UNIFORM TREATMENT OF JENSEN TYPE INEQUALITIES II

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The main purpose of this paper is to give the uniform treatment of the Jensen type inequalities using some new Green functions. Using the same method, we also give adequate results for the converses of the Jensen inequality. As a consequence, the results concerning the Hermite-Hadamard inequalities are also presented.

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1. INTRODUCTION AND PRELIMINARY RESULTS

Consider the function $G_1[\alpha,\beta]\times[\alpha,\beta]\to\mathbb{R}$, the Green function of the boundary value problem

$$z''=0, z(\alpha)=z'(\beta)=0,$$

defined by

$$G_1(t,s)=\begin{cases}\alpha-s, & \text{for } \alpha\le s\le t,\\ \alpha-t, & \text{for } t\le s\le\beta.\end{cases} \tag{1.1}$$

The function G_1 is convex and continuous with respect to both s and t.

Similarly, we introduce some new types of Green functions, and define the functions $G_p:[\alpha,\beta]\times[\alpha,\beta]\to\mathbb{R}$, $(p=2,3,4)$ as follows:

$$G_2(t,s)=\begin{cases}t-\beta, & \text{for } \alpha\le s\le t,\\ s-\beta, & \text{for } t\le s\le\beta.\end{cases} \tag{1.2}$$

$$G_3(t,s)=\begin{cases}t-\alpha, & \text{for } \alpha\le s\le t,\\ s-\alpha, & \text{for } t\le s\le\beta.\end{cases} \tag{1.3}$$

$$G_4(t,s)=\begin{cases}\beta-s, & \text{for } \alpha\le s\le t,\\ \beta-t, & \text{for } t\le s\le\beta.\end{cases} \tag{1.4}$$

All these functions are convex and continuous with respect to both s and t, and the following Lemma holds.

LEMMA 1.1. *For every function $\varphi : [\alpha, \beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha, \beta])$, the following identities hold:*

$$\varphi(x) = \varphi(\alpha) + (x - \alpha)\varphi'(\beta) + \int_\alpha^\beta G_1(x, s)\varphi''(s)\mathrm{d}s, \tag{1.5}$$

$$\varphi(x) = \varphi(\beta) + (x - \beta)\varphi'(\alpha) + \int_\alpha^\beta G_2(x, s)\varphi''(s)\mathrm{d}s, \tag{1.6}$$

$$\varphi(x) = \varphi(\beta) - (\beta - \alpha)\varphi'(\beta) + (x - \alpha)\varphi'(\alpha) + \int_\alpha^\beta G_3(x, s)\varphi''(s)\mathrm{d}s, \tag{1.7}$$

$$\varphi(x) = \varphi(\alpha) + (\beta - \alpha)\varphi'(\alpha) - (\beta - x)\varphi'(\beta) + \int_\alpha^\beta G_4(x, s)\varphi''(s)\mathrm{d}s, \tag{1.8}$$

where the functions G_p $(p = 1, 2, 3, 4)$ are defined as above in (1.1)–(1.4).

Proof. Integrating by parts gives us

$$\begin{aligned}
\int_\alpha^\beta G_1(x, s)\varphi''(s)\mathrm{d}s &= \int_\alpha^x G_1(x, s)\varphi''(s)\mathrm{d}s + \int_x^\beta G_1(x, s)\varphi''(s)\mathrm{d}s \\
&= \int_\alpha^x (\alpha - s)\varphi''(s)\mathrm{d}s + \int_x^\beta (\alpha - x)\varphi''(s)\mathrm{d}s \\
&= (\alpha - s) \cdot \varphi'(s)|_\alpha^x + \varphi(s)|_\alpha^x + (\alpha - x)\varphi'(s)|_x^\beta \\
&= \varphi(x) - \varphi(\alpha) + (\alpha - x)\varphi'(\beta).
\end{aligned}$$

The other three identities are proved analogously. □

Remark 1.1. The result (1.5) given in the previous Lemma represents a special case of the representation of the function φ using the so-called 'two-point right focal' interpolating polynomial in case when $n = 2$ and $p = 0$ (see [1]).

Using the results from the previous lemma, in this paper we derive some interesting results concerning the Jensen type inequalities, giving the conditions on the real Stieltjes measure $d\lambda$, such that $\lambda(a) \neq \lambda(b)$, that for continuous convex function φ the Jensen inequality or the converse of the Jensen inequality holds, allowing that the measure can also be negative, but using the condition on previously defined Green functions.

The results given here represent the continuation of the investigation already published in this journal in [15].

2. MAIN RESULTS

In order to simplify the notation, for the functions g and λ we denote

$$\overline{g} = \frac{\int_a^b g(x) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}.$$

THEOREM 2.1. *Let $g : [a, b] \to \mathbb{R}$ be continuous function and $[\alpha, \beta]$ an interval such that the image of g is a subset of $[\alpha, \beta]$. Let $\lambda : [a, b] \to \mathbb{R}$ be continuous function or the function of bounded variation, such that $\lambda(a) \neq \lambda(b)$ and $\overline{g} \in [\alpha, \beta]$. Let the functions $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ $(p = 1, 2, 3, 4)$ be as defined in (1.1)–(1.4). Then the following two statements are equivalent:*

(1) *For every continuous convex function $\varphi : [\alpha, \beta] \to \mathbb{R}$*

$$\varphi\left(\overline{g}\right) \leq \frac{\int_a^b \varphi\left(g(x)\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \tag{2.1}$$

holds.

(2) *For all $s \in [\alpha, \beta]$*

$$G_p\left(\overline{g}, s\right) \leq \frac{\int_a^b G_p\left(g(x), s\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \tag{2.2}$$

holds.

Furthermore, the statements (1) and (2) are also equivalent if we reverse the sign of inequality in both (2.1) and (2.2).

Proof. Consider the Green function G_1 defined by (1.1).

(1) $\Rightarrow$ (2): Let (1) hold. As the function $G_1(\cdot, s)$ $(s \in [\alpha, \beta])$ is also continuous and convex, it follows that also for this function (2.1) holds, *i.e.* it holds (2.2) for $p = 1$.

(2) $\Rightarrow$ (1): Let (2) hold. As we already know from Lemma 1.1 that we can represent every function $\varphi : [\alpha, \beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha, \beta])$, in the form (1.5), where the function G_1 is defined in (1.1), by some calculation we can easily get that

$$\begin{aligned} &\varphi\left(\overline{g}\right) - \frac{\int_a^b \varphi\left(g(x)\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \\ &= \int_\alpha^\beta \left[G_1\left(\overline{g}, s\right) - \frac{\int_a^b G_1\left(g(x), s\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right] \varphi''(s) \mathrm{d}s. \end{aligned} \tag{2.3}$$

If the function φ is also convex, then $\varphi''(s) \geq 0$ for all $s \in [\alpha, \beta]$. So, if for every $s \in [\alpha, \beta]$ holds (2.2) for $p = 1$, then it follows that for every convex function $\varphi : [\alpha, \beta] \to \mathbb{R}$, with $\varphi \in C^2([\alpha, \beta])$, inequality (2.1) holds.

Additionally, note that it is not necessary to demand the existence of the second derivative of the function φ (see [16, p. 172]). The differentiability condition can be directly eliminated by using the fact that it is possible to approximate uniformly a continuous convex function by convex polynomials.

