GENERALIZED APOSTOL-TYPE POLYNOMIAL MATRIX AND ITS ALGEBRAIC PROPERTIES

YAMILET QUINTANA*, WILLIAM RAMÍREZ and ALEJANDRO URIELES

Communicated by Alexandru Zaharescu

The aim of this paper is to introduce the generalized Apostol-type polynomial matrix $\mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$ and the generalized Apos-tol-type matrix $\mathscr{W}^{[m-1,\alpha]}(c,a;\lambda;\mu;\nu)$. Using some properties of the generalized Apostol-type polynomials and numbers, we deduce a product formula for $\mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$ and provide some factorizations of the Apostol-type polynomial matrix $\mathscr{W}^{[m-1]}(x;c,a;\lambda;\mu;\nu)$, involving the generalized Pascal matrix, Fibonacci and Lucas matrices, respectively.

AMS 2010 Subject Classification: 11B68, 11B83, 11C08, 11B39, 33B99.

Key words: generalized Apostol-type polynomials, generalized Apostol-type matrix, admissible generalized Apostol-type matrix, generalized Pascal matrix, generalized Fibonacci matrix, Lucas matrix.

1. INTRODUCTION

In many settings (see *e.g.* [2–4,6,8,10,11,19,20,24–29]), a number of interesting and useful identities involving binomial (or q-binomial) coefficients can be obtained from a matrix representation of a particular counting sequence. Such a matrix representation provides a powerful computational tool for deriving identities and an explicit formula related to the sequence. Also, it is possible to obtain from a matrix representation of a particular counting sequence, an identity and an explicit formula for the general term of the sequence.

Particularly interesting are those contexts in which such a matrix representation is related to special classes of polynomials, namely, Bernoulli polynomials, Euler polynomials, Bell polynomials, Jacobi polynomials, Laguerre polynomials their generalizations and q-analogues, and so on.

Having in mind these facts and motivated by [19, 28], the main purpose of the present paper is to introduce the generalized Apostol-type polynomial matrix, provide some of its algebraic properties, as well as, determine explicit expressions for it, which connect it with Pascal, Fibonacci and Lucas matrices, respectively.

* Corresponding author.

So, we begin our study with the definition of a unified version of the generalized Apostol-type polynomials. For $m \in \mathbb{N}$, $\alpha, \lambda, \mu, \nu \in \mathbb{C}$ and a, c positive real numbers, the generalized Apostol-type polynomials in the variable x, parameters c, a, λ, μ, ν, order α and level m, are defined by means of the following generating function [7].

$$\left(E_{1,m+1}^{(c,a;\lambda;\mu;\nu)}(z)\right)^{\alpha} c^{xz} = \sum_{n=0}^{\infty} \mathscr{Q}_n^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu) \frac{z^n}{n!}, \tag{1}$$

where $|z| < 2\pi$ when $\lambda = 1$, $|z| < \pi$ when $\lambda = -1$, $\left(\left|z \ln\left(\frac{c}{a}\right)\right| < |\log(-\lambda)|\right)$ when $\lambda \in \mathbb{C} \setminus \{-1, 1\}$, $1^{\alpha} := 1$, and $E_{1,m+1}^{(c,a;\lambda;\mu;\nu)}(z)$ is the Mittag-Leffler type function given by

$$E_{1,m+1}^{(c,a;\lambda;\mu;\nu)}(z) := \frac{(2^{\mu} z^{\nu})^m}{\lambda c^z + \sum\limits_{l=0}^{m-1} \frac{(z \ln a)^l}{l!}}, \quad m \in \mathbb{N},\ a, c \in \mathbb{R}^+,\ \lambda, \mu, \nu \in \mathbb{C}.$$

This class of polynomials has been introduced recently in [7] and it provides a unified presentation of the generalized Apostol-type polynomials and the generalized Apostol-Bernoulli polynomials, Apostol-Euler polynomials and Apostol-Genocchi polynomials in the variable x, parameters λ, a, c, order α and level m, (cf. [13, 14, 17]).

The numbers given by

$$\mathscr{Q}_n^{[m-1,\alpha]}(c, a; \lambda; \mu; \nu) := \mathscr{Q}_n^{[m-1,\alpha]}(0; c, a; \lambda; \mu; \nu),$$

denote the corresponding unified presentation of the generalized Apostol-type numbers of parameters $\lambda \in \mathbb{C}$, $a, c \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level $m \in \mathbb{N}$.

From (1), it is easily observed that the following addition theorem of the argument is satisfied.

$$\begin{aligned} &\mathscr{Q}_n^{[m-1,\alpha+\beta]}(x + y; c, a; \lambda; \mu; \nu) \\ &\qquad = \sum_{k=0}^{n} \binom{n}{k} \mathscr{Q}_k^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu) \mathscr{Q}_{n-k}^{[m-1,\beta]}(y; c, a; \lambda; \mu; \nu). \end{aligned} \tag{2}$$

Since $\mathscr{Q}_n^{[m-1,0]}(x; c, a; \lambda; \mu; \nu) = (x \ln(c))^n$, upon setting $\beta = 0$ in addition theorem of the argument (2) and interchanging x and y, we obtain

$$\mathscr{Q}_n^{[m-1,\alpha]}(x + y; c, a; \lambda; \mu; \nu) = \sum_{k=0}^{n} \binom{n}{k} \mathscr{Q}_k^{[m-1,\alpha]}(y; c, a; \lambda; \mu; \nu)(x \ln c)^{n-k}. \tag{3}$$

The outline of the paper is as follows. Section 2 contains the basic background about the generalized Apostol-type polynomials in the variable x, parameters c, a, λ, μ, ν, order α and level m, and some other auxiliary results which

will be used throughout the paper. In Section 3, we introduce the generalized Apostol-type polynomial matrix, derive a product formula for it and give some factorizations for such a matrix, which involve summation matrices and the generalized Pascal matrix of first kind in base c, respectively. Finally, Section 4 shows several factorizations of the generalized Apostol-type matrix in terms of the Fibonacci and Lucas matrices, respectively (cf. Theorems 8 and 9).

