DYNAMICS OF A FISH MODEL IN A MULTILAYER ENVIRONMENT AND DISCONTINUOUS DIFFUSIVITY

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We consider a marine population that feeds mainly on phytoplankton, zooplankton and living in a stratified environment composed of n layers. The life cycle of the population is divided in two stages, juveniles and adults. Each stage is modeled by an advection-diffusion reaction equation. Thus, we have to deal with a nonlinear partial differential system with jumping diffusion. These layers are arranged according to the light and temperature received by the habitat. We formulate the model as a suitable Cauchy problem, and make use of the m-accretive operators theory to establish well-posedness of the system. Moreover, global solution is examined in L^2 space.

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1. INTRODUCTION

It is well known that the fluctuations in anchovy stock is not only due to fishing pressure but it also strongly relies on environmental conditions [3]. Movement and transport of fish by hydrodynamics (marine currents) is important because it determines the distribution of the stock in a given region [11]. Spatial heterogeneity may be of striking features of the fish population and their exploitation.

There is a relation between the distribution of nutrients and anchovy abundance [8]. This fact is clearly observed during El Niño season where there is a drop of photosynthesis rates which in turn impacts the food supply of phytoplancton, and fish disperse in deep water [26].

The water column is divided into distinct layers [30]. The upper layer is sunlit and photosynthesis is limited by the supply of nutrients. The lower layer is dark and nutrients are abundant but photosynthesis is limited by the lack of light. For instance in the Mediterranean sea, summer is characterized by a marked stratification in the water column.

Dominated by vertical movement as a response to food availability and light, fish migrate upward during the afternoon, reside close the upper layer and descend early in the morning [30]. Neglecting vertical migration may lead to errors in estimates of the stock.

1.1. THE MATHEMATICAL MODEL

In this section, we give the equations of our model. Let D be an open bounded domain with smooth boundary in R^2. The x -axis is West-East positively oriented. The y -axis is South-North positively oriented. The vertical coordinate is oriented downwards and the sea surface corresponds to $z = 0$.

The random walk depends on the depth of the column water and the fish population never swim under a threshold value $z = z^*$, called the thermocline. Eggs and larvae are found above the thermocline zone. Eggs production is not possible in the cold layer. The movement is restricted to vertical one, going from one to several ten of meters. The horizontal diffusion exhibits several orders of magnitude higher ranging up to several kilometers. Hence, reducing both scales to the same order of magnitude, horizontal diffusion has been neglected.

The column water $(0, z^*)$ is partitioned into n layers (z_{i-1}, z_i). Let

$$(0, z^*) = \cup_{i=1}^{i=n}(z_{i-1}, z_i).$$

The population is living in a habitat given by

$$\Omega = \cup_{i=1}^{i=n} D \times (z_{i-1}, z_i).$$

1.1.1. **Velocity**

In each layer i, the field (w_1^i, w_2^i, w_3^i) represents velocity of marine currents and is deduced from circulation model applied to a small volume of water, see [24]. For simplicity, we investigate the model when the velocity of the horizontal direction w_1^i, w_2^i , $1 \leq i \leq n$, does not depend on the variable z.

1.1.2. **Modeling the distribution of the anchovy**

The cycle of life of anchovy is divided in two stages: juveniles and adults. The quantity (u^i, v^i) represents the population spreading in $\Omega_i = D \times (z_{i-1}, z_i)$ where u^i,v^i are respectively the adult and juvenile density for each layer i. We assume that the change of density occurs as a result of demographic, competition process and spatial movement of population. The dynamics of such species can be written in terms of reaction-diffusion-advection equations.

In each layer i $(1 \leq i \leq n)$, we have the following problem:

$$\begin{cases} \frac{\partial}{\partial t}u^i - \mathrm{d}_1^i \frac{\partial^2 u^i}{\partial z^2} - w_1^i \frac{\partial u^i}{\partial x} - w_2^i \frac{\partial u^i}{\partial y} - w_3^i \frac{\partial u^i}{\partial z} = \sigma v^i - eu^i - cu^i(u^i + v^i), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega_i \times (0,T), \\ \frac{\partial}{\partial t}v^i - \mathrm{d}_2^i \frac{\partial^2 v^i}{\partial z^2} - w_1^i \frac{\partial v^i}{\partial x} - w_2^i \frac{\partial v^i}{\partial y} - w_3^i \frac{\partial v^i}{\partial z} = bu^i - fv^i - dv^i(u^i + v^i), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega_i \times (0,T). \end{cases}$$

Here the time runs in $(0,T)$, T is a fixed time. The parameter σ gives the rate at which juveniles become adult. The constant b corresponds to the birth rate. The parameters e, and f, reflect the natural mortality respectively of adult and juveniles. The constants c and d measure the competition between adult and juveniles. All the parameters are positive (see [20] for more details). The model has a diffusion term with dispersion rate in each layer d_a^i, $a = 1, 2$.

The equations have to be completed by appropriate initial and boundary conditions.

1.1.3. **Vertical boundary conditions**

We suppose a no-flux condition on the top and bottom layers

$$\begin{cases} \frac{\partial u^1}{\partial z}(t,x,y,0) = \frac{\partial u^n}{\partial z}(t,x,y,z^*) = 0, \\ \frac{\partial v^1}{\partial z}(t,x,y,0) = \frac{\partial v^n}{\partial z}(t,x,y,z^*) = 0. \end{cases}$$

There is no flux of biomass across the sea surface and thermocline level.

1.1.4. **Lateral boundary conditions**

The lateral boundary is the part of the physical boundary from the sea surface and the bottom surface represented by the thermocline. The system does not show any lateral boundary conditions. Choosing the right conditions is a difficult problem that we avoid by assuming that the initial values have a compact support in the interior of the domain.

1.1.5. **Interface conditions**

The flux at the interface of each layer i $(1 \leq i \leq n)$ and the continuity of the solution give the supplementary conditions

$$\begin{cases} \mathrm{d}_1^i \frac{\partial u^i}{\partial z}(t,x,y,z_i) - w_3^i u^i = \mathrm{d}_1^{i+1} \frac{\partial u^{i+1}}{\partial z}(t,x,y,z_i) - w_3^{i+1} u^{i+1}, \\ \mathrm{d}_2^i \frac{\partial v^i}{\partial z}(t,x,y,z_i) - w_3^i v^i = \mathrm{d}_2^{i+1} \frac{\partial v^{i+1}}{\partial z}(t,x,y,z_i) - w_3^{i+1} v^{i+1}, \\ u^{i+1}(t,x,y,z_i) = u^i(t,x,y,z_i), \\ v^{i+1}(t,x,y,z_i) = v^i(t,x,y,z_i). \end{cases}$$

These conditions impose continuity of the densities and the flux at each interface.

1.1.6. **Initial conditions**

At the beginning of the year, $t = 0$, the initial densities are $u_0^i(x, y, z)$ and $v_0^i(x, y, z)$

$$u^i(0, x, y, z) = u_0^i(x, y, z), v^i(0, x, y, z) = v_0^i(x, y, z), \text{in } \Omega_i.$$

Since the diffusion takes place only in the vertical direction, the above system is degenerate.

The application of advection-diffusion models to fish population dynamics has been the subject of many studies, only few analytical results are obtained for these models, and the main research tool is computer simulation.

