ \mathbf{ij} = \mathbf{k} = -\mathbf{ji},\ \mathbf{jk} = \mathbf{i} = -\mathbf{kj},\ \text{and } \mathbf{ki} = \mathbf{j} = -\mathbf{ik},$$

and the conjugate of quaternion q is defined as $\overline{q} = a - \mathbf{i}b - \mathbf{j}c - \mathbf{k}d$ in [3]. Quaternions are first observed by William R. Hamilton in 1843. Hamilton called the pure real term a scalar and the imaginary part a vector. The group of quaternions are denoted as $\mathbb{H}$. Addition is closed over the quaternion group $\mathbb{H}$ and addition is commutative. But quaternion multiplication is not commutative.

Generalized Fibonacci quaternion Q_n is defined by

$$Q_n = U_n + \mathbf{i}U_{n+1} + \mathbf{j}U_{n+2} + \mathbf{k}U_{n+3},\ n \geq 0$$

where U_n is the nth generalized Fibonacci number and the generalized Lucas quaternion K_n is defined by

$$K_n = V_n + \mathbf{i}V_{n+1} + \mathbf{j}V_{n+2} + \mathbf{k}V_{n+3},\ n \geq 0$$

where V_n is the nth generalized Lucas number in [4,6].

In [8], Ramirez considered $k-$Fibonacci quaternions and $k-$Lucas quaternions with $k-$Fibonacci and $k-$Lucas number components. He proved Cassini's identity for $k-$Fibonacci quaternions. Then he gave a conjecture related to Catalan identity of $k-$Fibonacci quaternions. In [9], the authors proved

Ramirez's conjecture. In this paper, we prove Catalan identity for the generalized Fibonacci and Lucas quaternions. If someone takes $p = k$ and $q = 1$ in the components of generalized Fibonacci quaternion Q_n and Lucas quaternion K_n, then he/she gets $k-$Fibonacci quaternions and $k-$Lucas quaternions. Therefore $k-$Fibonacci and $k-$Lucas quaternions are the special cases of the generalized Fibonacci quaternion Q_n and Lucas quaternion K_n. Thus the following theorems are more general forms of the theorems given in [8] by Ramirez and in [9, 10] by Polatlı and Kesim.

The Binet formulae for the generalized Fibonacci quaternions Q_n and the generalized Lucas quaternions K_n were discovered by Iakin in [5]. We will prove these formulae in the following theorem.

THEOREM 1. *For all* $n \in \mathbb{Z}$,

$$Q_n = \frac{\widehat{\alpha}\alpha^n - \widehat{\beta}\beta^n}{\alpha - \beta}$$

and

$$K_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n$$

where $\widehat{\alpha} = \mathbf{1} + \mathbf{i}\alpha + \mathbf{j}\alpha^2 + \mathbf{k}\alpha^3$ *and* $\widehat{\beta} = \mathbf{1} + \mathbf{i}\beta + \mathbf{j}\beta^2 + \mathbf{k}\beta^3$.

Proof. If we consider Binet formulae for generalized Fibonacci number U_n and generalized Lucas number V_n, then we obtain

$$\begin{aligned}
Q_n &= U_n + \mathbf{i}U_{n+1} + \mathbf{j}U_{n+2} + \mathbf{k}U_{n+3} \\
&= \frac{\alpha^n - \beta^n}{\alpha - \beta} + \mathbf{i}\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + \mathbf{j}\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} + \mathbf{k}\frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta} \\
&= \frac{\left(\mathbf{1} + \mathbf{i}\alpha + \mathbf{j}\alpha^2 + \mathbf{k}\alpha^3\right)\alpha^n}{\alpha - \beta} - \frac{\left(\mathbf{1} + \mathbf{i}\beta + \mathbf{j}\beta^2 + \mathbf{k}\beta^3\right)\beta^n}{\alpha - \beta} \\
&= \frac{\widehat{\alpha}\alpha^n - \widehat{\beta}\beta^n}{\alpha - \beta}
\end{aligned}$$