2. BACKGROUND AND PREVIOUS RESULTS

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ the sets of natural, nonnegative integer, real, positive real and complex numbers, respectively. All matrices are in $M_{n+1}(\mathbb{K})$, the set of all $(n+1)\times(n+1)$ matrices over the field $\mathbb{K}$, with $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Also, for i,j any nonnegative integers we adopt the following convention

$$\binom{i}{j}=0,\ \text{whenever } j>i.$$

Definition 1. Let x be any nonzero real number. For $c\in\mathbb{R}^+$, the generalized Pascal matrix of first kind in base c $P_c[x]$, is an $(n+1)\times(n+1)$ matrix whose entries are given by

$$p_{i,j,c}(x):=\begin{cases}\binom{i}{j}(x\ln c)^{i-j}, & i\geq j,\\ 0, & \text{otherwise.}\end{cases}$$

It is clear that when $c=e$, the matrix $P_c[x]$ coincides with the generalized Pascal matrix of first kind $P[x]$. Furthermore, if we adopt the convention $0^0=1$, then $P_c[0]=I_{n+1}$, with $I_{n+1}=\operatorname{diag}(1,1,\ldots,1)$.

Following [2], a useful alternative expression for $P_c[x]$ can be deduced: suppose there is a matrix L_c such that $P_c[x]=e^{xL_c}$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}P_c[x]=\frac{\mathrm{d}}{\mathrm{d}x}e^{xL_c}=L_ce^{xL_c}=L_cP_c[x],$$

so,

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}P_c[x]\right|_{x=0}=L_cP_c[0]=L_cI_{n+1}=L_c.$$

Thus, there is at most one matrix L_c such that $P_c[x]=e^{xL_c}$. Indeed, such a matrix L_c is given by

$$(L_c)_{i,j}=\begin{cases}j\ln c, & i=j+1,\\ 0, & \text{otherwise.}\end{cases}$$

Note that for every $k \in \mathbb{N}$, the entries of the matrix $L_c^k := (L_c)^k$ are given by the formula

$$\left(L_c^k\right)_{i,j} = \begin{cases} \frac{i!}{j!}(\ln c)^{i-j}, & i = j+k, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, for $k \geq n+1$ we have $\left(L_c^k\right)_{i,j} = 0$, which implies that the infinite series for e^{xL_c} reduces to the finite sum

$$e^{xL_c} = I + x\ln(c)L_c + \frac{(x\ln(c))^2}{2!}L_c^2 + \cdots + \frac{(x\ln c)^n}{n!}L_c^n.$$

Clearly e^{xL_c} is a lower triangular matrix with diagonal entries equal to 1. If we assume $i > j$ and let $k = i - j$, then the only matrix in the sum above which has a nonzero $(i,j)-$th entry is $\frac{(x\ln(c))^k}{k!}L_c$, so

$$\left(e^{xL_c}\right)_{i,j} = \frac{(x\ln c)^k}{k!}\left(L_c^k\right)_{i,j} = \frac{i!}{j!(i-j)!}(x\ln c)^{i-j} = \binom{i}{j}(x\ln c)^{i-j} = p_{i,j,c}(x).$$

An immediate consequence of the remarks above is the following proposition.

Proposition 1 (Addition theorem of the argument). *For $x, y \in \mathbb{R}$ we have*

$$P_c[x+y] = P_c[x]P_c[y].$$

Taking into account that the affine transformation $x \mapsto x\ln c$ implies the identity $P_c[x] = P[x\ln c]$, where $P[y]$ denotes the generalized Pascal matrix of first kind, the following algebraic and differential properties of $P_c[x]$ can be derived.

Proposition 2. *For $c \in \mathbb{R}^+$, let $P_c[x]$ be the generalized Pascal matrix of first kind in base c and order $n+1$. Then the following statements hold.*

(a) *$P_c[x]$ is an invertible matrix and its inverse is given by*

$$P_c^{-1}[x] := (P_c[x])^{-1} = P_c[-x].$$

(b) *Differential relation (Appell type polynomial entries). $P_c[x]$ satisfies the following differential equation*

$$D_x P_c[x] = \mathfrak{L}_c P_c[x] = P_c[x]\mathfrak{L}_c,$$

where $D_x P_c[x]$ is the matrix resulting from taking the derivative with respect to x of each entry of $P_c[x]$ and the entries of the $(n+1)\times(n+1)$

matrix $\mathfrak{L}$ *are given by*

$$\mathrm{l}_{i,j,c} = \begin{cases} p'_{i,j,c}(0), & i \geq j, \\ 0, & \textit{otherwise}, \end{cases}$$
$$= \begin{cases} (j+1)\ln c, & i = j+1, \\ 0, & \textit{otherwise}. \end{cases}$$

(e) *The matrix* $P_c[x]$ *can be factorized as follows.*

$$P_c[x] = G_{n,c}[x]G_{n-1,c}[x]\cdots G_{1,c}[x], \tag{4}$$

where $G_{k,c}[x]$ *is the* $(n+1)\times(n+1)$ *summation matrix given by*

$$G_{k,c}[x] = \begin{cases} \begin{bmatrix} I_{n-k} & 0 \\ 0 & S_{k,c}[x] \end{bmatrix}, & k = 1,\ldots,n-1, \\ S_{n,c}[x], & k = n, \end{cases}$$

being $S_{k,c}[x]$ *the* $(k+1)\times(k+1)$ *matrix whose entries* $S_{k,c}(x;i,j)$ *are given by*

$$S_{k,c}(x;i,j,c) = \begin{cases} (x\ln c)^{i-j}, & i \geq j, \\ 0, & j > i, \end{cases} \qquad (0 \leq i,j \leq k).$$

Another necessary structured matrices in what follows, are the Fibonacci and Lucas matrices. Below, we recall the definitions of each one of them.