The last part of our theorem can be proved analogously.

Cases when $p = 2, 3, 4$ are treated analogously. □

Remark 2.1. Let the conditions from the previous theorem hold. Then the following two statements are equivalent:

(1′) For every continuous concave function $\varphi : [\alpha, \beta] \to \mathbb{R}$ the reverse inequality in (2.1) holds.

(2′) For all $s \in [\alpha, \beta]$ inequality (2.2) holds.

Moreover, the statements (1′) and (2′) are also equivalent if we change the sign of inequality in both statements (1′) and (2′).

Remark 2.2. If the function λ in Theorem 2.1 is increasing and bounded, with $\lambda(a) \neq \lambda(b)$, then inequality (2.1) becomes Jensen's integral inequality.

In the following two theorems, we give the conditions on the real Stieltjes measure $\mathrm{d}\lambda$, with $\lambda(a) \neq \lambda(b)$, so that for the functions of the class C^2, the Lagrange and Cauchy type theorems hold.

THEOREM 2.2. *Let $g : [a, b] \to \mathbb{R}$ be continuous function and $\varphi : [\alpha, \beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha, \beta])$, where the image of g is a subset of $[\alpha, \beta]$. Let $\lambda : [a, b] \to \mathbb{R}$ be continuous function or the function of bounded variation, such that $\lambda(a) \neq \lambda(b)$ and $\overline{g} \in [\alpha, \beta]$. Let the functions $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ $(p = 1, 2, 3, 4)$ be as defined in* (1.1)–(1.4). *If for all $s \in [\alpha, \beta]$ the inequality* (2.2) *holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in* (2.2) *holds, for any $p \in \{1, 2, 3, 4\}$, then there exists $\xi \in [\alpha, \beta]$ such that*

$$\varphi\left(\overline{g}\right) - \frac{\int_a^b \varphi\left(g(x)\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} = \frac{1}{2}\varphi''(\xi)\left[\left(\overline{g}\right)^2 - \frac{\int_a^b \left(g(x)\right)^2 \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right]. \tag{2.4}$$

Proof. Consider the Green function G_1 defined by (1.1).

Following the assumptions of our theorem, we have that the function φ'' is continuous and

$$G_1\left(\overline{g}, s\right) - \frac{\int_a^b G_1\left(g(x), s\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}$$

doesn't change its positivity on $[\alpha, \beta]$. For our function φ equality (2.3) holds, and now applying the integral mean-value theorem we get that there exists some $\xi \in [\alpha, \beta]$ such that

$$\varphi(\overline{g}) - \frac{\int_a^b \varphi(g(x))\, \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} = \varphi''(\xi) \int_\alpha^\beta \left[G_1(\overline{g}, s) - \frac{\int_a^b G_1(g(x), s)\, \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \right] \mathrm{d}s. \tag{2.5}$$

It can be easily checked that for the function G_1 it holds

$$\begin{aligned} \int_\alpha^\beta G_1(t, s)\mathrm{d}s &= \int_\alpha^t (\alpha - s)\mathrm{d}s + \int_t^\beta (\alpha - t)\mathrm{d}s \\ &= \alpha \int_\alpha^t \mathrm{d}s - \int_\alpha^t s\mathrm{d}s + (\alpha - t) \int_t^\beta \mathrm{d}s \\ &= \frac{1}{2}(t - \alpha)(t + \alpha - 2\beta). \end{aligned} \tag{2.6}$$

Calculating the integral on the right side in (2.5) we get

$$\begin{aligned} &\varphi(\overline{g}) - \frac{\int_a^b \varphi(g(x))\, \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \\ &= \varphi''(\xi) \left[\int_\alpha^\beta G_1(\overline{g}, s)\, \mathrm{d}s - \frac{1}{\int_a^b \mathrm{d}\lambda(x)} \int_a^b \int_\alpha^\beta G_1(g(x), s)\, \mathrm{d}s\, \mathrm{d}\lambda(x) \right] \\ &= \varphi''(\xi) \left[\frac{1}{2}(\overline{g} - \alpha)(\overline{g} + \alpha - 2\beta) - \frac{1}{\int_a^b \mathrm{d}\lambda(x)} \int_a^b \frac{1}{2}(g(x) - \alpha)(g(x) + \alpha - 2\beta)\, \mathrm{d}\lambda(x) \right] \\ &= \frac{1}{2}\varphi''(\xi) \left[\overline{g}^{\,2} - \frac{\int_a^b (g(x))^2\, \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \right], \end{aligned}$$

which concludes our proof.

Cases when $p = 2, 3, 4$ are proved analogously. □

THEOREM 2.3. *Let $g : [a, b] \to \mathbb{R}$ be continuous function and $\varphi, \psi : [\alpha, \beta] \to \mathbb{R}$, $\varphi, \psi \in C^2([\alpha, \beta])$, where the image of g is a subset of $[\alpha, \beta]$. Let $\lambda : [a, b] \to \mathbb{R}$ be continuous function or the function of bounded variation, such that $\lambda(a) \neq \lambda(b)$ and $\overline{g} \in [\alpha, \beta]$. Let the functions $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ ($p = 1, 2, 3, 4$) be as defined in (1.1)–(1.4). If for all $s \in [\alpha, \beta]$ the inequality (2.2) holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in (2.2) holds, for any $p \in \{1, 2, 3, 4\}$, then there exists $\xi \in [\alpha, \beta]$ such that*

$$\frac{\varphi\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) - \frac{\int_a^b \varphi(g(x))\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}}{\psi\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) - \frac{\int_a^b \psi(g(x))\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}} = \frac{\varphi''(\xi)}{\psi''(\xi)}, \tag{2.7}$$

provided that the denominator of the left-hand side is nonzero.

Proof. Consider the function χ defined by

$$\chi(t) = \left[\psi\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) - \frac{\int_a^b \psi\left(g(x)\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right]\cdot\varphi(t)$$
$$- \left[\varphi\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) - \frac{\int_a^b \varphi\left(g(x)\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right]\cdot\psi(t).$$

Function χ is the linear combination of functions φ and ψ, so it is also defined on the segment $[\alpha, \beta]$, it is continuous and χ'' is continuous on $[\alpha, \beta]$. Now we can apply the previous theorem on function χ and it follows that there exists some $\xi \in [\alpha, \beta]$ such that the following is valid

$$\chi\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) - \frac{\int_a^b \chi\left(g(x)\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}$$
$$= \frac{1}{2}\chi''(\xi)\left[\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)^2 - \frac{\int_a^b \left(g(x)\right)^2\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right]. \tag{2.8}$$

After a short calculation we get that the left-hand side of this equation equals to zero. The term in the square brackets on the right-hand side of (2.8) is different from 0, because otherwise, from the Theorem 2.2 applied on the function ψ, we would have that the denominator on the left-hand side of (2.7) equals to 0, which contradicts our assumption from our theorem (Theorem 2.3). It follows that

$$\chi''(\xi) = 0,$$

and the assertion of our theorem follows directly. □

Remark 2.3. Note that setting the function ψ as $\psi(x) = x^2$ in Theorem 2.3, we get the statement of Theorem 2.2.

Similarly as in the paper [15], the previous two theorems allow us to easily prove some further results where we give explicit conditions on the functions g and λ, for (2.4) and (2.7) to hold.