and

$$\begin{aligned}
K_n &= V_n + \mathbf{i}V_{n+1} + \mathbf{j}V_{n+2} + \mathbf{k}V_{n+3} \\
&= (\alpha^n + \beta^n) + \mathbf{i}\left(\alpha^{n+1} + \beta^{n+1}\right) + \mathbf{j}\left(\alpha^{n+2} + \beta^{n+2}\right) + \mathbf{k}\left(\alpha^{n+3} + \beta^{n+3}\right) \\
&= \left(\mathbf{1} + \mathbf{i}\alpha + \mathbf{j}\alpha^2 + \mathbf{k}\alpha^3\right)\alpha^n + \left(\mathbf{1} + \mathbf{i}\beta + \mathbf{j}\beta^2 + \mathbf{k}\beta^3\right)\beta^n \\
&= \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n. \quad \square
\end{aligned}$$

In the proofs of the following theorems we will need the formulas $\widehat{\alpha}\,\widehat{\beta}$, $\widehat{\beta}\,\widehat{\alpha}$, $\widehat{\alpha}\,\widehat{\beta} - \widehat{\beta}\,\widehat{\alpha}$, $\widehat{\alpha}\,\widehat{\beta} + \widehat{\beta}\widehat{\alpha}$, $\widehat{\alpha}\,\widehat{\alpha}$, $\widehat{\beta}\,\widehat{\beta}$, $\widehat{\alpha}\,\widehat{\beta}\beta^r - \widehat{\beta}\,\widehat{\alpha}\alpha^r$, $\widehat{\beta}\alpha - \widehat{\alpha}\beta$, $\widehat{\beta}\alpha + \widehat{\alpha}\beta$, $\widehat{\beta}\alpha^m - \widehat{\alpha}\beta^m$

and $\widehat{\beta}\alpha^m + \widehat{\alpha}\beta^m$ for all $m, r \in \mathbb{Z}$. So, we list these elementary results without proof by using the definitions of $\widehat{\alpha}$ and $\widehat{\beta}$ as follows:

$$\widehat{\alpha}\,\widehat{\beta} = A + q\sqrt{\Delta}B, \tag{2.1}$$

$$\widehat{\beta}\,\widehat{\alpha} = A - q\sqrt{\Delta}B, \tag{2.2}$$

$$\widehat{\alpha}\,\widehat{\beta} + \widehat{\beta}\,\widehat{\alpha} = 2A \tag{2.3}$$

$$\widehat{\alpha}\,\widehat{\beta} - \widehat{\beta}\,\widehat{\alpha} = 2q\sqrt{\Delta}B, \tag{2.4}$$

$$\widehat{\alpha}\,\widehat{\alpha} = 2\,\widehat{\alpha} - \left(1+\alpha^2\right)\left(1+\alpha^4\right), \tag{2.5}$$

$$\widehat{\beta}\,\widehat{\beta} = 2\,\widehat{\beta} - \left(1+\beta^2\right)\left(1+\beta^4\right), \tag{2.6}$$

$$\widehat{\alpha}\,\widehat{\beta}\beta^r - \widehat{\beta}\,\widehat{\alpha}\alpha^r = \sqrt{\Delta}\left(qBV_r - AU_r\right), \tag{2.7}$$

$$\widehat{\beta}\alpha - \widehat{\alpha}\beta = q\sqrt{\Delta}Q_{-1}, \tag{2.8}$$

$$\widehat{\beta}\alpha + \widehat{\alpha}\beta = -qK_{-1}, \tag{2.9}$$

$$\widehat{\beta}\alpha^m - \widehat{\alpha}\beta^m = -\left(-q\right)^m\sqrt{\Delta}Q_{-m}, \tag{2.10}$$

$$\widehat{\beta}\alpha^m + \widehat{\alpha}\beta^m = \left(-q\right)^m K_{-m} \tag{2.11}$$

where $A = K_0 - (1-q)(1+q^2)$ and $B = \mathbf{i}(-q) + \mathbf{j}(-p) + \mathbf{k}$.