Definition 2. Let $\{F_n\}_{n\geq 1}$ be the Fibonacci sequence, *i.e.*, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with initial conditions $F_0 = 0$ and $F_1 = 1$. The Fibonacci matrix $\mathscr{F}$ is an $(n+1)\times(n+1)$ matrix whose entries are given by [11]:

$$f_{i,j} = \begin{cases} F_{i-j+1}, & i-j+1 \geq 0, \\ 0, & i-j+1 < 0. \end{cases}$$

Let $\mathscr{F}^{-1}$ be the inverse of $\mathscr{F}$ and denote by $\tilde{f}_{i,j}$ the entries of $\mathscr{F}^{-1}$. In [11] the authors obtained the following explicit expression for $\mathscr{F}^{-1}$.

$$\tilde{f}_{i,j} = \begin{cases} 1, & i = j, \\ -1, & i = j+1, j+2, \\ 0, & \text{otherwise}. \end{cases}$$

Definition 3. Let $\{L_n\}_{n\geq 1}$ be the Lucas sequence, *i.e.*, $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$ with initial conditions $L_1 = 1$ and $L_2 = 3$. The Lucas matrix $\mathscr{L}$ is an $(n+1)\times(n+1)$ matrix whose entries are given by [29]:

$$l_{i,j} = \begin{cases} L_{i-j+1}, & i-j \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathscr{L}^{-1}$ be the inverse of $\mathscr{L}$ and denote by $\tilde{l}_{i,j}$ the entries of $\mathscr{L}^{-1}$. In [29, Theorem 2.2] the authors obtained the following explicit expression for $\mathscr{L}^{-1}$.

$$\tilde{l}_{i,j} = \begin{cases} 1, & i = j, \\ -3, & i = j+1, \\ 5(-1)^{i-j}2^{i-j-2}, & i \geq j+2, \\ 0, & \text{otherwise.} \end{cases}$$

For x any nonzero real number, using [29, Theorem 3.1] we can deduce the following relation between the matrices $P_c[x]$ and $\mathscr{L}$.

$$P_c[x] = \mathscr{L}\mathscr{G}_c[x] = \mathscr{H}_c[x]\mathscr{L}, \tag{5}$$

where the entries of the $(n+1)\times(n+1)$ matrices $\mathscr{G}_c[x]$ and $\mathscr{H}_c[x]$ are given by

$$\begin{aligned} g_{i,j,c}(x) &= (x\ln c)^{-j-1}\left[(x\ln c)^{i+1}\binom{i}{j} - 3(x\ln c)^{i}\binom{i-1}{j}\right. \\ &\quad \left. + 5(-1)^{i+1}2^{i-1}m_{i-1,j+1}\left(\frac{x\ln c}{2}\right)\right], \\ h_{i,j,c}(x) &= (x\ln c)^{-j-1}\left[(x\ln c)^{i+1}\binom{i}{j} - 3(x\ln c)^{i}\binom{i}{j+1}\right. \\ &\quad \left. + (-1)^{j+1}\frac{5(x\ln c)^{i+j+2}}{2^{j+3}}n_{i+1,j+3}\left(\frac{2}{x\ln c}\right)\right], \end{aligned}$$

respectively, with

$$m_{i,j}(x) := \begin{cases} \sum_{k=j}^{i}(-1)^k\binom{k}{j}(x\ln c)^k, & i \geq j, \\ 0, & \text{otherwise,} \end{cases}$$

$$n_{i,j}(x) := \begin{cases} \sum_{k=j}^{i}(-1)^k\binom{i}{k}(x\ln c)^k, & i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

3. THE GENERALIZED APOSTOL-TYPE POLYNOMIAL MATRIX

Definition 4. For $m \in \mathbb{N}$, $\alpha, \lambda, \mu, \nu \in \mathbb{C}$ and a, c positive real numbers, the generalized $(n+1)\times(n+1)$ Apostol-type polynomial matrix $\mathscr{W}^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu)$ is defined by

$$\mathscr{W}_{i,j}^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu) = \begin{cases} \binom{i}{j}\mathscr{Q}_{i-j}^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu), & i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

While, the matrices

$$\begin{aligned} \mathscr{W}^{[m-1]}(x; c, a; \lambda; \mu; \nu) &:= \mathscr{W}^{[m-1,1]}(x; c, a; \lambda; \mu; \nu), \\ \mathscr{W}^{[m-1]}(c, a; \lambda; \mu; \nu) &:= \mathscr{W}^{[m-1]}(0; c, a; \lambda; \mu; \nu) \end{aligned}$$

are called the Apostol-type polynomial matrix and the Apostol-type matrix, respectively.