With no attempt to be exhaustive, let us mention the following works: in absence of stratification and without a preferred direction for diffusion, the model is considered in [20]. In [4], the author describes the evolution of phytoplankton by a scalar equation with vertical migration. In [1], the authors introduce a mathematical model for the coupled dynamics for phytoplankton and its nutrient in the sea. Similar model with vertical diffusion and constant diffusivity has been carried out in the context of atmospheric pollutants, in [10]. All the previous works assumed incompressibility condition of the fluid

$$\operatorname{div}(w_1^i, w_2^i, w_3^i) = 0, \quad i = 1, n.$$

In [15], the authors investigated global solution of the model considered in [10] without incompressibility condition and distinct diffusivities. Most spatial models are studied by simulations [5, 6, 18, 28] and [17].

The present paper differs from previous studies in two aspects. First, the diffusion coefficients depend on the age of the population, the depth of the ocean and may have jumps across the interfaces. Secondly, the fluid velocity changes with the layers. Since diffusion is ignored in horizontal direction, the system (1.1) is ultraparabolic and classical approaches are not directly applied. Our strategy is to use a change of variables to get rid of degeneracies of the system. In the new coordinates, the problem is reduced to a one dimensional non degenerate parabolic problem to which we can use semi-group theory as in [16] and [14]. Similar methods are used with stratified domain [2].

We give an abstract setting to our problem allowing other nonlinearities satisfying conditions below. Hence, we turn our attention to the more general

system

(1.1)
$$\begin{cases} \frac{\partial}{\partial t}u^i - \mathrm{d}_1^i \frac{\partial^2 u^i}{\partial z^2} - w_1^i \frac{\partial u^i}{\partial x} - w_2^i \frac{\partial u^i}{\partial y} - w_3^i \frac{\partial u^i}{\partial z} = f_1(u^i, v^i), \text{in } \Omega_i \times (0,T), 1 \le i \le n, \\ \frac{\partial}{\partial t}v^i - \mathrm{d}_2^i \frac{\partial^2 v^i}{\partial z^2} - w_1^i \frac{\partial v^i}{\partial x} - w_2^i \frac{\partial v^i}{\partial y} - w_3^i \frac{\partial v^i}{\partial z} = f_2(u^i, v^i), \text{in } \Omega_i \times (0,T), 1 \le i \le n, \\ \frac{\partial u^1}{\partial z}(t,x,y,0) = \frac{\partial u^n}{\partial z}(t,x,y,z^*) = 0, \\ \frac{\partial v^1}{\partial z}(t,x,y,0) = \frac{\partial v^n}{\partial z}(t,x,y,z^*) = 0, \\ \mathrm{d}_1^i \frac{\partial u^i}{\partial z}(t,x,y,z_i) - w_3^i u^i = \mathrm{d}_1^{i+1} \frac{\partial u^{i+1}}{\partial z}(t,x,y,z_i) - w_3^{i+1} u^{i+1}, 1 \le i \le n-1, \\ \mathrm{d}_2^i \frac{\partial v^i}{\partial z}(t,x,y,z_i) - w_3^i v^i = \mathrm{d}_2^{i+1} \frac{\partial v^{i+1}}{\partial z}(t,x,y,z_i) - w_3^{i+1} v^{i+1}, 1 \le i \le n-1, \\ u^i(t,x,y,z_i) = u^{i+1}(t,x,y,z_i), v^i(t,x,y,z_i) = v^{i+1}(t,x,y,z_i), 1 \le i \le n-1, \\ u^i(0,x,y,z) = u_0^i(x,y,z), v^i(0,x,y,z) = v_0^i(x,y,z), \text{in } \Omega_i. \end{cases}$$

The main goal of this work is to assess the mathematical well-posedness of the system (1.1).

1.2. ASSUMPTIONS AND COMMENTS

A1) Throughout the paper, we suppose that the nonlinearity $f = (f_1, f_2)$ is continuously differentiable.

A2) To preserve positiveness of the solution, we assume that f is quasi-positive which means that

$$\begin{aligned} \forall v &\in \mathbb{R}_+, \ f_1(0,v) \ge 0 \ , \\ \forall u &\in \mathbb{R}_+, \ f_2(u,0) \ge 0. \end{aligned} \tag{1.2}$$

A3) To avoid blow-up of the solution, we will suppose that the nonlinearity is at most linear in the positive quadrant, this implies that there exist positive constants a^i, b^i such that

$$\forall (u,v) \in \mathbb{R}_+^2 \colon \ f_i(u,v) \le a^i u + b^i v, i = 1,2. \tag{1.3}$$

A4) We now turn to the current velocity, and we make the assumptions:

When the water column is divided into small layers, it is reasonable as a first approximation to assume that inside each layer, the horizontal velocity is independent of the vertical variable z, this assumption is also discussed in [15].

For all $i = 1, ...n$, we suppose that w_1^i,$w_2^i : [0, T] \times \overline{D} \to \mathbb{R}$ are continuous, and for fixed t, we assume that

$$w_1^i(t, .), \quad w_2^i(t, .) \in C^1(D).$$

Moreover, for any i, we assume that the vertical velocity w_3^i :$D \times [z_{i-1}, z_i] \to \mathbb{R}$ is continuously differentiable. We assume for $(x, y) \in D$ that $w_3^1(x, y, 0) = w_3^n(x, y, z^*) = 0$ and

$$w_3^i(x, y, z_i) = w_3^{i+1}(x, y, z_i),\ 1 \leq i \leq n - 1.$$

A5) The initial values are nonnegative,

$$(u_0^i, v_0^i) \in C(\overline{\Omega_i}) \times C(\overline{\Omega_i}),$$

with horizontal projection of the support inside a compact of the interior of D.

Remark 1. All the assumptions are satisfied by the nonlinearity

$$\begin{cases} f_1(u^i, v^i) = \sigma v^i - eu^i - cu^i(u^i + v^i), \\ f_2(u^i, v^i) = bu^i - fv^i - dv^i(u^i + v^i). \end{cases}$$

The condition (1.3) is consistent with quadratic nonlinearity, for instance, for $(u, v) \in \mathbb{R}_+^2$ then

$$\sigma v - eu - cu(u + v) \leq \sigma v - eu \leq au + bv.$$

for some positive constants a, b.

The organization of the paper is as follows: Section 2 deals with local existence. Section 3 is devoted to positiveness of solution. In Section 4, we are concerned with global solution. Section 5 contains the main results of the paper.

2. LOCAL EXISTENCE

Before starting with the treatment of the problem (1.1), we perform a change of variables. The problem is solved along the horizontal components, on the characteristic lines. On these lines, the system is reduced to one dimensional parabolic equations with vertical components.

2.1. CHANGE OF VARIABLES

For each layer (z_{i-1}, z_i), we define the characteristic lines $(x^i(t, x_0, y_0), y^i(t, x_0, y_0))$ as solutions of the Cauchy problem:

$$\begin{cases} \frac{\mathrm{d}x^i}{\mathrm{d}t} = -w_1^i(t, x^i, y^i), \\ \frac{\mathrm{d}y^i}{\mathrm{d}t} = -w_2^i(t, x^i, y^i), \\ x^i(0) = x_0, y^i(0) = y_0. \end{cases}$$

From assumptions made on the functions w_k^i for $k = 1, 2$, these solutions are defined on $(0, T)$ and are C^1 in all variables. Let

$$I_t = \left\{(x_0, y_0) \in \mathbb{R}^2 : (x^i(t, x_0, y_0), y^i(t, x_0, y_0)) \in D\right\}.$$