THEOREM 2. *Let $n \geq r \geq 1$ be integers. Then*

$$Q_{n-r}Q_{n+r} - Q_n^2 = (-q)^{n-r}U_r\left(qBV_r - AU_r\right)$$

and

$$K_{n-r}K_{n+r} - K_n^2 = (-q)^{n-r}\Delta U_r\left(AU_r - qBV_r\right).$$

Proof. If we apply Binet formula for generalized Fibonacci quaternion Q_n, then we get

$$\begin{aligned}
&Q_{n-r}Q_{n+r} - Q_n^2 \\
&= \left(\frac{\widehat{\alpha}\alpha^{n-r} - \widehat{\beta}\beta^{n-r}}{\alpha-\beta}\right)\left(\frac{\widehat{\alpha}\alpha^{n+r} - \widehat{\beta}\beta^{n+r}}{\alpha-\beta}\right) - \left(\frac{\widehat{\alpha}\alpha^{n} - \widehat{\beta}\beta^{n}}{\alpha-\beta}\right)^2 \\
&= \frac{(\alpha\beta)^n}{(\alpha-\beta)^2}\left[\widehat{\alpha}\,\widehat{\beta}\left(1-\frac{\beta^r}{\alpha^r}\right) + \widehat{\beta}\,\widehat{\alpha}\left(1-\frac{\alpha^r}{\beta^r}\right)\right] \\
&= \frac{(\alpha\beta)^{n-r}\left(\alpha^r-\beta^r\right)}{(\alpha-\beta)}\left(\frac{\widehat{\alpha}\,\widehat{\beta}\beta^r - \widehat{\beta}\,\widehat{\alpha}\alpha^r}{\alpha-\beta}\right) \\
&= (-q)^{n-r}U_r\left(qBV_r - AU_r\right),
\end{aligned}$$

using the facts that $\alpha\beta = -q$, $\alpha - \beta = \sqrt{\Delta}$ and (2.7).

Similarly, applying Binet formula for generalized Lucas quaternion K_n and using the facts $\alpha\beta = -q$, $\alpha - \beta = \sqrt{\Delta}$ and (2.7), we obtain

$$
\begin{aligned}
&K_{n-r}K_{n+r}-K_n^2\\
&\quad=\left(\widehat{\alpha}\alpha^{n-r}+\widehat{\beta}\beta^{n-r}\right)\left(\widehat{\alpha}\alpha^{n+r}+\widehat{\beta}\beta^{n+r}\right)-\left(\widehat{\alpha}\alpha^{n}+\widehat{\beta}\beta^{n}\right)^2\\
&\quad=(\alpha\beta)^n\left(\alpha^r-\beta^r\right)\left(\frac{\widehat{\beta}\,\widehat{\alpha}\alpha^r-\widehat{\alpha}\,\widehat{\beta}\beta^r}{(\alpha\beta)^r}\right)\\
&\quad=-(\alpha\beta)^{n-r}(\alpha-\beta)^2\frac{(\alpha^r-\beta^r)}{(\alpha-\beta)}\left(\frac{\widehat{\alpha}\,\widehat{\beta}\beta^r-\widehat{\beta}\,\widehat{\alpha}\alpha^r}{\alpha-\beta}\right)\\
&\quad=(-q)^{n-r}\Delta U_r\left(AU_r-qBV_r\right).\quad\square
\end{aligned}
$$

Since the following corollary can be proved by Theorem 2, we omit its proof.

Corollary 1. *Let $n,r\in\mathbb{Z}$ and $n\geq r\geq 1$. Then*

$$K_{n-r}K_{n+r}-K_n^2=-\Delta\left[Q_{n-r}Q_{n+r}-Q_n^2\right].$$

Theorem 3. *Let $n\in\mathbb{N}$ and $m,r\in\mathbb{Z}$. Then*

$$\sum_{i=0}^{n}Q_{mi+r}=\frac{(-q)^m[Q_{mn+r}-Q_{r-m}]-Q_{mn+m+r}+Q_r}{1-V_m+(-q)^m}$$

and

$$\sum_{i=0}^{n}K_{mi+r}=\frac{(-q)^m[K_{mn+r}-K_{r-m}]-K_{mn+m+r}+K_r}{1-V_m+(-q)^m}.$$