Since $\mathscr{Q}_n^{[m-1,0]}(x; c, a; \lambda; \mu; \nu) = (x\ln(c))^n$, we have

$$\mathscr{W}^{[m-1,0]}(x; c, a; \lambda; \mu; \nu) = P_c[x].$$

It is clear that (3) yields the following matrix identity:

$$\mathscr{W}^{[m-1,\alpha]}(x+y; c, a; \lambda; \mu; \nu) = \mathscr{W}^{[m-1,\alpha]}(y; c, a; \lambda; \mu; \nu)P_c[x].$$

The next result is an immediate consequence of Definition 4 and the addition formula (2).

THEOREM 3. *The generalized Apostol-type polynomial matrix* $\mathscr{W}^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu)$ *satisfies the following product formula.*

$$\begin{aligned} (6)\quad & \mathscr{W}^{[m-1,\alpha+\beta]}(x+y; c, a; \lambda; \mu; \nu) \\ &= \mathscr{W}^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu)\, \mathscr{W}^{[m-1,\beta]}(y; c, a; \lambda; \mu; \nu) \\ &= \mathscr{W}^{[m-1,\beta]}(x; c, a; \lambda; \mu; \nu)\, \mathscr{W}^{[m-1,\alpha]}(y; c, a; \lambda; \mu; \nu) \\ &= \mathscr{W}^{[m-1,\alpha]}(y; c, a; \lambda; \mu; \nu)\, \mathscr{W}^{[m-1,\beta]}(x; c, a; \lambda; \mu; \nu). \end{aligned}$$

Proof. We proceed as in the proof of [19, Theorem 3.1], making the corresponding modifications. Let $A_{i,j,c}^{[m-1,\alpha,\beta]}(a; \lambda; \mu; \nu)(x, y)$ be the (i, j)-th entry of the matrix product $\mathscr{W}^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu)\, \mathscr{W}^{[m-1,\beta]}(y; c, a; \lambda; \mu; \nu)$, then by the addition formula (2) we have

$$\begin{aligned} & A_{i,j,c}^{[m-1,\alpha,\beta]}(a; \lambda; \mu; \nu)(x, y) \\ &\quad = \sum_{k=0}^{n} \binom{i}{k} \mathscr{Q}_{i-k}^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu) \binom{k}{j} \mathscr{Q}_{k-j}^{[m-1,\beta]}(y; c, a; \lambda; \mu; \nu) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=j}^{i} \binom{i}{k} \mathscr{Q}_{i-k}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu) \binom{k}{j} \mathscr{Q}_{k-j}^{[m-1,\beta]}(y;c,a;\lambda;\mu;\nu) \\
&= \sum_{k=j}^{i} \binom{i}{j} \binom{i-j}{i-k} \mathscr{Q}_{i-k}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu) \mathscr{Q}_{k-j}^{[m-1,\beta]}(y;c,a;\lambda;\mu;\nu) \\
&= \binom{i}{j} \sum_{k=0}^{i-j} \binom{i-j}{k} \mathscr{Q}_{i-j-k}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu) \mathscr{Q}_{k}^{[m-1,\beta]}(y;c,a;\lambda;\mu;\nu) \\
&= \binom{i}{j} \mathscr{Q}_{i-j}^{[m-1,\alpha+\beta]}(x+y;c,a;\lambda;\mu;\nu),
\end{aligned}
$$

which implies the first equality of the theorem. The second and third equalities can be derived in a similar way. □

The next result establishes the relation between the generalized Apostol-type polynomial matrix and the generalized Pascal matrix of first kind in base c.

Corollary 4. *The generalized Apostol-type matrix* $\mathscr{W}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)$ *satisfies the following relation.*

$$
\begin{aligned}
\mathscr{W}^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu) &= \mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)P_c[y] \\
&= P_c[x]\mathscr{W}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu) \\
&= \mathscr{W}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)P_c[x].
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\mathscr{W}^{[m-1]}(x+y;c,a;\lambda;\mu;\nu) &= P_c[x]\mathscr{W}^{[m-1]}(y;c,a;\lambda;\mu;\nu) \\
&= P_c[y]\mathscr{W}^{[m-1]}(x;c,a;\lambda;\mu;\nu).
\end{aligned}
$$

Proof. The substitution $\beta = 0$ into (6) yields

$$
\mathscr{W}^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu) = \mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\mathscr{W}^{[m-1,0]}(y;c,a;\lambda;\mu;\nu).
$$

Since $\mathscr{W}^{[m-1,0]}(y;c,a;\lambda;\mu;\nu) = P_c[y]$, we obtain

$$
\mathscr{W}^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu) = \mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)P_c[y]. \tag{7}
$$

A similar argument allows to show that

$$
\begin{aligned}
\mathscr{W}^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu) &= P_c[x]\mathscr{W}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu) \\
&= \mathscr{W}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)P_c[x].
\end{aligned}
$$

Finally, the substitution $\alpha = 1$ into (7) and its combination with the previous equations complete the proof. □

Using the relation (4) and Corollary 4 we obtain the following factorization for $\mathscr{W}^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu)$ in terms of summation matrices.

$$\begin{aligned}&\mathscr{W}^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu)\\&\qquad=\mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)G_{n,c}[y]G_{n-1,c}[y]\cdots G_{1,c}[y].\end{aligned}$$

Also, for any x nonzero real number, using (5) and Corollary 4 we obtain the following factorizations for $\mathscr{W}^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu)$ in terms of the Lucas matrix.

$$\begin{aligned}\mathscr{W}^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu) &= \mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\mathscr{L}\mathscr{G}_c[y]\\ &= \mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\mathscr{H}_c[y]\mathscr{L}.\end{aligned}$$