For each i, and for fixed t, define the map $\Psi_t^i : I_t \to D$ by

$$\Psi_t^i(x_0, y_0) = (x^i, y^i).$$

The map Ψ_t^i defines a diffeomorphism. Indeed, let (x^i, y^i) be fixed in D, then there exists a unique (x_0, y_0) defined by

$$(x_0, y_0) = \Psi_{-t}^i(x^i, y^i).$$

Writing the solution (u^i, v^i) in the new coordinates (t, x_0, y_0, z), we have

$$\varphi_1^i(t, x_0, y_0, z) = u^i(t, \Psi_t^i(x_0, y_0), z),$$

and

$$\varphi_2^i(t, x_0, y_0, z) = v^i(t, \Psi_t^i(x_0, y_0), z).$$

For notational convenience, we will occasionally omit the reference to (x_0, y_0). For each (x_0, y_0), we associate the following system:

$$\begin{cases} \frac{\partial}{\partial t}\varphi_1^i - \mathrm{d}_1^i \frac{\partial^2 \varphi_1^i}{\partial z^2} - w_3^i \frac{\partial \varphi_1^i}{\partial z} = f_1(\varphi_1^i, \varphi_2^i), \text{in } \ (z_{i-1}, z_i), 1 \le i \le n, \\ \frac{\partial}{\partial t}\varphi_2^i - \mathrm{d}_2^i \frac{\partial^2 \varphi_2^i}{\partial z^2} - w_3^i \frac{\partial \varphi_2^i}{\partial z} = f_2(\varphi_1^i, \varphi_2^i), \text{in } (z_{i-1}, z_i), 1 \le i \le n, \\ \mathrm{d}_1^1 \frac{\partial \varphi_k^1}{\partial z}(t, 0) = \frac{\partial \varphi_k^n}{\partial z}(t, z^*) = 0, \text{for } \ z = 0, z^*, k = 1, 2, \\ \mathrm{d}_k^i \frac{\partial \varphi_k^i}{\partial z}(t, z_i) - w_3^i \varphi_k^i(t, z_i) = \mathrm{d}_k^{i+1} \frac{\partial \varphi_k^{i+1}}{\partial z}(t, z_i) - w_3^{i+1} \varphi_k^{i+1}(t, z_i), \\ \qquad\qquad 1 \le i \le n-1, k = 1, 2, \\ \varphi_k^i(t, z_i) = \varphi_k^{i+1}(t, z_i), k = 1, 2, 1 \le i \le n-1, \\ \varphi_1^i(0, z) = u_0^i(z), \varphi_2^i(0, z) = v_0^i(z). \end{cases} \tag{2.1}$$

Here $w_3^i(\Psi_t^i(x_0,y_0),z) = \widetilde{w}_3^i(z)$. For writing simplicity, we shall no longer indicate the superscript "$\sim$" but we keep the same notation w_3^i.

We need to define the following functions, for $k = 1, 2$

$$\varphi_k = \begin{cases} \varphi_k^1 \text{ in } (z_0, z_1), \\ \varphi_k^2 \text{ in } (z_1, z_2), \\ \qquad \dots \\ \varphi_k^n \text{ in } (z_{n-1}, z_n). \end{cases}$$

For the initial conditions, we have

$$\varphi_1^0 = \begin{cases} u_0^1 \text{ in } (z_0, z_1), \\ u_0^2 \text{ in } (z_1, z_2), \\ \qquad \dots \\ u_0^n \text{ in } (z_{n-1}, z_n), \end{cases}$$

and

$$\varphi_2^0 = \begin{cases} v_0^1 \text{ in } (z_0, z_1), \\ v_0^2 \text{ in } (z_1, z_2), \\ \qquad \dots \\ v_0^n \text{ in } (z_{n-1}, z_n). \end{cases}$$

For $k = 1, 2$, let

$$\mathrm{d}_k = \begin{cases} \mathrm{d}_k^1 \text{ in } (z_0, z_1), \\ \mathrm{d}_k^2 \text{ in } (z_1, z_2), \\ \qquad \dots \\ \mathrm{d}_k^n \text{ in } (z_{n-1}, z_n). \end{cases}$$

Similarly, the map w_3 is defined by

$$w_3 = \begin{cases} w_3^1 \text{ in } (z_0, z_1), \\ w_3^2 \text{ in } (z_1, z_2), \\ \qquad \dots \\ w_3^n \text{ in } (z_{n-1}, z_n). \end{cases}$$

From the conditions on $w_3^i, 1 \le i \le n$, the assumptions on the interfaces will be just

$$\mathrm{d}_k^i \frac{\partial \varphi_k^i}{\partial z}(t, z_i) = \mathrm{d}_k^{i+1} \frac{\partial \varphi_k^{i+1}}{\partial z}(t, z_i), 1 \le i \le n-1, k = 1, 2.$$

Instead of (2.1), we will consider the system

$$
(2.2)\qquad \begin{cases} \frac{\partial}{\partial t}\varphi_1 - \frac{\partial}{\partial z}(\mathrm{d}_1(z)\frac{\partial \varphi_1}{\partial z}) - w_3\frac{\partial \varphi_1}{\partial z} = f_1(\varphi_1,\varphi_2) \text{ in } (0,z^*)\times(0,T), \\ \\ \frac{\partial}{\partial t}\varphi_2 - \frac{\partial}{\partial z}(\mathrm{d}_2(z)\frac{\partial \varphi_2}{\partial z}) - w_3\frac{\partial \varphi_2}{\partial z} = f_2(\varphi_1,\varphi_2), \text{in } (0,z^*)\times(0,T), \\ \varphi_1(0,z) = \varphi_1^0(z), \varphi^2(0,z) = \varphi_0^2(z), \\ \\ \frac{\partial \varphi_1}{\partial z} = \frac{\partial \varphi_2}{\partial z} = 0, \text{for } z = 0, z^*. \end{cases}
$$

We limit the study of this problem in a finite time interval when the species remains in the water column and does not reach the lateral boundary. For any (u_0^i, v_0^i), $1 \le i \le n$, let

$$
T^i_{(u_0^i,v_0^i)} = \sup \left\{ \begin{array}{c} t > 0 : \varphi_1^i(s,x_0,y_0,z) \in D, \forall\,(x_0,y_0,z) \in \text{support } u_0^i, \\ \forall s \in [0,t[\,, \text{ and } \varphi_2^i(s,x_0,y_0,z) \in D, \\ \forall\,(x_0,y_0,z) \in \text{support } v_0^i,\ \forall s \in [0,t[\end{array} \right\}.
$$

Then, we define the time of observation by

$$
T_o = \min_{1\le i\le n} T^i_{(u_0^i,v_0^i)}.
$$

Remark 2. Since D is open, then for $t \in (0,T_o)$, $\Psi_t^i(x_0,y_0) \notin \partial D$, and the support of the solutions does not cross the lateral boundaries. The quantity T_o is the time for which no material goes through the lateral boundaries. Hence, we do not need lateral boundaries in the sequel.