Proof. If we use Binet formulae for the quaternions Q_n and K_n, then we get

$$
\begin{aligned}
\sum_{i=0}^{n}Q_{mi+r}&=\sum_{i=0}^{n}\frac{\widehat{\alpha}\alpha^{mi+r}-\widehat{\beta}\beta^{mi+r}}{\alpha-\beta}=\frac{1}{\alpha-\beta}\left[\widehat{\alpha}\alpha^r\sum_{i=0}^{n}\alpha^{mi}-\widehat{\beta}\beta^r\sum_{i=0}^{n}\beta^{mi}\right]\\
&=\frac{1}{\alpha-\beta}\left[\widehat{\alpha}\alpha^r\left(\frac{\alpha^{mn+m}-1}{\alpha^m-1}\right)-\widehat{\beta}\beta^r\left(\frac{\beta^{mn+m}-1}{\beta^m-1}\right)\right]\\
&=\frac{1}{\alpha-\beta}\left[\frac{(\alpha\beta)^m\left(\widehat{\alpha}\alpha^{mn+r}-\widehat{\beta}\beta^{mn+r}\right)-\left(\widehat{\alpha}\alpha^{mn+m+r}-\widehat{\beta}\beta^{nm+m+r}\right)}{(\alpha\beta)^m-(\alpha^m+\beta^m)+1}\right.\\
&\qquad\left.\frac{-(\alpha\beta)^k\left(\widehat{\alpha}\alpha^{r-m}-\widehat{\beta}\beta^{r-m}\right)+\left(\widehat{\alpha}\alpha^{r}-\widehat{\beta}\beta^{r}\right)}{(\alpha\beta)^m-(\alpha^m+\beta^m)+1}\right]\\
&=\frac{1}{1-v_m+(-q)^m}\left[(\alpha\beta)^m\frac{\widehat{\alpha}\alpha^{mn+r}-\widehat{\beta}\beta^{mn+r}}{\alpha-\beta}-(\alpha\beta)^m\frac{\widehat{\alpha}\alpha^{r-m}-\widehat{\beta}\beta^{r-m}}{\alpha-\beta}\right.
\end{aligned}
$$

$$-\frac{\widehat{\alpha}\alpha^{mn+m+r}-\widehat{\beta}\beta^{mn+m+r}}{\alpha-\beta}+\frac{\widehat{\alpha}\alpha^{r}-\widehat{\beta}\beta^{r}}{\alpha-\beta}\Bigg]$$

$$=\frac{(\alpha\beta)^m Q_{mn+r}-(\alpha\beta)^m Q_{r-m}-Q_{mn+m+r}+Q_r}{1-V_m+(-q)^m}$$

$$=\frac{(-q)^m\left[Q_{mn+r}-Q_{r-m}\right]-Q_{mn+m+r}+Q_r}{1-V_m+(-q)^m}$$

and

$$\sum_{i=0}^{n}K_{mi+r}=\sum_{i=0}^{n}\left(\widehat{\alpha}\alpha^{mi+r}+\widehat{\beta}\beta^{mi+r}\right)=\widehat{\alpha}\alpha^{r}\sum_{i=0}^{n}\alpha^{mi}+\widehat{\beta}\beta^{r}\sum_{i=0}^{n}\beta^{mi}$$

$$=\widehat{\alpha}\alpha^{r}\left(\frac{\alpha^{mn+m}-1}{\alpha^{m}-1}\right)+\widehat{\beta}\beta^{r}\left(\frac{\beta^{mn+m}-1}{\beta^{m}-1}\right)$$

$$=\frac{\widehat{\alpha}\alpha^{r}\left(\alpha^{mn+m}-1\right)\left(\beta^{m}-1\right)+\widehat{\beta}\beta^{r}\left(\beta^{mn+m}-1\right)\left(\alpha^{m}-1\right)}{\left(\alpha^{m}-1\right)\left(\beta^{m}-1\right)}$$