Under the appropriate choice on the parameters, level and order, it is possible to provide some illustrative examples of the generalized Apostol-type polynomial matrices:

Example 5. For $m\in\mathbb{N}$, $c=a=e=\exp(1)$, $\alpha=\nu=1$, $\lambda=-1$ and $\mu=0$, let us consider the polynomials $\mathscr{Q}_k^{[m-1,1]}(x;e,e;-1;0;1)$, $k=0,1,2,3$. From the relation (cf. [7, Eq. (15)]):

$$\mathscr{Q}_n^{[m-1,1]}(x;e,e;-1;0;1)=-B_n^{[m-1]}(x),\quad n\in\mathbb{N}_0,$$

where $B_n^{[m-1]}(x)$ is the n-th generalized Bernoulli polynomial of level m (see [20] and the references therein).

Hence, for $n=3$ we have

$$\begin{aligned}&\mathscr{W}^{[m-1,1]}(x;e,e;-1;0;1)\\&\quad=\begin{bmatrix}-m! & 0 & 0 & 0\\ -m!\left(x-\frac{1}{m+1}\right) & -m! & 0 & 0\\ -B_2^{[m-1]}(x) & -2m!\left(x-\frac{1}{m+1}\right) & -m! & 0\\ -B_3^{[m-1]}(x) & -3B_2^{[m-1]}(x) & -3m!\left(x-\frac{1}{m+1}\right) & -m!\end{bmatrix},\end{aligned}$$

with

$$B_2^{[m-1]}(x)=m!\left(x^2-\frac{2}{m+1}x+\frac{2}{(m+1)^2(m+2)}\right),$$

$$\begin{aligned}&B_3^{[m-1]}(x)\\&\quad=m!\left(x^3-\frac{3}{m+1}x^2+\frac{6}{(m+1)^2(m+2)}x+\frac{6(m-1)}{(m+1)^2(m+2)(m+3)}\right).\end{aligned}$$

Example 6. For $m=\lambda=\mu=1$, $c=a=e=\exp(1)$ and $\nu=0$, let us consider the polynomials $\mathscr{Q}_k^{[1,\alpha]}(x;e,e;1;1;0)=E_k^{(\alpha)}(x)$, where $E_k^{(\alpha)}(x)$ is the k-th generalized Euler polynomial, $k=0,1,2,3$. Then, for $n=3$ we have

$$\mathscr{W}^{[m-1,1]}(x;e,e;1;1;0)=\mathscr{E}^{(\alpha)}(x),$$

where $\mathscr{E}^{(\alpha)}(x)$ is the generalized Euler matrix given by (cf. [19]):

$$\mathscr{E}^{(\alpha)}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x-\frac{\alpha}{2} & 1 & 0 & 0 \\ x^2-\alpha x+\frac{\alpha(\alpha-1)}{4} & 2x-\alpha & 1 & 0 \\ x^3-\frac{3\alpha}{2}x^2+\frac{3\alpha(\alpha-1)}{4}x-\frac{3\alpha^2(\alpha-1)}{8} & 3\left(x^2-\alpha x+\frac{\alpha(\alpha-1)}{4}\right) & 3\left(x-\frac{\alpha}{2}\right) & 1 \end{bmatrix}.$$

Example 7. For $n=3$, $m=c=\mu=2$, $a=3$, $\alpha=\frac{1}{2}$, $\nu=5$ and any $\lambda\in\mathbb{C}\setminus\{-1,1\}$, we have (see [7, Example 4]):

$$\begin{aligned} \mathscr{Q}_j^{\left[1,\frac{1}{2}\right]}(x;2,3;\lambda;2;5) &= 0, \quad j=0,1, \\ \mathscr{Q}_2^{\left[1,\frac{1}{2}\right]}(x;2,3;\lambda;2;5) &= \frac{32}{\sqrt{1+\lambda}}, \\ \mathscr{Q}_3^{\left[1,\frac{1}{2}\right]}(x;2,3;\lambda;2;5) &= \frac{48}{(1+\lambda)^{\frac{3}{2}}}\left[(2\ln 2)(\lambda+1)x-(\ln 2)\lambda-\ln 3\right]. \end{aligned}$$

Therefore,

$$\mathscr{W}^{\left[1,\frac{1}{2}\right]}(x;2,3;\lambda;2;5) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{32}{\sqrt{1+\lambda}} & 0 & 0 & 0 \\ a_{3,0}(x,\lambda) & \frac{96}{\sqrt{1+\lambda}} & 0 & 0 \end{bmatrix},$$

where $a_{3,0}(x,\lambda)=\mathscr{Q}_3^{\left[1,\frac{1}{2}\right]}(x;2,3;\lambda;2;5)$.

Remark 1. Note that the examples above say that the generalized Apostol-type polynomial matrices are not invertible matrices in general. Hence, extensions of classical factorization theorems for this family of polynomial matrices do not make sense in general.

However, Remark 1 suggests the following definition.

Definition 5. Let $\left\{\mathscr{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\right\}_{n\geq 0}$ be a sequence of generalized Apostol-type polynomials in the variable x, parameters c,a,λ,μ,ν, order α and level m. We say that $\left\{\mathscr{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\right\}_{n\geq 0}$ admits an inversion

if there exists a sequence $\{a_n(c,a;\lambda;\mu;\nu)\}_{n\geq 0}$ of nonzero complex (or real) numbers, such that

$$a_n(c,a;\lambda;\mu;\nu)(x\ln(c))^n = \sum_{k=0}^{n}\binom{n}{k}\mathscr{Q}_{n-k}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu), \quad n\geq 0.$$

When the polynomial sequence $\left\{\mathscr{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\right\}_{n\geq 0}$ admits an inversion, we call to its corresponding generalized Apostol-type polynomial matrix, admissible generalized Apostol-type matrix.