2.2. CAUCHY PROBLEM FORMULATION

In this section, we formulate the initial value-problem which will be studied. We introduce the functional spaces which we use in the remainder of this work. Let

$$
Y = L^2(0,z^*) \times L^2(0,z^*).
$$

endowed with the standard norm. To transform (2.2) to a Cauchy problem, we define the operator $(k = 1,2)$

$$
A_k\varphi = -\frac{\partial}{\partial z}(\mathrm{d}_k(z)\frac{\partial \varphi}{\partial z}) - w_3\frac{\partial \varphi}{\partial z}
$$

where

$$
D(A_k) = \left\{ \varphi \in L^2(0,z^*) \cap H^2(z_{i-1},z_i),\ A_k\varphi \in L^2(0,z^*), \frac{\partial \varphi}{\partial z} = 0 \text{ for } z = 0, z^* \right\}
$$

Here $H^2(z_{i-1},z_i)$ is the Sobolev space with standard norm. Let $A : D(A) \subset Y \to Y$ be the operator defined by

$$A(\varphi_1,\varphi_2) = (A_1\varphi_1, A_2\varphi_2)$$
$$= (-\frac{\partial}{\partial z}(\mathrm{d}_1(z)\frac{\partial\varphi_1}{\partial z}) - w_3\frac{\partial\varphi_1}{\partial z}, -\frac{\partial}{\partial z}(\mathrm{d}_2(z)\frac{\partial\varphi_2}{\partial z}) - w_3\frac{\partial\varphi_2}{\partial z}),$$

where

$$D(A) = \left\{\varphi \in Y \cap (H^2(z_{i-1},z_i))^2, A\varphi \in Y, \frac{\partial\varphi}{\partial z} = 0 \text{ for } z = 0, z^*\right\}$$
$$= D(A_1) \times D(A_2).$$

Let $\varphi = (\varphi_1,\varphi_2)$ and $F(\varphi) = (f_1(\varphi_1,\varphi_2), f_2(\varphi_1,\varphi_2))$. We are led to the Cauchy problem

$$\left\{\begin{array}{c} \frac{\partial}{\partial t}\varphi + A\varphi = F(\varphi), \\ \varphi(0) = \varphi_0. \end{array}\right. \tag{2.3}$$

We perform an analytical study of (2.3), by using m-accretive operators.

Proposition 1. *The operator A is quasi accretive on Y.*

Proof. We prove that there exists $\lambda > 0$ such that

$$(\lambda u + A_1 u, u)_{L^2(0,z^*)} \geq 0, \ \forall u \in D(A_1).$$

To see this, let

$$c_1 = \min d_i, 1 \leq i \leq n,$$

then

$$\begin{aligned}(\lambda u + A_1 u, u)_{L^2(0,z^*)} &= \int_0^{z^*}\left[\lambda u^2 + \mathrm{d}_1(z)(\frac{\partial u}{\partial z})^2 - w_3(z)\frac{\partial u}{\partial z}u\right]\mathrm{d}z \\ &\geq \lambda\int_0^{z^*} u^2\mathrm{d}z + c_1\int_0^{z^*}(\frac{\partial u}{\partial z})^2\mathrm{d}z - \int_0^{z^*} w_3(z)\frac{\partial u}{\partial z}u\ \mathrm{d}z.\end{aligned}$$

Young's inequality gives

$$\int_0^{z^*} w_3(z)\frac{\partial u}{\partial z}u\ \mathrm{d}z \leq \|w_3\|_\infty\,(\rho\int_0^{z^*}(\frac{\partial u}{\partial z})^2\mathrm{d}z + \frac{1}{\rho}\int_0^{z^*} u^2\mathrm{d}z),$$

for some $\rho > 0$.

We arrive at

$$\begin{aligned}(\lambda u + A_1 u, u)_{L^2(0,z^*)} &\geq \lambda\int_0^{z^*} u^2\mathrm{d}z + c_1\int_0^{z^*}(\frac{\partial u}{\partial z})^2\mathrm{d}z \\ &\quad - \|w_3\|_\infty\,(\rho\int_0^{z^*}(\frac{\partial u}{\partial z})^2\mathrm{d}z + \frac{1}{\rho}\int_0^{z^*} u^2\mathrm{d}z) \\ &= (\lambda - \|w_3\|_\infty/\rho)\int_0^{z^*} u^2\mathrm{d}z + (c_1 - \|w_3\|_\infty\rho)\end{aligned}$$

$$\int_0^{z^*} (\frac{\partial u}{\partial z})^2 \mathrm{d}z \geq 0.$$

provided that λ, ρ verify

$$(\lambda - \|w_3\|_\infty / \rho) > 0, (c_1 - \|w_3\|_\infty \rho) > 0.$$

Similarly, we have

$$(\lambda u + A_2 u, u)_{L^2(0,z^*)} \geq 0, \ \forall u \in D(A_2).$$

We conclude that there exists $\lambda > 0$ such that:

$$(\lambda \varphi + A\varphi, \varphi)_Y \geq 0, \forall \varphi \in D(A).$$

Consequently, A is quasi-accretive. $\square$

To prove that A is quasi m-accretive on Y, we need to show that the range $R(\lambda I + A) = Y$. For convenience, we denote by c a general positive constant.

Proposition 2. *The operator A is quasi m-accretive on Y.*

Proof. There exists $\lambda > 0$ such that for all $f \in L^2(0, z^*)$, there exists $u \in D(A_1)$ such that

$$\lambda u + A_1 u = f.$$

Indeed, for $u, v \in H^1(0, z^*)$, let

$$a(u, v) = \lambda \int_0^{z^*} uv \mathrm{d}z + \int_0^{z^*} \mathrm{d}_1(z) \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \mathrm{d}z - \int_0^{z^*} w_3(z) \frac{\partial u}{\partial z} v \mathrm{d}z,$$

and

$$l(v) = \int_0^{z^*} fv \mathrm{d}z.$$

It is easy to show that

$$|a(u, v)| \leq c \|u\|_{H^1(0,z^*)} \|v\|_{H^1(0,z^*)},$$

and

$$|l(v)| \leq c \|v\|_{H^1}.$$

Furthermore

$$\begin{aligned} a(u, u) &= (\lambda u + A_1 u, u)_{L^2(0,z^*)} \\ &\geq (\lambda - \|w_3\|_\infty / \rho) \int_0^{z^*} u^2 \mathrm{d}z + (c_1 - \|w_3\|_\infty \rho) \int_0^{z^*} (\frac{\partial u}{\partial z})^2 \mathrm{d}z \\ &\geq c \|u\|^2_{H^1(0,z^*)}, \end{aligned}$$

provided that λ, ρ verify

$$(\lambda - \|w_3\|_\infty / \rho) > 0, \ (c_1 - \|w_3\|_\infty \rho) > 0.$$

The bilinear form a is then coercive and the Lax Milgram Theorem implies that the equation $a(u,v) = l(v)$ has a unique solution $u \in H^1(0, z^*)$.

It remains to verify that

$$\lambda u + A_1 u = f.$$

For that, let $v \in H^1(0, z^*)$, we obtain

$$\int_0^{z^*} \mathrm{d}_1(z) \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \mathrm{d}z = -\lambda \int_0^{z^*} uv \mathrm{d}z + \int_0^{z^*} w_3(z) \frac{\partial u}{\partial z} v \mathrm{d}z + \int_0^{z^*} fv \mathrm{d}z. \tag{2.4}$$

In particular, for all $v \in C_c^1(0, z^*)$

$$\left| \int_0^{z^*} \mathrm{d}_1(z) \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \mathrm{d}z \right| \leq \lambda \int_0^{z^*} |uv| \, \mathrm{d}z + \int_0^{z^*} \left| w_3(z) \frac{\partial u}{\partial z} v \right| \mathrm{d}z + \int_0^{z^*} |fv| \, \mathrm{d}z.$$

Holder's inequality gives

$$\left| \int_0^{z^*} \mathrm{d}_1(z) \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \mathrm{d}z \right| \leq c \, \|v\|_{L^2(0,z^*)} .$$

A classical result (see for instance [21], p. 124) implies that

$$\mathrm{d}_1(z) \frac{\partial u}{\partial z} \in H^1(0, z^*).$$

Now, by choosing $v \in C_c^1(z_{i-1}, z_i)$, in (2.4), $1 \leq i \leq n$, we obtain

$$\int_{z_{i-1}}^{z_i} \mathrm{d}_1^i \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \mathrm{d}z = -\lambda \int_{z_{i-1}}^{z_i} uv \mathrm{d}z + \int_{z_{i-1}}^{z_i} w_3(z) \frac{\partial u}{\partial z} v \mathrm{d}z + \int_{z_{i-1}}^{z_i} fv \mathrm{d}z,$$

hence

$$\left| \int_{z_{i-1}}^{z_i} \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \mathrm{d}z \right| \leq c \, \|v\|_{L^2(z_{i-1}, z_i)} , 1 \leq i \leq n,$$

it follows that

$$\frac{\partial u}{\partial z} \in H^1(z_{i-1}, z_i),$$

i.e., $u \in H^2(z_{i-1}, z_i), 1 \leq i \leq n$. We conclude that $u \in D(A)$.