Corollary 2.1. *Let $g : [a, b] \to \mathbb{R}$ be continuous function, and $\varphi, \psi : [\alpha, \beta] \to \mathbb{R}$, where the image of g is a subset of $[\alpha, \beta]$. Let $a = y_0 < y_1 < ... < y_k < ... < y_{n-1} < y_n = b$. Let $\lambda : [a, b] \to \mathbb{R}$ be continuous function or the function of bounded variation, such that $\lambda(a) \le \lambda(x_1) \le \lambda(y_1) \le \lambda(x_2) \le ... \le \lambda(y_{n-1}) \le \lambda(x_n) \le \lambda(b)$ for all $x_k \in (y_{k-1}, y_k)$, and $\lambda(b) > \lambda(a)$ and let g be monotonic function in each of the n intervals (y_{k-1}, y_k).*

(i) *If $\varphi \in C^2([\alpha, \beta])$, then there exists $\xi \in (\alpha, \beta)$ such that (2.4) holds.*

(ii) *If $\varphi, \psi \in C^2([\alpha, \beta])$, then there exists $\xi \in (\alpha, \beta)$ such that (2.7) holds.*

Proof. Consider any of the functions $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ $(p = 1, 2, 3, 4)$ defined in (1.1)–(1.4). They are all continuous and convex, so by the Jensen-Boas inequality (see [3] or [16]) we have that under the conditions of this corollary, the inequality (2.2) holds for all $s \in [\alpha, \beta]$. Now the statement (i) (respectively (ii)) of this Corollary follows directly from Theorem 2.2 (respectively Theorem 2.3). □

COROLLARY 2.2. *Let g be continuous and increasing function and $\varphi, \psi : [\alpha, \beta] \to \mathbb{R}$, where the image of g is a subset of $[\alpha, \beta]$. Let λ be continuous function or the function of bounded variation, and $\lambda(b) > \lambda(a)$. Let for all $x \in [a, b]$*

$$\int_a^x (g(x) - g(t))\,\mathrm{d}\lambda(t) \geq 0 \text{ and } \int_x^b (g(x) - g(t))\,\mathrm{d}\lambda(t) \leq 0 \tag{2.9}$$

hold.

(i) *If $\varphi \in C^2([\alpha, \beta])$, then there exists $\xi \in (\alpha, \beta)$ such that (2.4) holds.*
(ii) *If $\varphi, \psi \in C^2([\alpha, \beta])$, then there exists $\xi \in (\alpha, \beta)$ such that (2.7) holds.*

Proof. Consider any of the functions $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ $(p = 1, 2, 3, 4)$ defined in (1.1)–(1.4). They are all continuous and convex, so by the Jensen-Brunk inequality (see [4, 16]) we have that under the conditions of this corollary the inequality (2.2) holds for all $s \in [\alpha, \beta]$. The statement (i) (respectively (ii)) of this Corollary follows directly from Theorem 2.2 (respectively Theorem 2.3). □

Remark 2.4. As the similar result is also valid for decreasing function g, assuming that the function g is monotonic, we can substitute the condition (2.9) with the following condition

$$0 \leq \int_a^x |g(x) - g(t)|\,\mathrm{d}\lambda(t) \leq \int_a^b |g(x) - g(t)|\,\mathrm{d}\lambda(t). \tag{2.10}$$

Analogous results can be derived using the Jensen or Jensen-Steffensen inequality or the generalization of the Jensen-Steffensen inequality given in [11] (see also [16, p. 62]), or using the reverse Jensen or reverse Jensen-Steffensen inequality or the reverse Jensen-Brunk or the reverse Jensen-Boas inequality, all given in [12] (see also [16, p. 86]).

3. DISCRETE CASE

In this section, we give the results for discrete case. The proofs are similar to those in the integral case given in Section 2, so we give these results here without the proofs.

In discrete Jensen's inequality we have that p_i $(i = 1, ..., n)$ are positive real numbers. Here we give the generalization of that result, allowing that p_i can also be negative, with the sum different from 0, but with a supplementary demand on p_i, x_i given using the Green functions $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ $(p = 1, 2, 3, 4)$ defined in (1.1)–(1.4).

For p_i, x_i $(i = 1, ..., n)$ we use the common notation: $P_k = \sum_{i=1}^{k} p_i$, $\overline{P_k} = P_n - P_{k-1}$ $(k = 1, ..., n)$, and $\overline{x} = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i$.

Using the representation of any function $f : [\alpha, \beta] \to \mathbb{R}$, $f \in C^2([\alpha, \beta])$, in the form as given in Lemma 1.1, where the functions G_p $(p = 1, 2, 3, 4)$ are defined in (1.1)–(1.4), by some calculation we get that the following holds:

$$f(\overline{x}) - \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) = \int_{\alpha}^{\beta} \left(G_p(\overline{x}, s) - \frac{1}{P_n} \sum_{i=1}^{n} p_i G_p(x_i, s) \right) f''(s) \mathrm{d}s.$$

We have the following result.

Theorem 3.1. *Let $x_i \in [a, b] \subseteq [\alpha, \beta]$, $p_i \in \mathbb{R}$ $(i = 1, ..., n)$, be such that $P_n \neq 0$ and $\overline{x} \in [\alpha, \beta]$. Let the functions $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ $(p = 1, 2, 3, 4)$ be as defined in* (1.1)–(1.4). *Then the following two statements are equivalent:*

(1) *For every continuous convex function $f : [\alpha, \beta] \to \mathbb{R}$*

$$f(\overline{x}) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \tag{3.1}$$

holds.

(2) *For all $s \in [\alpha, \beta]$*

$$G_p(\overline{x}, s) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i G_p(x_i, s) \tag{3.2}$$

holds.

Moreover, the statements (1) *and* (2) *are also equivalent if we reverse the sign of inequality in both inequalities, in* (3.1) *and in* (3.2).

Remark 3.1. Let the conditions from the previous theorem hold. Then the following two statements are equivalent:

(1′) For every continuous concave function $f : [\alpha, \beta] \to \mathbb{R}$ the reverse inequality in (3.1) holds.

(2′) For all $s \in [\alpha, \beta]$ inequality (3.2) holds.

Moreover, the statements (1′) and (2′) are also equivalent if we change the sign of inequality in both statements (1′) and (2′).

Remark 3.2. Note that in the case when all $p_i > 0$ $(i = 1, ..., n)$, inequality (3.1) becomes discrete Jensen's inequality.