Remark 2. The following facts are straightforward consequences of Definition 5:

(a) It is clear that the definition above splits the class of the sequences of generalized Apostol-type polynomials into two disjoint subclasses, namely, the polynomial sequences which admit an inversion and the polynomial sequences which do not admit an inversion.

(b) If a sequence of generalized Apostol-type polynomials admits an inversion, then it is a basis of the space of polynomials and its corresponding generalized Apostol-type polynomial matrix is a nonsingular matrix.

(c) Extensions of the classical factorization theorems will take place in the framework of the admissible generalized Apostol-type polynomial matrices.

(d) Examples 5 and 6 show admissible generalized Apostol-type matrices, while Example 7 shows a non-admissible generalized Apostol-type matrix.

(e) As a consequence of Corollary 4, if $\mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$ is an admissible generalized Apostol-type matrix, then its inverse matrix can be factorized as follows.

$$\left(\mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\right)^{-1} = P_c[-x]\left(\mathscr{W}^{[m-1,\alpha]}(c,a;\lambda;\mu;\nu)\right)^{-1}.$$

So, factorizations for the matrix $\left(\mathscr{W}^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu)\right)^{-1}$ in terms of the inverses of summation matrices or in terms of the inverse of the Lucas matrix also will take place, respectively.

4. FACTORIZATIONS OF THE GENERALIZED APOSTOL-TYPE MATRICES *VIA* FIBONACCI AND LUCAS MATRICES

For $m\in\mathbb{N}$, a,c positive real numbers, $\lambda,\mu,\nu\in\mathbb{C}$, α a real or complex number and $0\leq i,j\leq n$, let $\mathscr{M}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$ be the $(n+1)\times(n+1)$ matrix whose entries are given by (cf. [19]):

$$\begin{aligned}
&\tilde{m}_{i,j}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\\
&\quad=\binom{i}{j}\mathscr{Q}_{i-j}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)-\binom{i-1}{j}\mathscr{Q}_{i-j-1}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\\
&\qquad-\binom{i-2}{j}\mathscr{Q}_{i-j-2}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu).
\end{aligned}$$

We denote $\mathscr{M}^{[m-1]}(x)=\mathscr{M}^{[m-1,1]}(x;e,e;1;0;1)$ and $\mathscr{M}^{[m-1]}=\mathscr{M}^{[m-1]}(0)$.

Similarly, let $\mathscr{N}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$ be the $(n+1)\times(n+1)$ matrix whose entries are given by (cf. [19]):

$$\begin{aligned}
&\tilde{n}_{i,j}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\\
&\quad=\binom{i}{j}\mathscr{Q}_{i-j}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)-\binom{i}{j+1}\mathscr{Q}_{i-j-1}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\\
&\qquad-\binom{i}{j+2}\mathscr{Q}_{i-j-2}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu).
\end{aligned}$$

We denote $\mathscr{N}^{[m-1]}(x)=\mathscr{N}^{[m-1,1]}(x;e,e;1;0;1)$ and $\mathscr{N}^{[m-1]}=\mathscr{N}^{[m-1]}(0)$.

From the definitions of $\mathscr{M}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$ and $\mathscr{N}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$, we see that

$$\begin{aligned}
\tilde{m}_{0,0}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu) &= \tilde{m}_{1,1}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)=\tilde{n}_{0,0}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\\
&= \tilde{n}_{1,1}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)=\mathscr{Q}_{0}^{[m-1,\alpha]}(c,a;\lambda;\mu;\nu),\\
\tilde{m}_{0,j}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu) &= \tilde{n}_{0,j}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)=0,\quad j\geq 1,\\
\tilde{m}_{1,0}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu) &= \tilde{n}_{1,0}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\\
&= \mathscr{Q}_{1}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)-\mathscr{Q}_{0}^{[m-1,\alpha]}(c,a;\lambda;\mu;\nu),\\
\tilde{m}_{1,j}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu) &= \tilde{n}_{1,j}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)=0,\quad j\geq 2,\\
\tilde{m}_{i,0}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu) &= \tilde{n}_{i,0}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\\
&= \mathscr{Q}_{i}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)-\mathscr{Q}_{i-1}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\\
&\quad -\mathscr{Q}_{i-2}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu),\quad i\geq 2.
\end{aligned}$$

For $m\in\mathbb{N}$, a,c positive real numbers, $\lambda,\mu,\nu\in\mathbb{C}$, α a real or complex number and $0\leq i,j\leq n$, let $\mathscr{L}_1^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$ be the $(n+1)\times(n+1)$ matrix whose entries are given by

$$\begin{aligned}
&\hat{l}_{i,j,1}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\\
&\quad=\binom{i}{j}\mathscr{Q}_{i-j}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)-3\binom{i-j}{j}\mathscr{Q}_{i-j-1}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)
\end{aligned}$$

$$+5\sum_{k=j}^{i-2}(-1)^{i-k}2^{i-k-2}\binom{k}{j}\mathscr{Q}_{k-j}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu).$$

We denote $\mathscr{L}_1^{[m-1]}(x)=\mathscr{L}_1^{[m-1,1]}(x;e,e;1;0;1)$ and $\mathscr{L}_1^{[m-1]}=\mathscr{L}_1^{[m-1]}(0)$.