$$=\frac{(\alpha\beta)^m\left(\widehat{\alpha}\alpha^{mn+r}+\widehat{\beta}\beta^{mn+r}\right)-\left(\widehat{\alpha}\alpha^{mn+m+r}+\widehat{\beta}\beta^{mn+m+r}\right)}{1-V_m+(-q)^m}$$

$$\frac{-(\alpha\beta)^k\left(\widehat{\alpha}\alpha^{r-m}+\widehat{\beta}\beta^{r-m}\right)+\left(\widehat{\alpha}\alpha^{r}+\widehat{\beta}\beta^{r}\right)}{1-V_m+(-q)^m}$$

$$=\frac{(\alpha\beta)^m\left[\left(\widehat{\alpha}\alpha^{mn+r}+\widehat{\beta}\beta^{mn+r}\right)-\left(\widehat{\alpha}\alpha^{r-m}+\widehat{\beta}\beta^{r-m}\right)\right]}{1-V_m+(-q)^m}$$

$$\frac{-\left(\widehat{\alpha}\alpha^{mn+m+r}+\widehat{\beta}\beta^{mn+m+r}\right)+\left(\widehat{\alpha}\alpha^{r}+\widehat{\beta}\beta^{r}\right)}{1-V_m+(-q)^m}$$

$$=\frac{(-q)^m\left[K_{mn+r}-K_{r-m}\right]-K_{mn+m+r}+K_r}{1-V_m+(-q)^m}. \quad \square$$

THEOREM 4. *Let $n\in\mathbb{N}$. Then*

$$\sum_{i=0}^{n}\binom{n}{i}p^iq^{n-i}Q_i=Q_{2n}$$

and

$$\sum_{i=0}^{n}\binom{n}{i}p^iq^{n-i}K_i=K_{2n}.$$

Proof. If we use Binet formulae for the quaternions Q_n and K_n, then we

get

$$\begin{aligned}
\sum_{i=0}^{n}\binom{n}{i}p^iq^{n-i}Q_i &= \sum_{i=0}^{n}\binom{n}{i}p^iq^{n-i}\left(\frac{\widehat{\alpha}\alpha^i-\widehat{\beta}\beta^i}{\alpha-\beta}\right)\\
&= \frac{\widehat{\alpha}}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}(p\alpha)^i q^{n-i}-\frac{\widehat{\beta}}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}(p\beta)^i q^{n-i}\\
&= \frac{\widehat{\alpha}}{\alpha-\beta}(p\alpha+q)^n-\frac{\widehat{\beta}}{\alpha-\beta}(p\beta+q)^n\\
&= \frac{\widehat{\alpha}\alpha^{2n}-\widehat{\beta}\beta^{2n}}{\alpha-\beta}=Q_{2n}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=0}^{n}\binom{n}{i}p^iq^{n-i}K_i &= \sum_{i=0}^{n}\binom{n}{i}p^iq^{n-i}\left(\widehat{\alpha}\alpha^i+\widehat{\beta}\beta^i\right)\\
&= \widehat{\alpha}\sum_{i=0}^{n}\binom{n}{i}(p\alpha)^i q^{n-i}+\widehat{\beta}\sum_{i=0}^{n}\binom{n}{i}(p\beta)^i q^{n-i}\\
&= \widehat{\alpha}(p\alpha+q)^n+\widehat{\beta}(p\beta+q)^n\\
&= \widehat{\alpha}\alpha^{2n}+\widehat{\beta}\beta^{2n}=K_{2n}. \quad \square
\end{aligned}$$

THEOREM 5. *The generating functions of the generalized Fibonacci and Lucas quaternions are given by*

$$G(x)=\frac{x+\mathbf{i}+\mathbf{j}\left[p+xq\right]+\mathbf{k}\left[\left(p^2+q\right)+qpx\right]}{1-px-qx^2}$$

and

(2.12)
$$J(x)=\frac{[2-px]+\mathbf{i}\left[p+2qx\right]+\mathbf{j}\left[\left(p^2+2q\right)+pqx\right]+\mathbf{k}\left[\left(p^3+3pq\right)+\left(p^2+2q\right)qx\right]}{1-px-qx^2},$$

respectively.