THEOREM 3.2. *Let $x_i \in [a,b] \subseteq [\alpha,\beta]$, $p_i \in \mathbb{R}$ $(i = 1,...,n)$, be such that $P_n \neq 0$ and $\overline{x} \in [\alpha,\beta]$, and let $f : [\alpha,\beta] \to \mathbb{R}$, $f \in C^2([\alpha,\beta])$. Let the functions $G_p : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ $(p = 1,2,3,4)$ be as defined in* (1.1)–(1.4)*. If for all $s \in [\alpha,\beta]$ the inequality* (3.2) *holds or if for all $s \in [\alpha,\beta]$ the reverse inequality in* (3.2) *holds, for any $p \in \{1,2,3,4\}$, then there exists $\xi \in [\alpha,\beta]$ such that*

$$f(\overline{x}) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) = \frac{1}{2} f''(\xi)\left[\overline{x}^2 - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^2\right]. \tag{3.3}$$

THEOREM 3.3. *Let $x_i \in [a,b] \subseteq [\alpha,\beta]$, $p_i \in \mathbb{R}$ $(i = 1,...,n)$, be such that $P_n \neq 0$ and $\overline{x} \in [\alpha,\beta]$, and let $f,g : [\alpha,\beta] \to \mathbb{R}$, $f,g \in C^2([\alpha,\beta])$. Let the functions $G_p : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ $(p = 1,2,3,4)$ be as defined in* (1.1)–(1.4)*. If for all $s \in [\alpha,\beta]$ the inequality* (3.2) *holds or if for all $s \in [\alpha,\beta]$ the reverse inequality in* (3.2) *holds, for any $p \in \{1,2,3,4\}$, then there exists $\xi \in [\alpha,\beta]$ such that*

$$\frac{f(\overline{x}) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i)}{g(\overline{x}) - \frac{1}{P_n}\sum_{i=1}^{n} p_i g(x_i)} = \frac{f''(\xi)}{g''(\xi)} \tag{3.4}$$

provided that the denominator of the left-hand side is nonzero.

As the consequence of the previous two theorems, now we give some further results in which we give explicit conditions on p_i, x_i $(i = 1,...,n)$ for (3.3) and (3.4) to hold, where using the properties of the functions G_p $(p = 1,2,3,4)$ we can skip the supplementary conditions on these functions.

COROLLARY 3.1. *Let $x_i \in [a,b] \subseteq [\alpha,\beta]$, $p_i \in \mathbb{R}$, $i = 1,...,n$, be such that*

$$p_1 > 0,\ p_2,...,p_n \leq 0 \text{ and } P_n > 0,$$

and let $\overline{x} \in [\alpha,\beta]$ and $f,g : [\alpha,\beta] \to \mathbb{R}$.

(i) *If $f \in C^2([\alpha,\beta])$, then there exists $\xi \in [a,b]$ such that* (3.3) *holds.*

(ii) *If $f,g \in C^2([\alpha,\beta])$, then there exists $\xi \in [a,b]$ such that* (3.4) *holds.*

Proof. The functions G_p $(p = 1,2,3,4)$ are convex, so having in mind the suppositions of the corollary, by the reverse Jensen's inequality we have that for all $s \in [\alpha,\beta]$ (3.2) holds with the reversed inequality. Now the statement of this corollary follows directly from Theorem 3.2 and Theorem 3.3. □

COROLLARY 3.2. *Let $(x_1,...,x_n)$ be monotonic $n-$tuple, $x_i \in [a,b] \subseteq [\alpha,\beta]$, $i = 1,...,n$, and let $(p_1,...,p_n)$ be real $n-$tuple such that there exists $m \in \{1,...,n\}$ such that $P_k \leq 0$ for $k < m$ and $\overline{P_k} \leq 0$ for $k > m$, where $P_n > 0$ and $\overline{x} \in [\alpha,\beta]$. Also let $f,g : [\alpha,\beta] \to \mathbb{R}$.*

(i) *If $f \in C^2([\alpha,\beta])$, then there exists $\xi \in [a,b]$ such that* (3.3) *holds.*

(ii) *If $f,g \in C^2([\alpha,\beta])$, then there exists $\xi \in [a,b]$ such that* (3.4) *holds.*

Proof. The functions G_p $(p = 1, 2, 3, 4)$ are convex, so by the reverse Jensen-Steffensen inequality we have that for all $s \in [\alpha, \beta]$ (3.2) holds with the reversed sign of inequality. Now the statement of this Corollary follows directly from Theorem 3.2 and Theorem 3.3. □

In [2] (see also [13] or [16]) one can find the result which is equivalent to the Jensen-Steffensen and the reverse Jensen-Steffensen inequality together. It is the so-called Jensen-Petrović inequality.

Here, without the proof, we give the adequate Corollary which uses that result. The proof goes the same way as in the previous two corollaries.

Corollary 3.3. *Let $x_i \in [a, b] \subseteq [\alpha, \beta]$, $i = 1, ..., n$, be such that $x_1 \leq x_2 \leq ... \leq x_n$. Let $p_i \in \mathbb{R}$, $i = 1, ..., n$ be such that $P_n = 1$ and let $f, g : [\alpha, \beta] \to \mathbb{R}$. Let any of the following conditions hold:*

(1) *$P_k \geq 0$ for $1 \leq k \leq n-1$ and $\overline{P_k} \geq 0$ for $2 \leq k \leq n$, where*

(2) *$\exists m \in \{1, ..., n\}$ such that $P_k \leq 0$ for $k < m$ and $\overline{P_k} \leq 0$ for $k > m$, and let $\overline{x} \in [\alpha, \beta]$.*

Then the following statements hold:

(i) *If $f \in C^2([\alpha, \beta])$, then there exists $\xi \in (\alpha, \beta)$ such that (3.3) holds.*

(ii) *If $f, g \in C^2([\alpha, \beta])$, then there exists $\xi \in (\alpha, \beta)$ such that (3.4) holds.*

Corollary 3.4. *Let $(x_1, ..., x_n)$ and $(p_1, ..., p_n)$ be real $n-$tuples, $n = 2m + 1, m \in \mathbb{N}$, such that $x_i \in [a, b] \subseteq [\alpha, \beta]$ for $i = 1, ..., n$, and $\frac{1}{\sum_{i=1}^{2k+1} p_i} \sum_{i=1}^{2k+1} p_i x_i \in [\alpha, \beta]$ for every $k = 1, ..., m$, and let $f, g : [\alpha, \beta] \to \mathbb{R}$. Let for every $k = 1, ..., m$ the following conditions hold:*

(1°) *$p_1 > 0$, $p_{2k} \leq 0$, $p_{2k} + p_{2k+1} \leq 0$, $\sum_{i=1}^{2k} p_i \geq 0$, $\sum_{i=1}^{2k+1} p_i > 0$;*

and

(2°) *$x_{2k} \leq x_{2k+1}$, $\sum_{i=1}^{2k} p_i(x_i - x_{2k+1}) \geq 0$.*

Then the following statements hold:

(i) *If $f \in C^2([\alpha, \beta])$, then there exists $\xi \in (\alpha, \beta)$ such that (3.3) holds.*

(ii) *$f, g \in C^2([\alpha, \beta])$, then there exists $\xi \in (\alpha, \beta)$ such that (3.4) holds.*

Proof. As the functions G_p $(p = 1, 2, 3, 4)$ are convex, by the inequality of P.M. Vasić and R.R. Janić given in [17] (see also [10]), under the suppositions of this corollary, (3.2) holds with reversed inequality for all $s \in [\alpha, \beta]$. Now the statement of this corollary follows directly from Theorem 3.2 and Theorem 3.3. □

Remark 3.3. If we change the conditions of this corollary (as in [10]) as follows:

(I) Let the condition (1°) hold and let the reverse inequalities in condition (2°) hold;

or

(II) Let instead of the conditions (1°) and (2°) the following ones hold
(3°) $p_1 > 0$, $p_{2k+1} \geq 0$, $p_{2k} + p_{2k+1} \geq 0$, $\sum_{i=1}^{2k} p_i \geq 0$, $\sum_{i=1}^{2k+1} p_i > 0$;
(4°) $x_{2k} \leq x_{2k+1}$, $\sum_{i=1}^{2k-1} p_i(x_i - x_{2k}) \leq 0$;

or

(III) Let the condition (3°) hold and let the reverse inequalities in condition (4°) hold.