Similarly, let $\mathscr{L}_2^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$ be the $(n+1)\times(n+1)$ matrix whose entries are given by

$$\begin{aligned}&\hat{l}_{i,j,2}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\\&\quad=\binom{i}{j}\mathscr{Q}_{i-j}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)-3\binom{i}{j+1}\mathscr{Q}_{i-j-1}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\\&\qquad+5\sum_{k=j+1}^{i}(-1)^{k-j}2^{k-j-2}\binom{i}{k}\mathscr{Q}_{i-k}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu).\end{aligned}$$

We denote $\mathscr{L}_2^{[m-1]}(x)=\mathscr{L}_2^{[m-1,1]}(x;e,e;1;0;1)$ and $\mathscr{L}_2^{[m-1]}=\mathscr{L}_2^{[m-1]}(0)$.

From the definitions of $\mathscr{L}_1^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$ and $\mathscr{L}_2^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$, we see that

$$\begin{aligned}\hat{l}_{i,i,1}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)&=\hat{l}_{i,i,2}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\\&=\mathscr{Q}_0^{[m-1,\alpha]}(c,a;\lambda;\mu;\nu),\quad i\geq 0,\end{aligned}$$

$$\hat{l}_{0,j,1}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)=\hat{l}_{0,j,2}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu),\quad j\geq 1,$$

$$\begin{aligned}\hat{l}_{1,0,1}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)&=\hat{l}_{1,0,2}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\\&=\mathscr{Q}_1^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)-3\mathscr{Q}_0^{[m-1,\alpha]}(c,a;\lambda;\mu;\nu),\end{aligned}$$

$$\hat{l}_{1,j,1}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)=\hat{l}_{1,j,2}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)=0,\quad j\geq 2.$$

The following results show some factorizations of $\mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$ in terms of Fibonacci and Lucas matrices, respectively.

THEOREM 8. *The generalized Apostol-type matrix $\mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$ can be factorized in terms of the Fibonacci matrix $\mathscr{F}$ as follows.*

$$\mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)=\mathscr{F}\mathscr{M}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu),\tag{8}$$

or,

$$\mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)=\mathscr{N}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\mathscr{F}.\tag{9}$$

In particular,

$$\mathscr{F}\mathscr{M}^{[m-1]}(x)=\mathscr{W}^{[m-1,\alpha]}(x;e,e;1;0;1)=\mathscr{N}^{[m-1]}(x)\mathscr{F}.\tag{10}$$

Proof. Since the relation (8) is equivalent to

$$\mathscr{F}^{-1}\mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)=\mathscr{M}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu),$$

it is possible to follow the proofs given in [19] or [28, Theorem 4.1], making the corresponding modifications, for obtaining (8). The relation (9) can be obtained using a similar procedure. The relation (10) is a straightforward consequence of (8) and (9). □

Also, the relations (8) and (9) allow us to deduce the following identity:

$$\mathscr{M}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)=\mathscr{F}^{-1}\mathscr{N}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\,\mathscr{F}.$$

An analogous reasoning as used in the proof of Theorem 8 allows us to prove the results below.

THEOREM 9. *The generalized Apostol-type matrix* $\mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$ *can be factorized in terms of the Lucas matrix* $\mathscr{L}$ *as follows.*

$$\mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)=\mathscr{L}\,\mathscr{L}_1^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu), \tag{11}$$

or,

$$\mathscr{W}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)=\mathscr{L}_2^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\mathscr{L}. \tag{12}$$

In particular,

$$\mathscr{L}\,\mathscr{L}_1^{[m-1]}(x)=\mathscr{W}^{[m-1]}(x;e,e;1;0;1)=\mathscr{L}_2^{[m-1]}(x)\mathscr{L}.$$

Also, the relations (11) and (12) allow us to deduce the following identity:

$$\mathscr{L}_1^{[m-1]}(x)=\mathscr{L}^{-1}\mathscr{L}_2^{[m-1]}(x)\,\mathscr{L}.$$

Remark 3. It is worthwhile to mention that

(a) If we consider $a\in\mathbb{C}$, $b\in\mathbb{C}\setminus\{0\}$ and $s=0,1$, then Theorems 8 and 9, as well as, their corollaries have corresponding analogous forms for generalized Fibonacci matrices of type s, $\mathscr{F}^{(a,b,s)}$, and for generalized Fibonacci matrices $\mathscr{U}^{(a,b,0)}$ with second order recurrent sequence $U_n^{(a,b)}$ subordinated to certain constraints. The reader may consult [24] in order to complete the details of this assertion.

(b) In the present article, all matrix identities have been expressed using finite matrices. Since such matrix identities involve lower triangular matrices, they have an analogue for infinite matrices.

Finally, we are only at the beginning of this subject, some possible paths to continue are to study further q-analogues of these matrices, (cf. [1, 8, 9, 11, 12, 19, 21–23]). Because of the close affinity to combinatorial identities and combinatorics, the q-analogues of the generalized Apostol-type polynomial

matrices could have many applications (depending on the notion of q-analogy chosen).

Acknowledgements. The work of the first author (YQ) has been supported by Decanato de Investigación y Desarrollo, Universidad Simón Bolívar, Venezuela, grant DID-USB (S1-IC-CB-003-16). Also, the first author thanks the hospitality of Coordinación de Matemáticas of Universidad del Atlántico during her visit for the Fall Semester 2016.

The work of the second author (WR) has been supported by Universidad del Atlántico, Colombia, grant Impacto Caribe (IC-002627-2015).