Since

$$\mathrm{d}_1(z) \frac{\partial u}{\partial z} \in H^1(0, z^*),$$

and

$$u \in H^2(z_{i-1}, z_i),$$

this yields that

$$u \in C^1[z_{i-1}, z_i], 1 \leq i \leq n.$$

Integrating by part (2.4), one can see that for all $v \in H^1(0, z^*)$,

$$(2.5)\quad -\int_0^{z^*}\frac{\partial}{\partial z}(\mathrm{d}_1(z)\frac{\partial u}{\partial z})v\mathrm{d}z+\mathrm{d}_1^n\frac{\partial u}{\partial z}(z^*)v(z^*)-\mathrm{d}_1^1\frac{\partial u}{\partial z}(0)v(0)$$
$$=-\lambda\int_0^{z^*}uv+\int_0^{z^*}w_3(z)\frac{\partial u}{\partial z}v+\int_0^{z^*}fv,$$

In particular, for all $v\in H_0^1(0,z^*)$, we obtain

$$-\int_0^{z^*}(\frac{\partial}{\partial z}(\mathrm{d}_1(z)\frac{\partial u}{\partial z})+\lambda u-w_3(z)\frac{\partial u}{\partial z}-f\ )v\mathrm{d}z=0,$$

hence

$$-\frac{\partial}{\partial z}(\mathrm{d}_1(z)\frac{\partial u}{\partial z})+\lambda u-w_3(z)\frac{\partial u}{\partial z}-f\ =0.$$

By (2.5), it follows that for all $v\in H^1(0,z^*)$,

$$\mathrm{d}_1^n\frac{\partial u}{\partial z}(z^*)v(z^*)-\mathrm{d}_1^1\frac{\partial u}{\partial z}(0)v(0)=0.$$

Now, by choosing $v_1(z)=\exp(z)$ and $v_2(z)=\frac{1}{\exp(z)}$ respectively in (2.5), we obtain an algebraic system, where the solution is

$$\frac{\partial u}{\partial z}(z^*)=\frac{\partial u}{\partial z}(0)=0.$$

We conclude that u satisfies

$$\begin{cases}\lambda u-\frac{\partial}{\partial z}(\mathrm{d}_1(z)\frac{\partial u}{\partial z})-w_3(z)\frac{\partial u}{\partial z}=f,\\ \frac{\partial u}{\partial z}(z^*)=\frac{\partial u}{\partial z}(0)=0.\end{cases}$$

That is

$$u\in D(A_1),\lambda u+A_1u=f.$$

Similarly, we show that A_2 is quasi m accretive. $\square$

Since A is quasi m-accretive in Y, then -A generates a quasi contractive semigroup $S(t)$ on Y with $\|S(t)\|_Y\leq e^{wt}$, (see [22]). Here $S(t)=(S_1(t),S_2(t))$ on Y, where $S_1(t),S_2(t)$ are semi groups generated respectively by $-A_1,-A_2$. The mild solution of the system

$$\begin{cases}\frac{\partial}{\partial t}\varphi+A\varphi=0,\\ \varphi(0)=\varphi_0\end{cases}$$

is given by

$$(\varphi_1,\varphi_2)=S(t)\varphi_0.$$

Since it is in general non trivial to show that the operator F leaves Y invariant, then it is convenient to work in the framework of continuous function. Let

$$X=C[0,z^*]\times C[0,z^*].$$

In the next lemma, we show that the part of A in X, considered as an operator on X, generates a C^0 semigroup on X. For self-completeness of the paper, we give a proof strongly inspired from [19].

LEMMA 1. *Let* $A^c = A\backslash_{C[0,z^*]}$, *the restriction of* A *to*

$$D(A^c) = \{u \in D(A) \cap X, A^c u \in X\}.$$

Then $-A^c$ *generates a* C^0 *semigroup* $S^c(t)$*on* X, *and for* $\varphi_0 \in X$, *the mild solution* $S(t)\varphi_0 \in C([0,\infty[, X)$.

For the convenience of the reader, we give a brief proof, see Appendix A.

We define $C([0,\delta], X)$ to be the set of continuous functions $\varphi(t)$ defined on $0 \leq t \leq \delta$, taking values in X. With this notation, we say that $\varphi(t) \in C([0,\delta], X)$ is a mild solution of (2.3) provided for every $0 \leq t \leq \delta$, we have

$$\varphi(t) = S^c(t)\varphi_0 + \int_0^t S^c(t-s)F(\varphi(s))\mathrm{d}s,\ 0 \leq t \leq \delta,$$

where $S^c(t)$ is the semigroup generated on X by A^c. We are ready now, to establish local existence of solution of (2.2).

PROPOSITION 3. *There exists* $T_{\max} > 0$ *such that for any initial data* $\varphi_0 \in D(A^c)$, *the problem (2.3) has a unique classical solution* $(\varphi_1, \varphi_2) \in (C([0, T_{\max}[, X) \cap C(]0, T_{\max}[, D(A^c)) \cap C^1(]0, T_{\max}[, X))$.*If* $T_{\max} < \infty$, *then*

$$\lim_{t \to T_{\max}} \|(\varphi_1, \varphi_2)\|_X = +\infty.$$

Proof. This result is well known and we give only a sketch of the proof. The first step is to convert the system using variation of constants formula, to an integral equation

$$\varphi(t) = S^c(t)\varphi_0 + \int_0^t S^c(t-s)F(\varphi(s))\mathrm{d}s.$$

Since $F : X \to X$ is continuously differentiable, by standard contraction arguments, one can prove existence of a local mild solution defined on a maximal interval $[0, T_{\max}[$.

The hypotheses on φ_0 and F implies that this solution is classical. □

3. POSITIVITY OF THE SOLUTION

The system (2.2) preserves positiveness. If initial conditions φ_1^0, φ_2^0 are positive, then the solution of system (2.2) is positive.

PROPOSITION 4. *If* $(\varphi_1^0, \varphi_2^0)$ *is positive, then* $(\varphi_1(t,.), \varphi_2(t,.))$ *is positive.*

Proof. Let $\Pi : \mathbb{R}^2 \to (\mathbb{R}^+)^2$ the orthogonal projection onto the positive cone $(\mathbb{R}^+)^2$ of $\mathbb{R}^2$.