Then the following inequalities are valid:

(i) If $f \in C^2([\alpha, \beta])$, then there exists $\xi \in (\alpha, \beta)$ such that (3.3) holds.

(ii) If $f, g \in C^2([\alpha, \beta])$, then there exists $\xi \in (\alpha, \beta)$ such that (3.4) holds.

(In the case (I), the proof is the same as of the previous Corollary, and in the cases (II) and (III) the statement follows from the fact that the statement (3.2) holds for all $s \in [\alpha, \beta]$ for the convex function G_p, $p = 1, 2, 3, 4$.)

Similar as in [15], analogous results can also be derived using the Jensen or the Jensen-Steffensen inequality.

4. CONVERSES OF THE JENSEN INEQUALITY

In this section, we give the results for the converse of the Jensen inequality to hold, giving the conditions on the real Stieltjes measure $\mathrm{d}\lambda$, such that $\lambda(a) \neq \lambda(b)$, allowing that the measure can also be negative, but using the condition on previously defined new Green functions. In our results, we use the similar method as in Section 2 of this paper.

THEOREM 4.1. *Let $g : [a, b] \to \mathbb{R}$ be continuous function and $[\alpha, \beta]$ be an interval such that the image of g is a subset of $[\alpha, \beta]$. Let $m, M \in [\alpha, \beta]$ $(m \neq M)$ be such that $m \leq g(t) \leq M$ for all $t \in [a, b]$. Let $\lambda : [a, b] \to \mathbb{R}$ be continuous function or the function of bounded variation, and $\lambda(a) \neq \lambda(b)$. Let the functions $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ $(p = 1, 2, 3, 4)$ be as defined in (1.1)–(1.4). Then the following two statements are equivalent:*

(1) *For every continuous convex function $\varphi : [\alpha, \beta] \to \mathbb{R}$*

$$\frac{\int_a^b \varphi\left(g(x)\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \leq \frac{M - \bar{g}}{M - m}\varphi(m) + \frac{\bar{g} - m}{M - m}\varphi(M) \tag{4.1}$$

holds.

(2) *For all $s \in [\alpha, \beta]$*

$$\frac{\int_a^b G_p\left(g(x), s\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \leq \frac{M - \bar{g}}{M - m}G_p(m, s) + \frac{\bar{g} - m}{M - m}G_p(M, s) \tag{4.2}$$

holds.

Furthermore, the statements (1) *and* (2) *are also equivalent if we reverse the sign of inequality in both* (4.1) *and* (4.2).

Proof. The idea of the proof is very similar to the proof of Theorem 2.1. Consider the Green function G_1 defined by (1.1).

(1) $\Rightarrow$ (2): Let (1) hold. As the function $G_1(\cdot, s)$ ($s \in [\alpha, \beta]$) is also continuous and convex, it follows that also for this function (4.1) holds, *i.e.* it holds (4.2) for $p = 1$.

(2) $\Rightarrow$ (1): Let (2) hold. We already know from Lemma 1.1 that we can represent every function $\varphi : [\alpha, \beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha, \beta])$, in the form (1.5), where the function G_1 is defined in (1.1), and by some calculation we can easily get that

$$\frac{\int_a^b \varphi\left(g(x)\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M - \overline{g}}{M - m}\varphi(m) - \frac{\overline{g} - m}{M - m}\varphi(M)$$

(4.3)

$$= \int_\alpha^\beta \left[\frac{\int_a^b G_1\left(g(x), s\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M - \overline{g}}{M - m} G_1(m, s) - \frac{\overline{g} - m}{M - m} G_1(M, s) \right] \varphi''(s) \mathrm{d}s.$$

If the function φ is also convex, then $\varphi''(s) \geq 0$ for all $s \in [\alpha, \beta]$. So, if for every $s \in [\alpha, \beta]$ holds (4.2) for $p = 1$, then it follows that for every convex function $\varphi : [\alpha, \beta] \to \mathbb{R}$, with $\varphi \in C^2([\alpha, \beta])$, inequality (4.1) holds.

At the end, as in the proof of Theorem 2.1, note that it is not necessary to demand the existence of the second derivative of the function φ (see [16, p. 172]). The differentiability condition can be directly eliminated by using the fact that it is possible to approximate uniformly a continuous convex function by convex polynomials.

The last part of our theorem can be proved analogously.

Cases when $p = 2, 3, 4$ are treated analogously. □

Remark 4.1. Let the conditions from the previous theorem hold. Then the following two statements are equivalent:

(1′) For every continuous concave function $\varphi : [\alpha, \beta] \to \mathbb{R}$ the reverse inequality in (4.1) holds.

(2′) For all $s \in [\alpha, \beta]$ inequality (4.2) holds.

Moreover, the statements (1′) and (2′) are also equivalent if we change the sign of inequality in both statements (1′) and (2′).

Remark 4.2. Note that in all the results in this section we allow that the mean value $\overline{g}$ goes out of the interval $[\alpha, \beta]$, while in the results from previous sections we demanded that $\overline{g} \in [\alpha, \beta]$.

For the converses of Jensen's inequality we can also give the mean value theorems of Lagrange and Cauchy type.

THEOREM 4.2. *Let $g : [a,b] \to \mathbb{R}$ be continuous function and $\varphi : [\alpha,\beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha,\beta])$, where the image of g is a subset of $[\alpha,\beta]$. Let $m, M \in [\alpha,\beta]$ $(m \neq M)$ be such that $m \leq g(t) \leq M$ for all $t \in [a,b]$. Let $\lambda : [a,b] \to \mathbb{R}$ be continuous function or the function of bounded variation, and $\lambda(a) \neq \lambda(b)$. Let the functions $G_p : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ $(p = 1,2,3,4)$ be as defined in (1.1)–(1.4). If for all $s \in [\alpha,\beta]$ the inequality (4.2) holds or if for all $s \in [\alpha,\beta]$ the reverse inequality in (4.2) holds, for any $p \in \{1,2,3,4\}$, then there exists $\xi \in [\alpha,\beta]$ such that*

$$\begin{aligned}
&\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\varphi(m) - \frac{\overline{g}-m}{M-m}\varphi(M) \\
&= \frac{1}{2}\varphi''(\xi)\left[\frac{\int_a^b (g(x))^2\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\cdot m^2 - \frac{\overline{g}-m}{M-m}\cdot M^2\right]. \qquad (4.4)
\end{aligned}$$

Proof. The idea of the proof is very similar to the proof of the Theorem 2.2. Consider the Green function G_1 defined by (1.1). Following the assumptions of our theorem, we have that the function φ'' is continuous and that

$$\frac{\int_a^b G_1(g(x),s)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}G_1(m,s) - \frac{\overline{g}-m}{M-m}G_1(M,s)$$

doesn't change its positivity on $[\alpha,\beta]$. For our function φ equality (4.3) holds, and now applying the integral mean-value theorem we get that there exists some $\xi \in [\alpha,\beta]$ such that

$$\begin{aligned}
&\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\varphi(m) - \frac{\overline{g}-m}{M-m}\varphi(M) \qquad (4.5)\\
&= \varphi''(\xi)\int_\alpha^\beta \left[\frac{\int_a^b G_1(g(x),s)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}G_1(m,s) - \frac{\overline{g}-m}{M-m}G_1(M,s)\right]\mathrm{d}s.
\end{aligned}$$

Calculating the integral on the right side in (4.5) and using (2.6), we get the statement (4.4) of our theorem.