The work of the third author (AU) has been partially supported by Universidad del Atlántico, Colombia, grant Impacto Caribe (IC-002627-2015).

REFERENCES

[1] T. Arakawa, T. Ibukiyama and M. Kaneko, *Bernoulli Numbers and Zeta Functions.* Springer Monogr. Math., Springer, Tokyo, Heidelberg, New York, Dordrecht and London, 2014.

[2] G.S. Call and D.J. Velleman, *Pascal's matrices.* Amer. Math. Monthly **100** (1993), 372–376.

[3] G.-S. Cheon and J.-S. Kim, *Stirling matrix via Pascal matrix.* Linear Algebra Appl. **329** (2001), 49–59.

[4] G.-S. Cheon and J.-S. Kim, *Factorial Stirling matrix and related combinatorial sequences.* Linear Algebra Appl. **357** (2002), 247–258.

[5] L. Comtet, *Advanced Combinatorics. The Art of Finite and Infinite Expansions.* D. Reidel Publishing Co., Dordrecht and Boston, 1974. (Translated from French by J.W. Nienhuys).

[6] T. Ernst, *A Comprehensive Treatment of q-Calculus.* Birkhäuser, Springer, Basel, Heidelberg, New York, Dordrecht and London, 2012.

[7] P. Hernández-Llanos, Y. Quintana and A. Urieles, *About extensions of generalized Apostol-type polynomials.* Results Math. **68** (2015), 203–225.

[8] G.I. Infante, J.L. Ramírez and A. Şahin, *Some results on q-analogue of the Bernoulli, Euler and Fibonacci matrices.* Math. Rep. (Bucur.) **19** (2017), *4*, 399–417.

[9] D.S. Kim and T. Kim, *Some identities of q-Euler polynomials arising from q-umbral calculus.* J. Inequal. Appl. **2014** (2014), Paper No. 1, 12 p.

[10] G.-Y. Lee and J.-S. Kim, *The linear algebra of the k-Fibonacci matrix.* Linear Algebra Appl. **373** (2003), 75–87.

[11] G.-Y. Lee, J.-S. Kim and S.-G. Lee, *Factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices.* Fibonacci Quart. **40** (2002), *3*, 203–211.

[12] G.-Y. Lee, J.-S. Kim and S.-H. Cho, *Some combinatorial identities via Fibonacci numbers.* Discrete Appl. Math. **130** (2003), *3*, 527–534.

[13] B. Kurt, *Some relationships between the generalized Apostol-Bernoulli and Apostol-Euler polynomials.* Turkish J. Anal. Number Theory **1** (2013), *1*, 54–58.

[14] D.-Q. Lu and Q.-M. Luo, *Some properties of the generalized Apostol-type polynomials.* Bound. Value Probl. **2013** (2013), Paper No. 64, 13 p.

[15] Q.-M. Luo and H.M. Srivastava, *Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials.* J. Math. Anal. Appl. **308** (2005), *1*, 290–302.

[16] Q.-M. Luo and H.M. Srivastava, *Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials.* Comput. Math. Appl. **51** (2006), 631–642.

[17] Q.-M. Luo and H.M. Srivastava, *Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind.* Appl. Math. Comput. **217** (2011), 5702–5728.

[18] H. Ozden, Y. Simsek and H.M. Srivastava, *A unified presentation of the generating functions of the generalized Bernolli, Euler and Genocchi polynomials.* Comput. Math. Appl. **60** (2010), 2779–2787.

[19] Y. Quintana, W. Ramírez and A. Urieles, *Euler matrices and their algebraic properties revisited.* Submitted. arXiv:1811.01455 [math.NT].

[20] Y. Quintana, W. Ramírez and A. Urieles, *On an operational matrix method based on generalized Bernoulli polynomials of level m.* Calcolo **55** (2018), *3*, Paper No. 30, 29 p.

[21] J. Riordan, *Combinatorial Identities.* Wiley, New York, London and Sydney, 1968.

[22] H.M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions.* Springer, Dordrecht, Netherlands, 2001.

[23] H.M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals.* Elsevier, London, 2012.

[24] P. Stanimirović, J. Nikolov and I. Stanimirović, *A generalization of Fibonacci and Lucas matrices.* Discrete Appl. Math. **156** (2008), *14*, 2606–2619.

[25] W. Wang and T. Wang, *Matrices related to the Bell polynomials.* Linear Algebra Appl. **422** (2007), 139–154.

[26] Z.Z. Zhang, *The linear algebra of generalized Pascal matrix.* Linear Algebra Appl. **250** (1997), 51–60.

[27] Z.Z. Zhang and M.X. Liu, *An extension of generalized Pascal matrix and its algebraic properties.* Linear Algebra Appl. **271** (1998), 169–177.

[28] Z. Zhang and J. Wang, *Bernoulli matrix and its algebraic properties.* Discrete Appl. Math. **154** (2006), 1622–1632.

[29] Z. Zhang and Y. Zhang, *The Lucas matrix and some combinatorial identities.* Indian J. Pure Appl. Math. **38** (2007), *5*, 457–465.

Received 11 January 2017

Universidad Simón Bolívar,
Departamento de Matemáticas Puras y Aplicadas,
Postal Code: 89000,
Caracas 1080 A, Venezuela
yquintana@usb.ve

Universidad de la Costa,
Departamento de Ciencias Básicas,
Barranquilla, Colombia
wramirez4@cuc.edu.co

Universidad del Atlántico,
Programa de Matemáticas,
Km 7 Vía Pto. Colombia, Barranquilla, Colombia
alejandrourieles@mail.uniatlantico.edu.co