We denote by (u, v) the solution of (2.2). From (1.2), it follows that

$$f_1(\Pi(u,v))u^- \geq 0, f_2(\Pi(u,v))v^- \geq 0.$$

Consider the modified system

$$(3.1)\quad \begin{cases} \frac{\partial}{\partial t}u - \frac{\partial}{\partial z}(\mathrm{d}_1(z)\frac{\partial u}{\partial z}) - w_3\frac{\partial u}{\partial z} = f_1(\Pi(u,v)), \text{on } (0,z^*)\times(0,T_{\max}), \\ \frac{\partial}{\partial t}v - \frac{\partial}{\partial z}(\mathrm{d}_2(z)\frac{\partial v}{\partial z}) - w_3\frac{\partial v}{\partial z} = f_2(\Pi(u,v)), \text{on } (0,z^*)\times(0,T_{\max}), \\ u(0,z) = u_0(z), v(0,z) = v_0(z), \text{in } (0,z^*), \\ \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \text{for } z = 0, z^*. \end{cases}$$

Multiplying the first equation of (3.1) by u^- and integrating over $(0, z^*)$, we have

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u^-(t)\right\|^2_{L^2(0,z^*)} - \int_0^{z^*}\mathrm{d}_1(z)(\frac{\partial u^-}{\partial z})^2\mathrm{d}z + \int_0^{z^*} w_3\frac{\partial u^-}{\partial z}u^-\mathrm{d}z \geq 0.$$

This implies that

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u^-(t)\right\|^2_{L^2(0,z^*)} - \int_0^{z^*}\mathrm{d}_1(z)(\frac{\partial u^-}{\partial z})^2\mathrm{d}z \geq -c_1\int_0^{z^*}\left|\frac{\partial u^-}{\partial z}u^-\right|\mathrm{d}z,$$

where $c_1 = \max_{[0,z^*]}|w_3(z)|$.

By Young's inequality, we get

$$\int_0^{z^*}\left|\frac{\partial u^-}{\partial z}u^-\right|\mathrm{d}z \leq \int_0^{z^*}(\rho(\frac{\partial u^-}{\partial z})^2 + \frac{1}{\rho}(u^-)^2)\mathrm{d}z,$$

for some $\rho > 0$.

So

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u^-(t)\right\|^2_{L^2(0,z^*)} - \int_0^{z^*}\mathrm{d}_1(z)(\frac{\partial u^-}{\partial z})^2\mathrm{d}z \geq -c_1\rho\int_0^{z^*}(\frac{\partial u^-}{\partial z})^2\mathrm{d}z - c_1\frac{1}{\rho}\int_0^{z^*}(u^-)^2\mathrm{d}z,$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u^-(t)\right\|^2_{L^2(0,z^*)} + \int_0^{z^*}\mathrm{d}_1(z)(\frac{\partial u^-}{\partial z})^2\mathrm{d}z - c_1\rho\int_0^{z^*}(\frac{\partial u^-}{\partial z})^2\mathrm{d}z - c_1\frac{1}{\rho}\int_0^{z^*}(u^-)^2\mathrm{d}z \leq 0,$$

consequently

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u^-(t)\right\|^2_{L^2(0,z^*)} + \int_0^{z^*}(\mathrm{d}_1(z)-c_1\rho)(\frac{\partial u^-}{\partial z})^2\mathrm{d}z + (c_2-\frac{c_1}{\rho})\int_0^{z^*}(u^-)^2\mathrm{d}z \\ \leq c_2\int_0^{z^*}(u^-)^2\mathrm{d}z,$$

provided that c_2, ρ satisfy

$$\min_{[0,z^*]}\mathrm{d}_1(z)-c_1\rho>0, \quad c_2-\frac{c_1}{\rho}>0.$$

We deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u^-(t)\right\|^2_{L^2(0,z^*)} \leq c_2\int_0^{z^*}(u^-)^2\mathrm{d}z.$$

Since $\varphi_1^0 \geq 0$, Gronwall's inequality implies that $u^- = 0$.

Similarly $v^- = 0$.

Since on the positive cone, $f_i(u,v) = f_i \circ \Pi(u,v)$, we deduce that the solution (u,v) of (2.2) is positive. $\square$

4. GLOBAL EXISTENCE OF THE SOLUTION

In order to show that the solution is global in L^2, it is sufficient to prove that the L^2-norm of the solution cannot tend to infinity in finite time.

Let (u,v) be a solution of (2.2), then

PROPOSITION 5. *For all* $t \in (0, T_{\max})$, $\|u(t)\|_{L^2(0,z^*)} + \|v(t)\|_{L^2(0,z^*)} \leq c\exp(t)$.

Proof. From (1.3),there exist $L>0$ such that $f_k(u,v) \leq L(u+v+1)$, $k=1,2$.

Let $\mathrm{Q}_t = [0,t]\times(0,z^*)$, $t\in(0,T_{\max})$, and $(\overline{u},\overline{v})$ be the solution of

$$\begin{cases} \frac{\partial}{\partial t}\overline{u} - \frac{\partial}{\partial z}(\mathrm{d}_1(z)\frac{\partial\overline{u}}{\partial z}) - w_3\frac{\partial\overline{u}}{\partial z} = L(u+v+1) & \text{in } (0,z^*)\times(0,t), \\ \frac{\partial}{\partial t}\overline{v} - \frac{\partial}{\partial z}(\mathrm{d}_2(z)\frac{\partial\overline{v}}{\partial z}) - w_3\frac{\partial\overline{v}}{\partial z} = L(u+v+1) & \text{in } (0,z^*)\times(0,t), \\ \overline{u}(0,z) = \overline{v}_0, \overline{v}(0,z) = \overline{v}_0, & \\ \frac{\partial\overline{u}}{\partial z} = \frac{\partial\overline{v}}{\partial z} = 0, & \text{for } z = 0, z^* \end{cases}$$

with $\overline{v}_0 \geq u_0, \overline{v}_0 \geq v_0$.

Since $f_1(u,v) \le L(u+v+1)$, $f_2(u,v) \le L(u+v+1)$, then $(w_1,w_2)=(\overline{u},\overline{v})-(u,v)$ satisfies

$$\begin{cases} \frac{\partial}{\partial t}w_1 - \frac{\partial}{\partial z}(\mathrm{d}_1(z)\frac{\partial w_1}{\partial z}) - w_3\frac{\partial w_1}{\partial z} \ge 0 & \text{in } (0,z^*)\times(0,t) \\ \frac{\partial}{\partial t}w_2 - \frac{\partial}{\partial z}(\mathrm{d}_2(z)\frac{\partial w_2}{\partial z}) - w_3\frac{\partial w_2}{\partial z} \ge 0 & \text{in } (0,z^*)\times(0,t) \\ \qquad w_1(0,z)\ge 0, w_2(0,z)\ge 0 & \\ \qquad \frac{\partial w_1}{\partial z} = \frac{\partial w_2}{\partial z} = 0, & \text{for } z=0,z^* \end{cases}$$

We show that (w_1,w_2) is positive by the same arguments given to prove positiveness.

From the Theorem of page 143 in [27], we have the estimates

$$\begin{aligned} \forall t \quad &\in \quad (0,T_{\max}), \|\overline{u}(t)\|_{L^2(0,z^*)} \le c + \int_0^t \|(u+v)(s)\|_{L^2(0,z^*)}\,\mathrm{d}s, \\ \forall t \quad &\in \quad (0,T_{\max}), \|\overline{v}(t)\|_{L^2(0,z^*)} \le c + \int_0^t \|(u+v)(s)\|_{L^2(0,z^*)}\,\mathrm{d}s, \end{aligned}$$

hence for all $t\in(0,T_{\max})$,

$$\begin{aligned} \|u(t)\|_{L^2(0,z^*)} + \|v(t)\|_{L^2(0,z^*)} \quad &\le \quad \|\overline{u}(t)\|_{L^2(0,z^*)} + \|\overline{v}(t)\|_{L^2(0,z^*)} \le \\ & c + \int_0^t (\|u(s)\|_{L^2(0,z^*)} + \|v(s)\|_{L^2(0,z^*)}). \end{aligned}$$

Gronwall's inequality implies that for all $t\in(0,T_{\max})$,

$$\|u(t)\|_{L^2(0,z^*)} + \|v(t)\|_{L^2(0,z^*)} \le c\exp(t). \quad \square$$

5. SOLVABILITY OF PROBLEM (2.1) and (1.1)

Let (x_0,y_0) be in the support of (u_0^i,v_0^i), then

PROPOSITION 6. *Suppose $(\varphi_0^1,\varphi_0^2)\in D(A^c)$. The system (2.1) has a unique positive solution (φ_1,φ_2) such that for each $1\le i\le n$*

$$\left(\varphi_1^i,\varphi_2^i\right) \in C([0,T_o],H^2(z_{i-1},z_i)) \cap C^1(]0,T_o],L^2(z_{i-1},z_i)).$$