Cases when $p = 2,3,4$ are proved analogously. □

Remark 4.3. Note that (4.4) can also be expressed as

$$\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\varphi(m) - \frac{\overline{g}-m}{M-m}\varphi(M)$$

$$= \frac{1}{2}\varphi''(\xi)\left[\frac{\int_a^b (g(x))^2\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \overline{g}(M+m) + Mm\right].$$

THEOREM 4.3. *Let $g : [a,b] \to \mathbb{R}$ be continuous function and $\varphi, \psi : [\alpha,\beta] \to \mathbb{R}$, $\varphi, \psi \in C^2([\alpha,\beta])$, where the image of g is a subset of $[\alpha,\beta]$. Let $m, M \in [\alpha,\beta]$ $(m \neq M)$ be such that $m \leq g(t) \leq M$ for all $t \in [a,b]$. Let $\lambda : [a,b] \to \mathbb{R}$ be continuous function or the function of bounded variation, and $\lambda(a) \neq \lambda(b)$. Let the functions $G_p : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ $(p = 1,2,3,4)$ be as defined in* (1.1)–(1.4)*. If for all $s \in [\alpha,\beta]$ the inequality* (4.2) *holds or if for all $s \in [\alpha,\beta]$ the reverse inequality in* (4.2) *holds, for any $p \in \{1,2,3,4\}$, then there exists $\xi \in [\alpha,\beta]$ such that*

$$\frac{\frac{\int_a^b \varphi(g(x))\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\varphi(m) - \frac{\overline{g}-m}{M-m}\varphi(M)}{\frac{\int_a^b \psi(g(x))\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\psi(m) - \frac{\overline{g}-m}{M-m}\psi(M)} = \frac{\varphi''(\xi)}{\psi''(\xi)}, \tag{4.6}$$

provided that the denominator of the left-hand side of (4.6) *is nonzero.*

Proof. The idea of the proof is very similar to the proof of Theorem 2.3. Again we define the function χ as a linear combination of functions φ and ψ.

$$\begin{aligned}\chi(t) = &\left[\frac{\int_a^b \psi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\psi(m) - \frac{\overline{g}-m}{M-m}\psi(M)\right]\cdot\varphi(t)\\ &- \left[\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\varphi(m) - \frac{\overline{g}-m}{M-m}\varphi(M)\right]\cdot\psi(t).\end{aligned}$$

Function χ is also defined on the segment $[\alpha,\beta]$, it is continuous and χ'' is continuous on $[\alpha,\beta]$. Now we can apply Theorem 4.2 on function χ and it follows that there exists some $\xi \in [\alpha,\beta]$ such that the following is valid

$$\begin{aligned}&\frac{\int_a^b \chi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\chi(m) - \frac{\overline{g}-m}{M-m}\chi(M)\\ &\qquad= \frac{1}{2}\chi''(\xi)\left[\frac{\int_a^b (g(x))^2\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\cdot m^2 - \frac{\overline{g}-m}{M-m}\cdot M^2\right].\end{aligned}$$

After a short calculation we get that the left-hand side of this equation equals to zero. The term in the square brackets on the right-hand side is different from 0, because otherwise, from Theorem 4.2 applied on the function ψ, we would have that the denominator on the left-hand side of (4.6) equals to 0. It follows that

$$\chi''(\xi) = 0.$$

Now the assertion of our theorem follows directly. □

5. DISCRETE FORM OF THE CONVERSES OF THE JENSEN INEQUALITY

In this section, we give the results for converses of Jensen's inequality in discrete case. The proofs are similar to those in the integral case given in the previous section, so we give these results here without the proofs.

Using the representation of any function $f : [\alpha, \beta] \to \mathbb{R}$, $f \in C^2([\alpha, \beta])$, in the form as given in Lemma 1.1, where the functions G_p $(p = 1, 2, 3, 4)$ are defined in (1.1)–(1.4), by some calculation we can easily show that the following holds:

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - \frac{b-\overline{x}}{b-a} f(a) - \frac{\overline{x}-a}{b-a} f(b)$$

$$= \int_{\alpha}^{\beta} \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i G_p(x_i, s) - \frac{b-\overline{x}}{b-a} G_p(a, s) - \frac{\overline{x}-a}{b-a} G_p(b, s) \right) f''(s) \mathrm{d}s.$$

We have the following result.

Theorem 5.1. *Let $x_i \in [a, b] \subseteq [\alpha, \beta]$, $a \neq b$, $p_i \in \mathbb{R}$ $(i = 1, ..., n)$ be such that $P_n \neq 0$. Let the functions $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ $(p = 1, 2, 3, 4)$ be as defined in (1.1)–(1.4). Then the following two statements are equivalent:*

(1) *For every continuous convex function $f : [\alpha, \beta] \to \mathbb{R}$*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \leq \frac{b-\overline{x}}{b-a} f(a) + \frac{\overline{x}-a}{b-a} f(b) \tag{5.1}$$

holds.

(2) *For all $s \in [\alpha, \beta]$*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i G_p(x_i, s) \leq \frac{b-\overline{x}}{b-a} G_p(a, s) + \frac{\overline{x}-a}{b-a} G_p(b, s) \tag{5.2}$$

holds.

Moreover, the statements (1) and (2) are also equivalent if we reverse the sign of inequality in both inequalities, in (5.1) and in (5.2).

Remark 5.1. If we set that all $p_i \in \mathbb{R}$ $(i = 1, ..., n)$ are positive, then (5.1) becomes classical converse of the Jensen inequality (see for example [13, p. 48]).

Remark 5.2. Let the conditions from the previous theorem hold. Then the following two statements are equivalent:

(1′) For every continuous concave function $\varphi : [\alpha, \beta] \to \mathbb{R}$ the reverse inequality in (5.1) holds.

(2′) For all $s \in [\alpha, \beta]$ inequality (5.2) holds.

Moreover, the statements $(1')$ and $(2')$ are also equivalent if we change the sign of inequality in both statements $(1')$ and $(2')$.

Setting $a = \alpha, b = \beta$ in Theorem 5.1 we get the following result.

Corollary 5.1. *Let $x_i \in [\alpha, \beta]$, $\alpha \neq \beta$, $p_i \in \mathbb{R}$ $(i = 1, ..., n)$ be such that $P_n \neq 0$. Let the functions $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ $(p = 1, 2, 3, 4)$ be as defined in (1.1)–(1.4). Then the following two statements are equivalent:*

(1) *For every continuous convex function $f : [\alpha, \beta] \to \mathbb{R}$*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \leq \frac{\beta - \overline{x}}{\beta - \alpha} f(\alpha) + \frac{\overline{x} - \alpha}{\beta - \alpha} f(\beta) \tag{5.3}$$

holds.

(2) *For all $s \in [\alpha, \beta]$*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i G_p(x_i, s) \leq 0 \tag{5.4}$$

holds.

Moreover, the statements (1) and (2) are also equivalent if we reverse the sign of inequality in both inequalities, in (5.3) and in (5.4).