Proof. Since the power series representation of the generating function for (Q_n) is

$$G(x)=\sum_{n=0}^{\infty}Q_nx^n=Q_0+xQ_1+x^2Q_2+x^3Q_3+x^4Q_4+... \tag{2.13}$$

we can write

$$-pxG(x)=-pxQ_0-px^2Q_1-px^3Q_2-px^4Q_3-px^5Q_4-... \tag{2.14}$$

and

$$-qx^2G(x) = -qx^2Q_0 - qx^3Q_1 - qx^4Q_2 - qx^5Q_3 - qx^6Q_4 - .... \tag{2.15}$$

If we take the sum of (2.13), (2.14) and (2.15), then we obtain

$$\left(1 - px - qx^2\right) G(x) = Q_0 + xQ_1 - pxQ_0.$$

Hence it follows that

$$\begin{aligned} G(x) &= \frac{Q_0 + x(Q_1 - pQ_0)}{1 - px - qx^2} = \frac{Q_0 + xqQ_{-1}}{1 - px - qx^2} \\ &= \frac{x + \mathbf{i} + \mathbf{j}\left[p + xq\right] + \mathbf{k}\left[\left(p^2 + q\right) + qpx\right]}{1 - px - qx^2}. \end{aligned}$$

The proof of (2.12) follows by using the similar steps. □

THEOREM 6. *Let $m, n, r \in \mathbb{Z}$. Then the generating function of the quaternions Q_{mn+r} and K_{mn+r} are given by*

$$\sum_{n=0}^{\infty} Q_{mn+r}x^n = \frac{Q_r - (-q)^m Q_{r-m}x}{1 - V_m x + (-q)^m x^2} \tag{2.16}$$

and

$$\sum_{n=0}^{\infty} K_{mn+r}x^n = \frac{K_r - (-q)^m V_{r-m}x}{1 - V_m x + (-q)^m x^2}, \tag{2.17}$$

respectively.

Proof. If we use Binet formula for Q_{mn+r}, then we get

$$\begin{aligned} \sum_{n=0}^{\infty} Q_{mn+r}x^n &= \sum_{n=0}^{\infty} \frac{\widehat{\alpha}\alpha^{mn+r} - \widehat{\beta}\beta^{mn+r}}{\alpha - \beta} x^n \\ &= \frac{1}{\alpha - \beta}\left[\widehat{\alpha}\alpha^r \sum_{n=0}^{\infty} (\alpha^m x)^n - \widehat{\beta}\beta^r \sum_{n=0}^{\infty} (\beta^m x)^n\right] \\ &= \frac{1}{\alpha - \beta}\left[\frac{\widehat{\alpha}\alpha^r}{1 - \alpha^m x} - \frac{\widehat{\beta}\beta^r}{1 - \beta^m x}\right] \\ &= \frac{1}{\alpha - \beta}\left[\frac{\left(\widehat{\alpha}\alpha^r - \widehat{\beta}\beta^r\right) - \left(\widehat{\alpha}\alpha^r\beta^m x - \widehat{\beta}\alpha^m\beta^r x\right)}{1 - (\alpha^m + \beta^m)x + \alpha^m\beta^m x^2}\right] \\ &= \frac{1}{\alpha - \beta}\left[\frac{\left(\widehat{\alpha}\alpha^r - \widehat{\beta}\beta^r\right) - (\alpha\beta)^m\left(\widehat{\alpha}\alpha^{r-m} - \widehat{\beta}\beta^{r-m}\right)x}{1 - V_m x + (-q)^m x^2}\right] \end{aligned}$$

$$= \frac{1}{1 - V_m x + (-q)^m x^2} \left[\frac{\left(\widehat{\alpha}\alpha^r - \widehat{\beta}\beta^r \right)}{\alpha - \beta} - \frac{(\alpha\beta)^m \left(\widehat{\alpha}\alpha^{r-m} - \widehat{\beta}\beta^{r-m} \right)}{\alpha - \beta} x \right]$$

$$= \frac{Q_r - (-q)^m Q_{r-m} x}{1 - V_m x + (-q)^m x^2}.$$

The proof of (2.17) is almost the same as the proof of (2.16). □

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