Proof. Let (φ_1,φ_2) the solution of the system

$$(5.1) \quad \begin{cases} \frac{\partial}{\partial t}\varphi_1 - \frac{\partial}{\partial z}(\mathrm{d}_1(z)\frac{\partial \varphi_1}{\partial z}) - w_3\frac{\partial \varphi_1}{\partial z} = f_1(\varphi_1,\varphi_2) \text{ in } (0,z^*)\times(0,T_0], \\ \frac{\partial}{\partial t}\varphi_2 - \frac{\partial}{\partial z}(\mathrm{d}_2(z)\frac{\partial \varphi_2}{\partial z}) - w_3\frac{\partial \varphi_2}{\partial z} = f_2(\varphi_1,\varphi_2), \text{in } (0,z^*)\times(0,T_0], \\ \qquad \varphi_1(0,z)=\varphi_1^0(z), \varphi^2(0,z)=\varphi_0^2(z), \\ \qquad \frac{\partial \varphi_1}{\partial z} = \frac{\partial \varphi_2}{\partial z} = 0, \text{for } z=0,z^*. \end{cases}$$

Let $\psi \in C_c^\infty(0, z^*)$, we have

$$\begin{cases} \int_0^{z^*} (\frac{\partial}{\partial t}\varphi_1 - \frac{\partial}{\partial z}(\mathrm{d}_1(z)\frac{\partial \varphi_1}{\partial z}) - w_3(z)\frac{\partial \varphi_1}{\partial z})\psi \mathrm{d}z = \int_0^{z^*} f_1(\varphi_1, \varphi_2)\psi \mathrm{d}z, \\ \int_0^{z^*} (\frac{\partial \varphi_2}{\partial t} - \frac{\partial}{\partial z}(\mathrm{d}_2(z)\frac{\partial \varphi_2}{\partial z}) - w_3\frac{\partial \varphi_2}{\partial z})\psi \mathrm{d}z = \int_0^{z^*} f_2(\varphi_1, \varphi_2)\psi \mathrm{d}z. \end{cases}$$

So

$$\begin{aligned} \sum_{i=1}^{i=n} \int_{z_{i-1}}^{z_i} (\frac{\partial}{\partial t}\varphi_1 - \frac{\partial}{\partial z}(\mathrm{d}_1(z)\frac{\partial \varphi_1}{\partial z}) - w_3(z)\frac{\partial \varphi_1}{\partial z})\psi &= \sum_{i=1}^{i=n} \int_{z_{i-1}}^{z_i} f_1(\varphi_1, \varphi_2)\psi, \\ \sum_{i=1}^{i=n} \int_{z_{i-1}}^{z_i} (\frac{\partial \varphi_2}{\partial t} - \frac{\partial}{\partial z}(\mathrm{d}_2(z)\frac{\partial \varphi_2}{\partial z}) - w_3\frac{\partial \varphi_2}{\partial z})\psi &= \sum_{i=1}^{i=n} \int_{z_{i-1}}^{z_i} f_2(\varphi_1, \varphi_2)\psi. \end{aligned}$$

For $\psi \in C_c^\infty(z_{i-1}, z_i), 1 \le i \le n$, we find that

$$\begin{aligned} \int_{z_{i-1}}^{z_i} (\frac{\partial}{\partial t}\varphi_1^i - \mathrm{d}_1^i\frac{\partial^2 \varphi_1^i}{\partial z^2} - w_3\frac{\partial \varphi_1^i}{\partial z})\psi \mathrm{d}z &= \int_{z_{i-1}}^{z_i} f_1(\varphi_1^i, \varphi_2^i)\psi \mathrm{d}z, 1 \le i \le n, \\ \int_{z_{i-1}}^{z_i} (\frac{\partial \varphi_2^i}{\partial t} - \mathrm{d}_2^i\frac{\partial^2 \varphi_2^i}{\partial z^2} - w_3\frac{\partial \varphi_2^i}{\partial z})\psi \mathrm{d}z &= \int_{z_{i-1}}^{z_i} f_2(\varphi_1^i, \varphi_2^i)\psi \mathrm{d}z, 1 \le i \le n, \end{aligned}$$

for $1 \le i \le n$, and in the distributional sense, we obtain

$$\begin{cases} \frac{\partial}{\partial t}\varphi_1^i - \mathrm{d}_1^i\frac{\partial^2 \varphi_1^i}{\partial z^2} - w_3(z)\frac{\partial \varphi_1^i}{\partial z} = f_1(\varphi_1^i, \varphi_2^i), \\ \frac{\partial}{\partial t}\varphi_2^i - \mathrm{d}_2^i\frac{\partial^2 \varphi_2^i}{\partial z^2} - w_3\frac{\partial \varphi_2^i}{\partial z} = f_2(\varphi_1^i, \varphi_2^i). \end{cases}$$

The transmission conditions between layers give

$$\frac{\partial \varphi_k^1}{\partial z}(t, 0) = \frac{\partial \varphi_k^n}{\partial z}(t, z^*) = 0, k = 1, 2.$$

Since (φ_1, φ_2) is continuous then

$$\varphi_k^i(t, z_i) = \varphi_k^{i+1}(t, z_i), k = 1, 2, \qquad 1 \le i \le n-1.$$

For $\psi \in H^1(0, z^*)$, we have

$$\begin{cases} \int_0^{z_i} (\frac{\partial}{\partial t}\varphi_1 - \frac{\partial}{\partial z}(\mathrm{d}_1(z)\frac{\partial \varphi_1}{\partial z}) - w_3(z)\frac{\partial \varphi_1}{\partial z})\psi \mathrm{d}z = \int_0^{z_i} f_1(\varphi_1, \varphi_2)\psi \mathrm{d}z, \\ \int_{z_i}^{z^*} (\frac{\partial \varphi_1}{\partial t} - \frac{\partial}{\partial z}(\mathrm{d}_2(z)\frac{\partial \varphi_1}{\partial z}) - w_3\frac{\partial \varphi_1}{\partial z})\psi \mathrm{d}z = \int_{z_i}^{z^*} f_1(\varphi_1, \varphi_2)\psi \mathrm{d}z. \end{cases}$$

The integration by part yields

$$\begin{cases} \int_0^{z_i} (\frac{\partial \varphi_1}{\partial t}\psi + \int_0^{z_i} \mathrm{d}_1(z)\frac{\partial \varphi_1}{\partial z}\frac{\partial \psi}{\partial z} - \int_0^{z_i} w_3(z)\frac{\partial \varphi_1}{\partial z}\psi \mathrm{d}z - \mathrm{d}_1^i\frac{\partial \varphi_1^i}{\partial z}(z_i)\psi(z_i) \\ \qquad = \int_0^{z_i} f_1(\varphi_1, \varphi_2)\psi, \\ \int_{z_i}^{z^*} (\frac{\partial \varphi_1}{\partial t}\psi + \int_{z_i}^{z^*} \mathrm{d}_1(z)\frac{\partial \varphi_1}{\partial z}\frac{\partial \psi}{\partial z} - \int_{z_i}^{z^*} w_3\frac{\partial \varphi_1}{\partial z}\psi \mathrm{d}z + \mathrm{d}_1^{i+1}\frac{\partial \varphi_1^{i+1}}{\partial z}(z_i)\psi(z_i) \\ \qquad = \int_{z_i}^{z^*} f_1(\varphi_1, \varphi_2)\psi. \end{cases}$$