Now we give the Lagrange and Cauchy type mean-value theorem for the converses of the Jensen inequality in discrete case.

Theorem 5.2. *Let $x_i \in [a, b] \subseteq [\alpha, \beta]$, $a \neq b$, $p_i \in \mathbb{R}$ $(i = 1, ..., n)$ be such that $P_n \neq 0$ and let $f : [\alpha, \beta] \to \mathbb{R}$, $f \in C^2([\alpha, \beta])$. Let the functions $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ $(p = 1, 2, 3, 4)$ be as defined in (1.1)–(1.4). If for all $s \in [\alpha, \beta]$ the inequality (5.2) holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in (5.2) holds, for any $p \in \{1, 2, 3, 4\}$, then there exists $\xi \in [\alpha, \beta]$ such that*

$$\begin{aligned}\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) &- \frac{b - \overline{x}}{b - a} f(a) - \frac{\overline{x} - a}{b - a} f(b) \\ &= \frac{1}{2} f''(\xi) \left[\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^2 - \frac{b - \overline{x}}{b - a} a^2 - \frac{\overline{x} - a}{b - a} b^2\right].\end{aligned} \tag{5.5}$$

Remark 5.3. Note that (5.5) can also be expressed as

$$\begin{aligned}\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) &- \frac{b - \overline{x}}{b - a} f(a) - \frac{\overline{x} - a}{b - a} f(b) \\ &= \frac{1}{2} f''(\xi) \left[\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^2 - \overline{x}(a + b) + ab\right].\end{aligned}$$

Theorem 5.3. *Let* $x_i \in [a,b] \subseteq [\alpha,\beta]$, $a \neq b$, $p_i \in \mathbb{R}$ $(i = 1, ..., n)$ *be such that* $P_n \neq 0$ *and let* $f, g : [\alpha,\beta] \to \mathbb{R}$, $f, g \in C^2([\alpha,\beta])$.

Let the functions $G_p : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ $(p = 1,2,3,4)$ *be as defined in* (1.1)–(1.4). *If for all* $s \in [\alpha,\beta]$ *the inequality* (5.2) *holds or if for all* $s \in [\alpha,\beta]$ *the reverse inequality in* (5.2) *holds, for any* $p \in \{1,2,3,4\}$, *then there exists* $\xi \in [\alpha,\beta]$ *such that*

$$\frac{\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) - \frac{b-\overline{x}}{b-a} f(a) - \frac{\overline{x}-a}{b-a} f(b)}{\frac{1}{P_n}\sum_{i=1}^n p_i g(x_i) - \frac{b-\overline{x}}{b-a} g(a) - \frac{\overline{x}-a}{b-a} g(b)} = \frac{f''(\xi)}{g''(\xi)}$$

provided that the denominator of the left-hand side is nonzero.

6. THE HERMITE-HADAMARD INEQUALITY

The classical Hermite-Hadamard inequality states that for a convex function $f : [a,b] \to \mathbb{R}$ the following estimation holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{6.1}$$

Its weighted form is proved by L. Fejér in [5], and A. M. Fink in [6] discussed on its generalization (separately looking its left and right side inequality) considering certain signed measures. In paper [7], authors gave complete characterization for the right side inequality ([7], Theorem 1), which in fact as special case follows from the result already given in [14].

As a consequence of our results given in Section 2 and 4, here we give the complete characterization for the left and the right side of the generalized Hermite-Hadamard inequality for the real Stieltjes measure.

As a consequence of our results given in Section 2, setting the function g as $g(x) = x$, follow the results for the left-side inequality of the generalized Hermite-Hadamard inequality.

In order to simplify the notation we denote:

$$\tilde{x} = \frac{\int_a^b x\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}.$$

Corollary 6.1. *Let* $\lambda : [a,b] \to \mathbb{R}$ *be continuous function or the function of bounded variation, and* $\lambda(a) \neq \lambda(b)$. *Let* $[\alpha,\beta] \subseteq \mathbb{R}$ *be such that* $[a,b] \subseteq [\alpha,\beta]$ *and* $\tilde{x} \in [\alpha,\beta]$. *Let the functions* $G_p : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ $(p = 1,2,3,4)$ *be as defined in* (1.1)–(1.4). *Then the following two statements are equivalent:*

(1) *For every continuous convex function* $f : [\alpha,\beta] \to \mathbb{R}$

$$f(\tilde{x}) \leq \frac{\int_a^b f(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \tag{6.2}$$

holds.

(2) *For all* $s \in [\alpha, \beta]$

$$G_p(\tilde{x}, s) \leq \frac{\int_a^b G_p(x, s)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}, \tag{6.3}$$

holds.

Furthermore, the statements (1) *and* (2) *are also equivalent if we reverse the signs of inequality in both* (6.2) *and* (6.3).

Analogous remark as Remark 2.1 here also holds.

Note that for the left-side inequality of the generalized Hermite-Hadamard inequality it is necessary to demand that $\tilde{x} \in [\alpha, \beta]$.

Remark 6.1. For $\lambda(x) = x$, it is $\int_a^b \mathrm{d}\lambda(x) = b - a$ and $\tilde{x} = \frac{a+b}{2}$, so (6.2) becomes the left inequality in the classical Hermite-Hadamard inequality (6.1).

For the left-hand side of the generalized Hermite-Hadamard inequality we can also derive adequate mean-value theorems of the Lagrange and Cauchy type.

Corollary 6.2. *Let* $\lambda : [a, b] \to \mathbb{R}$ *be continuous function or the function of bounded variation,* $\lambda(a) \neq \lambda(b)$*, and* $f : [\alpha, \beta] \to \mathbb{R}$*,* $f \in C^2([\alpha, \beta])$*, where* $[a, b] \subseteq [\alpha, \beta]$ *and* $\tilde{x} \in [\alpha, \beta]$*. Let the functions* $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ $(p = 1, 2, 3, 4)$ *be as defined in* (1.1)–(1.4)*. If for all* $s \in [\alpha, \beta]$ *the inequality* (6.3) *holds or if for all* $s \in [\alpha, \beta]$ *the reverse inequality in* (6.3) *holds, for any* $p \in \{1, 2, 3, 4\}$*, then there exists* $\xi \in [\alpha, \beta]$ *such that*

$$f(\tilde{x}) - \frac{\int_a^b f(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} = \frac{1}{2}\varphi''(\xi)\left[\tilde{x}^2 - \frac{\int_a^b x^2 \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right].$$

It's easy to see that for $\lambda(x) = x$, the condition (6.3) is always fulfilled. It is $\tilde{x} = \frac{a+b}{2} \in [a, b]$, so we can narrow the interval we are looking from $[\alpha, \beta]$ to $[a, b]$. In that case, the previous corollary gives us that for any function $f \in C^2([a, b])$ there exists $\xi \in [a, b]$ such that the following is valid

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \frac{1}{12}f''(\xi)\left(a^2 + 4ab + b^2\right).$$

Corollary 6.3. *Let* $\lambda : [a, b] \to \mathbb{R}$ *be continuous function or the function of bounded variation,* $\lambda(a) \neq \lambda(b)$*, and* $f, g : [\alpha, \beta] \to \mathbb{R}$*,* $f, g \in C^2([\alpha, \beta])$*, where* $[a, b] \subseteq [\alpha, \beta]$ *and* $\tilde{x} \in [\alpha, \beta]$*. Let the functions* $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$

$(p=1,2,3,4)$ *be as defined in* (1.1)–(1.4). *If for all* $s \in [\alpha, \beta]$ *the inequality* (6.3) *holds or if for all* $s \in [\alpha, \beta]$ *the reverse inequality in* (6.3) *holds, for any* $p \in \{1,2,3,4\}$, *then there exists* $\xi \in [\alpha, \beta]$ *such that*

$$\frac{f(\tilde{x}) - \frac{\int_a^b f(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}}{g(\tilde{x}) - \frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}} = \frac{f''(\xi)}{g''(\xi)},$$

provided that the denominator of the left-hand side is nonzero.