Summing all the inequalities, we find

$$\int_0^{z^*}(\tfrac{\partial\varphi_1}{\partial t}\psi\mathrm{d}z+\int_0^{z^*}(\mathrm{d}_1(z)\tfrac{\partial\varphi_1}{\partial z}\tfrac{\partial\psi}{\partial z}-w_3(z)\tfrac{\partial\varphi_1}{\partial z}\psi)\mathrm{d}z-\mathrm{d}_1^i\tfrac{\partial\varphi_1^i}{\partial z}(z_i)\psi(z_i)$$
$$+\mathrm{d}_1^{i+1}\tfrac{\partial\varphi_1^{i+1}}{\partial z}(z_i)\psi(z_i)=\int_0^{z^*}f_1(\varphi_1,\varphi_2)\psi\mathrm{d}z.$$

Integration by part gives

$$\int_0^{z^*}(\tfrac{\partial\varphi_1}{\partial t}-\tfrac{\partial}{\partial z}(\mathrm{d}_1(z)\tfrac{\partial\varphi_1}{\partial z})-w_3(z)\tfrac{\partial\varphi_1}{\partial z})\psi\mathrm{d}z-\mathrm{d}_1^i\tfrac{\partial\varphi_1^i}{\partial z}(z_i)\psi(z_i)$$
$$+\mathrm{d}_1^{i+1}\tfrac{\partial\varphi_1^{i+1}}{\partial z}(z_i)\psi(z_i)=\int_0^{z^*}f_1(\varphi_1,\varphi_2)\psi\mathrm{d}z.$$

Since φ_1 is a solution of (5.1), we deduce that

$$-\mathrm{d}_1^i\frac{\partial\varphi_1^i}{\partial z}(z_i)\psi(z_i)+\mathrm{d}_1^{i+1}\frac{\partial\varphi_1^{i+1}}{\partial z}(z_i)\psi(z_i)=0,\forall\psi\in H^1(0,z^*).$$

So

$$\mathrm{d}_1^{i+1}\frac{\partial\varphi_1^{i+1}}{\partial z}(z_i)=\mathrm{d}_1^i\frac{\partial\varphi_1^i}{\partial z}(z_i),\quad 1\le i\le n-1.$$

We proceed similarly for φ_2. $\square$

Proposition 7. *For all (x_0,y_0) in the support of (u_0^i,v_0^i), $1\le i\le n$, the system (1.1) has a unique positive solution (u^i,v^i), defined on the maximal time of observation $[0,T_o]$*

Proof. It suffices to see that for $t\in[0,T_o]$, $(x^i,y^i)\in D$ and $z\in[z_{i-1},z_i,]$,

$$(u^i(t,x^i,y^i,z),v^i(t,x^i,y^i,z))=(\varphi_1(t,\Psi^i_{-t}(x^i,y^i),z),\varphi_2(t,\Psi^i_{-t}(x^i,y^i),z)),$$
$$1\le i\le n.$$

Define a solution to system (1.1) by writing

$$u=\begin{cases} u^1 \text{ in } (z_0,z_1),\\ u^2 \text{ in } (z_1,z_2),\\ \dots\\ u^n \text{ in } (z_{n-1},z_n),\end{cases}$$

and

$$v=\begin{cases} v^1 \text{ in } (z_0,z_1),\\ v^2 \text{ in } (z_1,z_2),\\ \dots\\ v^n \text{ in } (z_{n-1},z_n).\end{cases}\quad\square$$

6. CONCLUDING REMARKS

The role of fishery on the economy of some countries is crucial. It is a source of foreign revenue and protein. The employment in fisheries is considerable. A few thousands people are employed in anchovy industry in Peru [7]. Thus the impact of fishery on political stability is dramatic.

For many countries, the ocean is an important source of food. That makes the study of marine ecosystems a question of crucial importance. Hence, mathematical modelling of marine ecology plays an essential role. The distribution of fish population is economic determinant for fishery management. Pytoplankton with zooplankton distributions occupy a central position in the variation of fish population. They are considered as marker of the presence of many marine species, and modulate the movement of fish in marine systems. Since phytoplankton concentration is limited by the availability of light and nutrients, photosynthesis provided by phytoplankton does not occur below thermocline zone.

Fish species dynamics, distribution and their abundance are of great importance. We develop a complex mathematical model describing the dynamics of fish population. Complexity can arise from the variability of biological and physical parameters on water column. The model includes hydrodynamics parameters and stage structure, it shows the response of the species through movement and biological reactions to an aquatic environment.

We assume that the column water is divided into several layers. Under reasonably restrictive assumptions, we establish existence and positiveness of solutions. It is difficult to solve empirically the system because spatial and temporal data are generally not well known. This is why it is necessary to carry on the modelling effort to study complex models. The model is applicable to a broad range of vertically stratified habitat.

Similar methods can be used for populations spreading in a region fragmented in patches with different life conditions.

Recently, in [25], the authors show that the ocean can be divided in hydrodynamic regions delimited by intense oceanic structures such as jets, fronts and eddies. The oceanic frontiers constitute a barrier for many fish species. Thus, on the lateral boundary, we can assume that the currents are inward, and using Fishera function see [23]. We can show that the boundary condition must not be given on lateral boundary, this point will be examined in a forthcoming paper.

There is a wealth of directions to be explored and improvements to be made. For instance, it is more realistic to include fishing effort in the model. Since the phytoplankton serves as a favorite food for the anchovy, it would

be interesting to investigate the system with another trophic level such phytoplankton.

The global solution is obtained in L^2 space. Even in one dimension, the Lp theory $(p \geq 2)$ for parabolic problem with piecewise constant coefficients is not completely understood (see for instance [12]). For these kinds of problems, Lp estimates are established only for the whole and half space see [9] and [13]. At our knowledge, the case of bounded domain and Neumann boundary conditions do not seem to have been systematically treated, and consequently, the existence of global solution in L^∞ remains open for our problem.

6.1. APPENDIX A

Proof of Lemma 1. It is clear that A^c is closed. In fact, let u_n be a sequence in X such that $u_n \to u$ and $A^c\ u_n \to v$ in X. Then the sequence converges in Y. Since A is the generator of quasi contractive semigroup $S(t)$ on Y then $u \in D(A)$ and $Au = v$. But since $u, v \in X$, then $u \in D(A^c)$. This implies that A^c is closed. Lemma 5.1.6 in [19] shows that A^c is densely defined on X. Since A is the generator of quasi contractive semigroup, then there exists $\omega_0 > 0$ such that $\lambda I + A$ is invertible for $\lambda > \omega_0$, then $\lambda I + A^c$ is injective. For $f \in X$ there exists $u \in D(A)$ such that $(\lambda I + A)\ u = f$. By elliptic regularity, $u \in X$. Hence $\lambda I + A^c$ is invertible by closed graph theorem. Using lemma 5.1.4 in [19], we derive that $\|S(t)u\|_X \leq e^{wt}\|u\|_X$ for all $u \in X$. Now the relation $R(\lambda, A)\, u = \int\limits_0^{+\infty} e^{-\lambda t} S(t) u dt$, gives that the nth derivative of the analytical function $\lambda \to R(\lambda, A^c)$, verifies that $\left\| R(\lambda, A^c)^{(n)}\, u \right\|_X \leq \frac{Mn!}{(\lambda - \omega_0)^{n+1}} \|u\|_X$ for all $u \in X$, see lemma 5.1.4 in [19]. The rest of the proof is a consequence of the theorem 2.7.2 in [19].

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