Similarly, from the results given in the fourth section we get the results for the right-side inequality of the generalized Hermite-Hadamard inequality. Here we allow that the mean value $\tilde{x}$ goes outside of the interval $[\alpha, \beta]$.

COROLLARY 6.4. *Let* $\lambda : [a,b] \to \mathbb{R}$ *be continuous function or the function of bounded variation, and* $\lambda(a) \neq \lambda(b)$. *Let* $[\alpha, \beta] \subseteq \mathbb{R}$ *be such that* $[a,b] \subseteq [\alpha, \beta]$. *Let the functions* $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ $(p=1,2,3,4)$ *be as defined in* (1.1)–(1.4). *Then the following two statements are equivalent:*

(1) *For every continuous convex function* $f : [\alpha, \beta] \to \mathbb{R}$

$$\frac{\int_a^b f(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \leq \frac{b-\tilde{x}}{b-a} f(a) + \frac{\tilde{x}-a}{b-a} f(b) \tag{6.4}$$

holds.

(2) *For all* $s \in [\alpha, \beta]$

$$\frac{\int_a^b G_p(x,s)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \leq \frac{b-\tilde{x}}{b-a} G_p(a,s) + \frac{\tilde{x}-a}{b-a} G_p(b,s) \tag{6.5}$$

holds.

Furthermore, the statements (1) *and* (2) *are also equivalent if we reverse the signs of inequality in both* (6.4) *and* (6.5).

Of course, here also analogous remark as Remark 4.1 holds.

Furthermore, setting $\alpha = a$ and $\beta = b$ in previous corollary, we get that the right side of the inequality (6.5) equals to zero, so (6.5) becomes

$$\frac{\int_a^b G_p(x,s)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \leq 0.$$

Remark 6.2. For $\lambda(x) = x$ inequality (6.4) becomes the right-side inequality in the classical Hermite-Hadamard inequality (6.1).

Also for the right-side of the generalized Hermite-Hadamard inequality we can derive adequate results of the Lagrange and Cauchy type.

Corollary 6.5. *Let $\lambda : [a,b] \to \mathbb{R}$ be continuous function or the function of bounded variation, $\lambda(a) \neq \lambda(b)$, and $f : [\alpha,\beta] \to \mathbb{R}$, $f \in C^2([\alpha,\beta])$, where $[a,b] \subseteq [\alpha,\beta]$. Let the functions $G_p : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ $(p = 1,2,3,4)$ be as defined in* (1.1)–(1.4). *If for all $s \in [\alpha,\beta]$ the inequality* (6.5) *holds or if for all $s \in [\alpha,\beta]$ the reverse inequality in* (6.5) *holds, for any $p \in \{1,2,3,4\}$, then there exists $\xi \in [\alpha,\beta]$ such that*

$$\frac{\int_a^b f(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{b-\tilde{x}}{b-a}f(a) - \frac{\tilde{x}-a}{b-a}f(b) = \frac{1}{2}f''(\xi)\left[\frac{\int_a^b x^2\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{b-\tilde{x}}{b-a}\cdot a^2 - \frac{\tilde{x}-a}{b-a}\cdot b^2\right]. \tag{6.6}$$

Note that (6.6) can also be expressed as

$$\frac{\int_a^b f(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{b-\tilde{x}}{b-a}f(a) - \frac{\tilde{x}-a}{b-a}f(b) = \frac{1}{2}f''(\xi)\left[\frac{\int_a^b x^2\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \tilde{x}(a+b) + ab\right].$$

For $\lambda(x) = x$ the condition (6.5) is always fulfilled. We can also narrow the interval we are looking from $[\alpha,\beta]$ to $[a,b]$. In that case, the previous corollary gives us that for any function $f \in C^2([a,b])$ there exists some $\xi \in [a,b]$ such that the following is valid

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{f(a)+f(b)}{2} = -\frac{1}{12}f''(\xi)\,(a-b)^2.$$

Corollary 6.6. *Let $\lambda : [a,b] \to \mathbb{R}$ be continuous function or the function of bounded variation, $\lambda(a) \neq \lambda(b)$, and $f,g : [\alpha,\beta] \to \mathbb{R}$, $f,g \in C^2([\alpha,\beta])$, where $[a,b] \subseteq [\alpha,\beta]$. Let the functions $G_p : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ $(p = 1,2,3,4)$ be as defined in* (1.1)–(1.4). *If for all $s \in [\alpha,\beta]$ the inequality* (6.5) *holds or if for all $s \in [\alpha,\beta]$ the reverse inequality in* (6.5) *holds, for any $p \in \{1,2,3,4\}$, then there exists $\xi \in [\alpha,\beta]$ such that*

$$\frac{\frac{\int_a^b f(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{b-\tilde{x}}{b-a}f(a) - \frac{\tilde{x}-a}{b-a}f(b)}{\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{b-\tilde{x}}{b-a}g(a) - \frac{\tilde{x}-a}{b-a}g(b)} = \frac{f''(\xi)}{g''(\xi)}, \tag{6.7}$$

provided that the denominator of the left-hand side of (6.7) *is nonzero.*

At the end, as a consequence of Corollary 6.1 and Corollary 6.4, we get the necessary and sufficient conditions on a real Stieltjes measure, given using the Green function G_p $(p = 1, 2, 3, 4)$ that for any convex function the generalization of the Hermite-Hadamard inequality holds.

Corollary 6.7. *Let $\lambda : [a, b] \to \mathbb{R}$ be continuous function or the function of bounded variation, and $\lambda(a) \neq \lambda(b)$. Let $[\alpha, \beta] \subseteq \mathbb{R}$ be such that $[a, b] \subseteq [\alpha, \beta]$ and $\tilde{x} \in [\alpha, \beta]$. Let the functions $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ $(p = 1, 2, 3, 4)$ be as defined in* (1.1)–(1.4)*. Then the following two statements are equivalent:*

(1) *For every continuous convex function $f : [\alpha, \beta] \to \mathbb{R}$*

$$f(\tilde{x}) \leq \frac{\int_a^b f(x)\, \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \leq \frac{b - \tilde{x}}{b - a} f(a) + \frac{\tilde{x} - a}{b - a} f(b) \tag{6.8}$$

holds.

(2) *For all $s \in [\alpha, \beta]$*

$$G_p(\tilde{x}, s) \leq \frac{\int_a^b G_p(x, s)\, \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \leq \frac{b - \tilde{x}}{b - a} G_p(a, s) + \frac{\tilde{x} - a}{b - a} G_p(b, s), \tag{6.9}$$

holds.

Furthermore, the statements (1) *and* (2) *are also equivalent if we reverse the signs of inequality in both* (6.8) *and* (6.9)*.*

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